International Journal of Applied Mathematical Research, 11 (2) (2022) 14-34

International Journal of Applied Mathematical Research

Website: www.sciencepubco.com/index.php/IJAMR

Research paper

No-regret Control for a degenerate population model in divergence form with incomplete data: a nonlinear case

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Abstract

We deal with a degenerate population equation in divergence form depending on time, on age and on space. In this model, the birth rate is unknown. We focus on the No-regret control and on the Low-regret control concepts of J. L. Lions treated in [*Contrôle à moindres regrets des systèmes distribués*, C. R. Acad. Sci.Paris Ser. I Math., SIAM J. Control Optim. ,1992, Vol 315, pp. 1253–1257] and in [*Duality arguments for multi agents least regret control*, Institut de France, 1999] to treat the problem. For this purpose, we prove first the existence of the Low-regret control and the No-regret control. And we use a suitable Hilbert space to show that the No-regret control is the limit of a family of adapted Low-regret controls defined by a quadratic pertubation and previously used by Nakoulima et al. in [*Perturbations à moindres regrets dans les systèmes distribués à données manquantes*, C. R. Acad. Sci.Paris Ser. I Math., 2000, Vol 330, pp. 801–806]. Then we give a singular optimality system for the family of adapted Low-regret controls and for the No-regret control.

Keywords: *Population dynamics, degenerate equation, incomplete data, Low-regret control, No-regret control.*

1. Introduction

We consider a degenerate population model in its divergence form with incomplete data:

$$\begin{cases} \dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}-(k(x)y_x)_x+\mu y = f+v\chi_\omega & \text{in } Q \\ y(t,a,1)=y(t,a,0) = 0 & \text{on } Q_{T,A} \\ y(0,a,x) = y^0(a,x) & \text{in } Q_{A,1} \\ y(t,0,x) = \displaystyle\int_0^A g(a)y(t,a,x)da & \text{in } Q_{T,1} \end{cases} \tag{1}$$

where $Q_{T,A}=(0,T)\times(0,A)$, $Q=Q_{T,A}\times(0,1)$, $Q_{A,1}=(0,A)\times(0,1)$, $Q_{T,1}=(0,T)\times(0,1)$, and $Q_\omega=Q_{T,A}\times\omega$ where the subset $\omega\subset(0,1)$ is the region where a control v is acting. This control corresponds to an external supply or to removal of individuals on the subdomain ω. In this model, $y(t,a,x)$ is the distribution of certain individuals at the point $x\in(0,1)$, at time $t\in(0,T)$, where T is fixed, and age $a\in(0,A)$, A being the life expectancy, μ denotes the natural rate of fertility, respectively. The formula $\int_0^A g(a)y(t,a,x)da$ is the proportion of newborns at time t and at location x. But this proportion is unknown because the unknown term $g=g(a)\in G$ denoting the natural rate of fertility where $G\subset L^2(0,A)$. In this model, χ_ω is the characteristic function of the control domain ω; the initial distribution of individuals $y^0=y^0(a,x)\in L^2(Q_{A,1})$; the given function $f\in L^2(Q)$ corresponds to an external supply. The function k denotes the dispersion coefficient and we assume that it depends on the space variable x and degenerates at the boundary.
In the follow, we define the following notions:

Definition 1.1. *We say that the function k is **weakly degenerate (W.D.)** if $k\in W^{1,1}([0,1])$, $k>0$ in (0,1) and $k(0)=k(1)=0$, for all $x\in[0,1]$, there exists two constants $\alpha,\delta\in(0,1)$ such that $xk'(x)\le\alpha k(x)$ and $(x-1)k'(x)\le\delta k(x)$.*

Definition 1.2. *We say that the function k is **Strongly degenerate (S.D.)** if $k\in W^{1,\infty}([0,1]),k>0$ in (0,1) and $k(0)=k(1)=0$, for all $x\in[0,1]$, there exists two constants $\alpha,\delta\in[1,2)$ such that $xk'(x)\le\alpha k(x)$ and $(x-1)k'(x)\le\delta k(x)$.*

During the recent years, population dynamics models have been widely studied by several authors from many points of view.
The majority of them have investigated the null controllability of the system for example [10], [6], [7], [8]. Indeed, y can represent the distribution of a damaging insect population or of a pest population, for example [21]. Thus it is important to control it. In [21], system (1) models an insect growth and the control corresponds to a removal of individuals by using pesticides. Authors in [10], are concerned with the null controllability of a population dynamics model with an interior degenerate diffusion. To this end, they proved first a new Carleman estimate for the full adjoint system and afterwards they deduce a suitable observability inequality which will be needed to establish the existence of a control acting on a subset of the space which lead the population to extinction in a finite time.G. Fragnelli in [7] and [8] deals with a degenerate model describing the dynamics of a population depending on time, on age and on space in divergence form. she assumes that the degeneracy can occur at the boundary or in the interior of the space domain and she focus on null controllability problem. To this aim, she proves first Carleman estimates for the associated adjoint problem, then, via cut off functions, she proves the existence of a null control function localized in the interior of the space domain in the both papers. In the second one, she considers two cases: either the control region contains the degeneracy point x_0, or the control region is the union of two intervals each of them lying on one side of x_0. Whereas in [6], the same previous research is done but on a degenerate population equation in non-divergence form. In [14], the authors study the null controllability by one control force of a nonlinear coupled of two models describing the dynamics of two species living together on the same spatial domain. Firstly they establish a Carleman type inequality for the adjoint system of an intermediate model. From this inequality, they derive an observability inequality. More, they prove the existence of a control acting on a subset of the domain that leads the uncontrolled species to extinction in a finite time, by a fixed point argument.
In such cases, it is well known from the general theory that all nontrivial solutions of the corresponding system (commonly named Lotka–McKendrick systems) are asymptotically exponentially growing or decaying, according to the size of a certain biological quantity (the so called net reproduction rate), see [19] and also [5] for related results concerning time-independent steady states. In [1], authors consider the control problem for a population dynamics model with age dependence, spatial structure, and a nonlocal birth process arising as a boundary condition. They examine the controllability at a given time T and show that approximate controllability holds for every fixed finite time T. As a consequence, a new uniqueness condition continuation result is proved.
As far as we know, the first controllability result for an age population dynamics model is established in [1], where the authors proved that a set of profiles is approximately reachable. Later, in [3], a local exact controllability was proved. In particular, in [2], the authors showed that, if the initial distribution is small enough, one can find a control that leads the population to extinction. In the last one, the null controllability is also studied for nonlinear diffusive dynamic populations when the fertility and the mortality rates depend on the total population. In [18], the authors considered a nonlinear distribution of newborns of the form $F\left(\int_0^A \beta(t,a,x)y(t,a,x)da\right)$.
However, in all the previous papers, the dispersion coefficient k is a constant or a strictly positive function.
To our best knowledge, [4] is the first paper where the dispersion coefficient, which depends on the space variable x, can degenerate. In particular, the authors assume that k degenerates at the boundary (for example $k(x) = x^r$, being $x \in (0,1)$ and $r > 0$). Using Carleman estimates for the adjoint problem, the authors prove null controllability for (1) under the condition $T \geq A$. However, this assumption is not realistic when A is too large. To overcome this problem in [20], the authors used Carleman estimates and a fixed point method via the Leray-Schauder Theorem.
Some authors are interested with systems with incomplete data. For example, B.Jacob and A.Omrane in [11] are concerned with the control for a linear age-structured population dynamics of incomplete initial data. More precisely, the initial population age distribution is supposed to be unknown. they here generalize the notion of No-regret control of J.L.Lions in [12] to such singular population dynamics, following the method by Nakoulima et al. as in [17]. They prove that the problem they are considering has a unique No-regret control that they characterize by a singular optimality system.
In the present paper, we are interested with the No-regret control of a degenerate population dynamics system describing a single species in divergence form with unknown information on the natural birth rate which to our knowledge has not been treated. In which, the dispersion k is not considered as a constant but as a function depending on space variable x. We consider the minimization of the following cost functional:

$$J(v;g) = \|y(v;g) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q_\omega)} \tag{2}$$

where $z_d \in L^2(Q)$ and $N > 0$ are given. We have to solve the following optimization problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} J(v;g)$$

But observing that we could have

$$\sup_{g\in L^2(0,A)} J(v;g) = +\infty,$$

we consider the following problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} (J(v;g) - J(0;g)) \tag{3}$$

One then looks for the control not making things worse with respect to a nominal control v_0 (or to than doing nothing, $v_0 = 0$ in our case), independently of the perturbations which may be of infinite number. Lions used the notions of No-regret control [12] in application to the control of systems having missing data. The No-regret concept was previously used in statistics by Savage [13]. The No-regret control which is related to incomplete data problems is difficult to characterize directly. We will use an approximate control: the Low-regret control. To achieve the No-regret control, we give the singular optimality system for the Low-regret control for problem (1)–(2), using a quadratic perturbation used by Nakoulima et al. in [17]. Then, we give a characterization of the No-regret control which appears clearly as the limit of a standard control problem.

The paper is organized as follows. In Section 2, we give well-posedness and some regularity results.We study the Low-regret and the adapted Low-regret control and its characterization in sections 3 and 4 respectively. Finally, we prove the existence of the No-regret control and its characterization in the last section.

2. Well-posedness result and preliminaries

In this section and for what below, we will assume that k satisfies the following hypotheses:

$$k \in C([0;1]) \cap C^1((0;1]) \ ; \ k > 0 \ \text{ in } \ (0;1], \ k(0) = 0; \tag{4}$$

there exists a constant $\alpha \in [0,1)$ such that $xk'(x) \leq \alpha k(x)$, $\forall x \in [0;1]$
In addition, we assume that the function μ satisfies the following conditions:

$$\mu \in C(\bar{Q}) \text{ and } \mu \geq 0 \text{ in } Q. \tag{5}$$

and the function $g = g(a)$ is not completely known but satisfies the following conditions:

$$g(a) \in C(\bar{Q}_{T,1}) \text{ and } g(a) \geq 0 \text{ in } Q_{T,1}, \forall a \in (0,A) \tag{6}$$

To prove that problem (1) is well posed, we introduce the following Sobolev spaces:

$$H_k^1 = \left\{ u \in L^2(0,1) | u \ \text{ absolutely continuous in } \ [0,1], \sqrt{k} u_x \in L^2(0,1) \ \text{ and } \ u(1) = u(0) = 0 \right\}$$

and

$$H_k^2 = \left\{ u \in H_k^1(0,1) | \ (ku_x)_x \in L^2(0,1) \right\}.$$

with their respective norms:

$$\|u\|^2_{H_k^1(0,1)} = \|u\|^2_{L^2(0,1)} + \|\sqrt{k} u_x\|^2_{L^2(0,1)} \quad \forall u \in H_k^1(0,1)$$

$$\|u\|^2_{H_k^2(0,1)} = \|u\|^2_{H_k^1(0,1)} + \|(ku_x)_x\|^2_{L^2(0,1)} \quad \forall u \in H_k^2(0,1)$$

Let the operator $A : D(A) = H_k^2(0,1) \to L^2(0,1)$ defined by $Au = \{(k(x)u_x)_x, \ u \in D(A)\}$, closed, self-adjoint and negative with dense domain in $L^2(0,1)$.
Using properties of the operator A, one can show as in [[15], [9]] the existence of a unique solution of the model (1) and that this solution is generated by a C_0-semigroup on the space $L^2((0,A) \times (0,1))$. Moreover, this solution has additional time, age and gene regularity. More precisely, the following well-posedness result holds.

Theorem 2.1. *Under the hypothesis (4)-(6) and for all $f \in L^2(Q)$, $y_0 \in L^2(Q_{A,1})$ and $v \in L^2(Q_\omega)$, the system (1) admits a unique solution*

$$y \in \mathscr{E} = \mathscr{C}([0,T], L^2(Q_{A,1})) \cap \mathscr{C}([0,A], L^2(Q_{T,1})) \cap L^2(Q_{T,A}, H_k^1(0,1))$$

and

$$\sup_{t \in [0,T]} \|y(t)\|^2_{L^2(Q_{A,1})} + \sup_{a \in [0,A]} \|y(a)\|^2_{L^2(Q_{T,1})} + \int_0^T \int_0^A \|\sqrt{k} y_x\|^2_{L^2(0,1)} \mathrm{d}a\mathrm{d}t \leq C\|y_0\|^2_{L^2(Q_{A,1})} + C\|f\|^2_{L^2(Q)} + C\|v\|^2_{L^2(Q_\omega)} \tag{7}$$

where C is a positive constant independent of k, y_0 and f.

The properties of operator A allow us also to define the root of the operator $B = -A$ denoted by $B^{\frac{1}{2}}$. Using the weighted spaces $H_k^1(0,1)$ and $H_k^2(0,1)$ introduced above, one can show that $D(B^{\frac{1}{2}}) = H_k^1(0,1)$.
The proof is analogous to that of [[16], Propositions 3.5.1 and 3.6.1]
Moreover, the following result is needed in the sequel.

Proposition 2.1. *The operator B defined above has a unique extension $B \in \mathscr{L}(H_k^1(0,1), H_k^{-1}(0,1))$ where $H_k^{-1}(0,1)$ denotes the dual space of $H_k^1(0,1)$ with respect to the pivot space $L^2(0,1)$.*

For the proof, see [[16], Corollary 3.4.6]

Lemma 1. *For any $y \in \mathscr{W}(T,A) = \left\{ y \in L^2(Q_{T,A}; H_k^1(0,1)) \text{ such that } \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} \in L^2(Q_{T,A}; H_k^{-1}(0,1)) \right\}$, one can define the trace at $t = t_0$ in $L^2(Q_{A,1})$. One can define also the trace at $a = a_0$ in $L^2(Q_{T,1})$. The applications “trace” are continuous for weak and strong topologies.*

For more details on the latter lemma, see Oumar in [Sur un problème de dynamique de populations(2003)]

Remark 1.
1. *The space $H_k^1(0,1)$ is compactly embedded in $L^2(0,1)$. See Alabau-Boussira in [Carleman estimates for degenerate parabolic operators with applications to null controllability(2006)]*
2. *$\mathscr{W}(T,A) \subset C([0,T], L^2(Q_{A,1}))$ and $\mathscr{W}(T,A) \subset C([0,A], L^2(Q_{T,1}))$. See Langlais in [Solutions fortes pour une classe de problèmes aux limites dégénérés(1979)]*

Proposition 2.2. *The mapping*

$$\begin{array}{rcl} y : L^2(Q_\omega) \times L^2(0,A) & \longrightarrow & L^2((0,T) \times (0,A); H_k^1(0,1)) \\ (v,g) & \longmapsto & y(v,g) \end{array}$$

is continuous.

Proof 1. *Let be $(v_0,g_0)\in L^2(Q_\omega)\times L^2(0,A)$. Let us show that $\lim\limits_{(v,g)\to(v_0,g_0)} y(v,g)=y(v_0,g_0)$. We set $\bar y=y(v,g)-y(v_0,g_0)$. then $\bar y$ is solution of :*

$$\begin{cases} \frac{\partial \bar y}{\partial t}+\frac{\partial \bar y}{\partial a}-(k(x)\bar y_x)_x+\mu\bar y &= (v-v_0)\chi\omega & \text{in} & Q\\ \bar y(t,a,1)=\bar y(t,a,0) &= 0 & \text{on} & Q_{T,A}\\ \bar y(0,a,x) &= 0 & \text{in} & Q_{A,1}\\ \bar y(t,0,x) &= \int_0^A [g(a)y(t,a,x;v,g)-g_0(a)y(t,a,x;v_0,g_0)]da & \text{in} & Q_{T,1}\end{cases}$$

If we set $z=e^{-rt}\bar y$ with $r>0$, we get that z is solution of:

$$\begin{cases} \frac{\partial z}{\partial t}+\frac{\partial z}{\partial a}-(k(x)z_x)_x+(\mu+r)z &= (v-v_0)e^{-rt}\chi\omega & \text{in} & Q\\ z(t,a,1)=z(t,a,0) &= 0 & \text{on} & Q_{T,A}\\ z(0,a,x) &= 0 & \text{in} & Q_{A,1}\\ z(t,0,x) &= e^{-rt}\int_0^A [g(a)y(t,a,x;v,g)-g_0(a)y(t,a,x;v_0,g_0)]da & \text{in} & Q_{T,1}\end{cases} \tag{8}$$

Multiply the first equation of (8) by z then integrate by parts on Q:

$$\frac{1}{2}\|z(T,.,.)\|^2_{L^2(Q_{A,1})}-\frac{1}{2}\|z(0,.,.)\|^2_{L^2(Q_{A,1})}+\frac{1}{2}\|z(.,A,.)\|^2_{L^2(Q_{T,1})}-\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})}+\|\sqrt{k(x)}z_x\|^2_{L^2(Q)}+\|\sqrt{r+\mu}z\|^2_{L^2(Q)}$$
$$=\int_{Q_\omega} e^{-rt}z(v-v_0)\text{dtdadx}$$

$$\Longrightarrow -\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})}+\|\sqrt{k(x)}z_x\|^2_{L^2(Q)}+\|\sqrt{r+\mu}z\|^2_{L^2(Q)}\le\int_{Q_\omega} z(v-v_0)\text{dtdadx}$$
$$\Longrightarrow\|\sqrt{k(x)}z_x\|^2_{L^2(Q)}+\|\sqrt{r+\mu}z\|^2_{L^2(Q)}\le\frac{1}{2}\|z(.,0,.)\|^2_{L^2(Q_{T,1})}+\frac{1}{2}\|z\|^2_{L^2(Q)}+\frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}$$

Now, observing that:

$$z(t,0,x)=e^{-rt}\int_0^A[g(a)-g_0(a)]y(t,a,x;v,g)da+\int_0^A g_0(a)z(t,a,x)da$$

then, we make a few calculations on this latter equation. We obtain this estimation:

$$\|z(.,0,.)\|^2_{L^2(Q_{T,1})}\le\|g(a)-g_0(a)\|^2_{L^2(0,A)}\|y\|^2_{L^\infty(0,A)}+\|g_0(a)\|^2_{L^2(0,A)}\|z\|^2_{L^2(Q)}$$

Hence,

$$\|\sqrt{k(x)}z_x\|^2_{L^2(Q)}+\|\sqrt{r+\mu}z\|^2_{L^2(Q)}\le\frac{1}{2}\|g(a)-g_0(a)\|^2_{L^2(0,A)}\|y\|^2_{L^\infty(0,A)}+\frac{1}{2}\|g_0(a)\|^2_{L^2(0,A)}\|z\|^2_{L^2(Q)}+\frac{1}{2}\|z\|^2_{L^2(Q)}+\frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}$$

$$\Longrightarrow\|\sqrt{k(x)}z_x\|^2_{L^2(Q)}+\left[r+\mu-\frac{1}{2}-\frac{1}{2}\|g_0(a)\|^2_{L^2(0,A)}\right]\|z\|^2_{L^2(Q)}\le\frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}+\frac{1}{2}\|g(a)-g_0(a)\|^2_{L^2(0,A)}\|y\|^2_{L^\infty(0,A)}$$

We will choose r such that $r+\mu>\frac{1+\|g_0(a)\|^2_{L^2(0,A)}}{2}$. Thus

$$\|z\|^2_{L^2((0,T)\times(0,A);H^1_k(0,1))}\le\frac{1}{2}\|v-v_0\|^2_{L^2(Q_\omega)}+\frac{1}{2}\|g(a)-g_0(a)\|^2_{L^2(0,A)}\|y\|^2_{L^\infty(0,A)}$$

We deduce

$$\|z\|_{L^2((0,T)\times(0,A);H^1_k(0,1))}\le\frac{\sqrt{2}}{2}\|v-v_0\|_{L^2(Q_\omega)}+\frac{\sqrt{2}}{2}\|g(a)-g_0(a)\|_{L^2(0,A)}\|y\|_{L^\infty(0,A)}$$

By substitution

$$\|\bar y\|_{L^2((0,T)\times(0,A);H^1_k(0,1))}\le\frac{\sqrt{2}}{2}e^{rT}\|v-v_0\|_{L^2(Q_\omega)}+\frac{\sqrt{2}}{2}e^{rT}\|g(a)-g_0(a)\|_{L^2(0,A)}\|y\|_{L^\infty(0,A)}$$

Now, passing to the limit in this latter inequality when $(v,g)\to(v_0,g_0)$, $\bar y\to 0$ strongly in $L^2(Q_{T,A};H^1_k(0,1))$. We obtain that $y(v,g)$ converges to $y(v_0,g_0)$ in $L^2(Q_{T,A};H^1_k(0,1))$.

Proposition 2.3. *Let $\lambda>0$. Let $g,h\in L^2(0,A)$ and $v,w\in L^2(Q\omega)$. Let $y=y(v,g)$ solution of system (1) and the application $(v,g)\mapsto y(v,g)$ continuous from $L^2(Q_\omega)\times L^2(0,A)$ onto $L^2(Q_{T,A};H^1_k(0,1))$. Set $\bar y_\lambda=\dfrac{y(v+\lambda w,g+\lambda h)-y(v,g)}{\lambda}$. Then $(\bar y_\lambda)$ converges strongly in $L^2(Q_{T,A};H^1_k(0,1))$ to a function $\bar y$ as $\lambda\to 0$ which verifies*

$$\begin{cases} \frac{\partial \bar y}{\partial t}+\frac{\partial \bar y}{\partial a}-(k(x)\bar y_x)_x+\mu\bar y &= w\chi\omega & \text{in} & Q\\ \bar y(t,a,1)=\bar y(t,a,0) &= 0 & \text{on} & Q_{T,A}\\ \bar y(0,a,x) &= 0 & \text{in} & Q_{A,1}\\ \bar y(t,0,x) &= \int_0^A g(a)\bar y da+\int_0^A h(a)y(t,a,x;v,g)da & \text{in} & Q_{T,1}\end{cases}$$

Proof 2. *Let $g,h \in L^2(0,A)$ and $v,w \in L^2(Q\omega)$. We define $\bar{y}_\lambda = \dfrac{y(v+\lambda w, g+\lambda h) - y(v,g)}{\lambda}$, then $\bar{y}_\lambda$ is solution of*

$$\begin{cases} \frac{\partial \bar{y}_\lambda}{\partial t} + \frac{\partial \bar{y}_\lambda}{\partial a} - (k(x)\bar{y}_{\lambda_x})_x + \mu \bar{y}_\lambda = w\chi_\omega & \text{in} \quad Q \\ \bar{y}_\lambda(t,a,1) = \bar{y}_\lambda(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ \bar{y}_\lambda(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ \bar{y}_\lambda(t,0,x) = \int_0^A g(a)\bar{y}_\lambda da + \int_0^A h(a)y(t,a,x;v+\lambda w, g+\lambda h)da & \text{in} \quad Q_{T,1} \end{cases}$$

Set $y_\lambda = \bar{y}_\lambda - \bar{y}$ where $\bar{y}$ is solution of

$$\begin{cases} \frac{\partial \bar{y}}{\partial t} + \frac{\partial \bar{y}}{\partial a} - (k(x)\bar{y}_x)_x + \mu \bar{y} = w\chi_\omega & \text{in} \quad Q \\ \bar{y}(t,a,1) = \bar{y}(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ \bar{y}(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ \bar{y}(t,0,x) = \int_0^A g(a)\bar{y} da + \int_0^A h(a)y(t,a,x;v,g)da & \text{in} \quad Q_{T,1} \end{cases}$$

Then y_λ is solution of

$$\begin{cases} \frac{\partial y_\lambda}{\partial t} + \frac{\partial y_\lambda}{\partial a} - (k(x)y_{\lambda_x})_x + \mu y_\lambda = 0 & \text{in} \quad Q \\ y_\lambda(t,a,1) = y_\lambda(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ y_\lambda(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ y_\lambda(t,0,x) = \int_0^A g(a)y_\lambda da + \int_0^A h(a)[y(t,a,x;v+\lambda w, g+\lambda h) - y(t,a,x;v,g)]da & \text{in} \quad Q_{T,1} \end{cases}$$

In the sequel, set $z_\lambda = e^{-rt}y_\lambda$ with $r>0$. Then z_λ is solution of:

$$\begin{cases} \frac{\partial z_\lambda}{\partial t} + \frac{\partial z_\lambda}{\partial a} - (k(x)z_{\lambda x})_x + (\mu+r)z_\lambda = 0 & \text{in} \quad Q \\ z_\lambda(t,a,1) = z_\lambda(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ z_\lambda(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ z_\lambda(t,0,x) = e^{-rt}\left(\int_0^A g(a)y_\lambda da + \int_0^A h(a)[y(t,a,x;v+\lambda w, g+\lambda h) - y(t,a,x;v,g)]da\right) & \text{in} \quad Q_{T,1} \end{cases}$$

which gives

$$\begin{cases} \frac{\partial z_\lambda}{\partial t} + \frac{\partial z_\lambda}{\partial a} - (k(x)z_{\lambda x})_x + (\mu+r)z_\lambda = 0 & \text{in} \quad Q \\ z_\lambda(t,a,1) = z_\lambda(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ z_\lambda(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ z_\lambda(t,0,x) = \int_0^A g(a)z_\lambda da + e^{-rt}\left(\int_0^A h(a)[y(t,a,x;v+\lambda w, g+\lambda h) - y(t,a,x;v,g)]da\right) & \text{in} \quad Q_{T,1} \end{cases} \tag{9}$$

Now, we multiply the first equation of (9) by z_λ then integrate by parts on Q:

$$\frac{1}{2}\|z_\lambda(T,.,.)\|^2_{L^2(Q_{A,1})} - \frac{1}{2}\|z_\lambda(0,.,.)\|^2_{L^2(Q_{A,1})} + \frac{1}{2}\|z_\lambda(.,A,.)\|^2_{L^2(Q_{T,1})} - \frac{1}{2}\|z_\lambda(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k(x)}z_{\lambda x}\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z_\lambda\|^2_{L^2(Q)} = 0$$

It comes:

$$\|\sqrt{k(x)}z_{\lambda x}\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z_\lambda\|^2_{L^2(Q)} = \frac{1}{2}\|z_\lambda(.,0,.)\|^2_{L^2(Q_{T,1})} - \frac{1}{2}\|z_\lambda(T,.,.)\|^2_{L^2(Q_{A,1})} - \frac{1}{2}\|z_\lambda(.,A,.)\|^2_{L^2(Q_{T,1})}$$

and,

$$\|\sqrt{k(x)}z_{\lambda x}\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z_\lambda\|^2_{L^2(Q)} \le \frac{1}{2}\|z_\lambda(.,0,.)\|^2_{L^2(Q_{T,1})}$$

while:

$$z_\lambda(t,0,x) = \int_0^A g(a)z_\lambda da + e^{-rt}\left(\int_0^A h(a)[y(t,a,x;v+\lambda w, g+\lambda h) - y(t,a,x;v,g)]da\right)$$

then,

$$\|z_\lambda(.,0,.)\|^2_{L^2(Q_{T,1})} \le \|g\|^2_{L^2(0,A)}\|z_\lambda\|^2_{L^2(Q)} + \|h\|^2_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|^2_{L^2(Q)}$$

In the sequel,

$$\|\sqrt{k(x)}z_{\lambda x}\|^2_{L^2(Q)} + \|\sqrt{r+\mu}z_\lambda\|^2_{L^2(Q)} \le \frac{1}{2}\|g\|^2_{L^2(0,A)}\|z_\lambda\|^2_{L^2(Q)} + \frac{1}{2}\|h\|^2_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|^2_{L^2(Q)}$$

Then,

$$\|\sqrt{k(x)}z_{\lambda x}\|^2_{L^2(Q)} + \left[r+\mu - \frac{1}{2}\|g\|^2_{L^2(0,A)}\right]\|z_\lambda\|^2_{L^2(Q)} \le \frac{1}{2}\|h\|^2_{L^2(0,A)}\|y(v+\lambda w, g+\lambda h) - y(v,g)\|^2_{L^2(Q)}$$

While $\lambda \to 0$, as the application $(v,g) \to y(v,g)$ is continuous, we deduce that $y_\lambda \to 0$ strongly in $L^2(Q_{T,A}; H^1_k(0,1))$. Then, the sequence $(\bar{y}_\lambda)$ converges to $\bar{y}$ strongly in $L^2(Q_{T,A}; H^1_k(0,1))$ when $\lambda \to 0$.

Proposition 2.4. *For any $g \in L^2(0,A)$, the application $v \mapsto \dfrac{\partial y}{\partial g}(v,g)$ is continuous from $L^2(Q_\omega)$ onto $\mathscr{L}(L^2(0,A);L^2(Q_{T,A};H_k^1(0,1)))$.*

Proof 3. *We set $\bar{y}(h) = \dfrac{\partial y}{\partial g}(v,g)(h)$. Then $\bar{y}(h)$ is solution of system*

$$\begin{cases} \dfrac{\partial \bar{y}(h)}{\partial t} + \dfrac{\partial \bar{y}(h)}{\partial a} - (k(x)\bar{y}(h)_x)_x + \mu \bar{y}(h) = 0 & \text{in} \quad Q \\ \bar{y}(h)(t,a,1) = \bar{y}(h)(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ \bar{y}(h)(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ \bar{y}(h)(t,0,x) = \displaystyle\int_0^A g(a)\bar{y}(h)da + \int_0^A h(a)y(t,a,x;v,g)da & \text{in} \quad Q_{T,1} \end{cases}$$

Let $v_1, v_2 \in L^2(Q_\omega)$. Set $\bar{y}_1(h) = \dfrac{\partial y}{\partial g}(v_1,g)(h)$, $\bar{y}_2(h) = \dfrac{\partial y}{\partial g}(v_2,g)(h)$ and take $\bar{z}(h) = e^{-rt}(\bar{y}_1(h) - \bar{y}_2(h))$ with $r > 0$. Then $\bar{z}(h)$ is solution to the problem:

$$\begin{cases} \dfrac{\partial \bar{z}(h)}{\partial t} + \dfrac{\partial \bar{z}(h)}{\partial a} - (k(x)\bar{z}(h)_x)_x + (\mu + r)\bar{z}(h) = 0 & \text{in} \quad Q \\ \bar{z}(h)(t,a,1) = \bar{z}(h)(t,a,0) = 0 & \text{on} \quad Q_{T,A} \\ \bar{z}(h)(0,a,x) = 0 & \text{in} \quad Q_{A,1} \\ \bar{z}(h)(t,0,x) = \displaystyle\int_0^A g(a)\bar{z}(h)da + e^{-rt}\int_0^A h(a)[y(v_1,g) - y(v_2,g)]da & \text{in} \quad Q_{T,1} \end{cases} \tag{10}$$

Multiply the first equation of (10) by $\bar{z}(h)$ then integrate by parts on Q:

$$\frac{1}{2}\|\bar{z}(h)(T,.,.)\|^2_{L^2(Q_{A,1})} - \frac{1}{2}\|\bar{z}(h)(0,.,.)\|^2_{L^2(Q_{A,1})} + \frac{1}{2}\|\bar{z}(h)(.,A,.)\|^2_{L^2(Q_{T,1})} - \frac{1}{2}\|\bar{z}(h)(.,0,.)\|^2_{L^2(Q_{T,1})} + \|\sqrt{k(x)}\bar{z}(h)_x\|^2_{L^2(Q)} + \|\sqrt{r+\mu}\,\bar{z}(h)\|^2_{L^2(Q)} = 0$$

Now

$$\|\bar{z}(h)(.,0,.)\|^2_{L^2(Q_{T,1})} \leq \|g\|^2_{L^2(0,A)}\|\bar{z}(h)\|^2_{L^2(Q)} + \|h\|^2_{L^2(0,A)}\|y(v_1,g) - y(v_2,g)\|^2_{L^2(Q)}$$

We choose r such that $r + \mu > \|g\|^2_{L^2(0,A)}$. Then we deduce:

$$\|\bar{z}(h)\|_{L^2((0,T)\times(0,A);H_k^1(0,1))} \leq \|h\|_{L^2(0,A)}\|y(v_1,g) - y(v_2,g)\|_{L^2(Q)}$$

By returning to $\bar{z}(h) = e^{-rt}(\bar{y}_1(h) - \bar{y}_2(h))$ with $r > 0$, one obtains :

$$\|\bar{y}_1(h) - \bar{y}_2(h)\|_{L^2((0,T)\times(0,A);H_k^1(0,1))} \leq \|h\|_{L^2(0,A)}e^{rT}\|y(v_1,g) - y(v_2,g)\|_{L^2(Q)}$$

Passing to the limit when $v_1 \to v_2$ and by using Proposition 2.2, we obtain that $\dfrac{\partial y}{\partial g}(v_1,g)$ converges to $\dfrac{\partial y}{\partial g}(v_2,g)$ in $\mathscr{L}(L^2(0,A);L^2(Q_{T,A};H_k^1(0,1)))$.

Proposition 2.5. *For any $v \in L^2(Q_\omega)$ the application $v \mapsto \xi(v,.)$ is continuous from $L^2(Q_\omega)$ to $L^2(Q_{T,A};H_k^1(0,1))$ where $\xi(v,.)$ is solution of*

$$\begin{cases} -\dfrac{\partial \xi}{\partial t} - \dfrac{\partial \xi}{\partial a} - (k(x)\xi_x)_x + \mu\xi &= y(v,0) - z_d & \text{in} \quad Q \\ \xi(t,a,1) = \xi(t,a,0) &= 0 & \text{on} \quad Q_{T,A} \\ \xi(T,a,x) &= 0 & \text{in} \quad Q_{A,1} \\ \xi(t,A,x) &= 0 & \text{in} \quad Q_{T,1} \end{cases} \tag{11}$$

Proof 4. *Let be $v_1, v_2 \in L^2(Q_\omega)$ and let be $\bar{\xi} = \xi(v_1) - \xi(v_2)$. Then $\bar{\xi}$ satifies the system:*

$$\begin{cases} -\dfrac{\partial \bar{\xi}}{\partial t} - \dfrac{\partial \bar{\xi}}{\partial a} - (k(x)\bar{\xi}_x)_x + (\mu + r)\bar{\xi} &= y(v_1,0) - y(v_2,0) & \text{in} \quad Q \\ \bar{\xi}(t,a,1) = \bar{\xi}(t,a,0) &= 0 & \text{on} \quad Q_{T,A} \\ \bar{\xi}(T,a,x) &= 0 & \text{in} \quad Q_{A,1} \\ \bar{\xi}(t,A,x) &= 0 & \text{in} \quad Q_{T,1} \end{cases}$$

If we set $\zeta = e^{-rt}\bar{\xi}$ with $r > 0$, we get that ζ is solution of:

$$\begin{cases} -\dfrac{\partial \zeta}{\partial t} - \dfrac{\partial \zeta}{\partial a} - (k(x)\zeta_x)_x + (\mu + r)\zeta &= (y(v_1,.) - y(v_2,.))e^{-rt} & \text{in} \quad Q \\ \zeta(t,a,1) = \zeta(t,a,0) &= 0 & \text{on} \quad Q_{T,A} \\ \zeta(T,a,x) &= 0 & \text{in} \quad Q_{A,1} \\ \zeta(t,A,x) &= 0 & \text{in} \quad Q_{T,1} \end{cases} \tag{12}$$

Multiply the first equation of (12) by ζ then integrate by parts on Q:

$$-\frac{1}{2}\|\zeta(T,.,.)\|_{Q_{A,1}}^2+\frac{1}{2}\|\zeta(0,.,.)\|_{Q_{A,1}}^2-\frac{1}{2}\|\zeta(.,A,.)\|_{Q_{T,1}}^2+\frac{1}{2}\|\zeta(.,0,.)\|_{Q_{T,1}}^2+\|\sqrt{k(x)}\zeta_x\|_Q^2+\|\sqrt{r+\mu}\zeta\|_Q^2$$
$$=\int_Q \zeta(y(v_1,.)-y(v_2,.))e^{-rt}\text{dtdadx}$$

Next,

$$\frac{1}{2}\|\zeta(0,.,.)\|_{Q_{A,1}}^2+\frac{1}{2}\|\zeta(.,0,.)\|_{Q_{T,1}}^2+\|\sqrt{k(x)}\zeta_x\|_Q^2+\|\sqrt{r+\mu}\zeta\|_Q^2=\int_Q \zeta(y(v_1,.)-y(v_2,.))e^{-rt}\text{dtdadx}$$
$$\Longrightarrow \frac{1}{2}\|\zeta(.,0,.)\|_{Q_{T,1}}^2+\|\sqrt{k(x)}\zeta_x\|_Q^2+\|\sqrt{r+\mu}\zeta\|_Q^2\le\int_Q \zeta(y(v_1,.)-y(v_2,.))e^{-rt}\text{dtdadx}$$
$$\Longrightarrow \frac{1}{2}\|\zeta(.,0,.)\|_{Q_{T,1}}^2+\|\sqrt{k(x)}\zeta_x\|_Q^2+(r+\mu)\|\zeta\|_Q^2\le\frac{1}{2}\|\zeta\|_Q^2+\frac{1}{2}\|y(v_1,.)-y(v_2,.)\|_Q^2$$
$$\Longrightarrow \frac{1}{2}\|\zeta(.,0,.)\|_{Q_{T,1}}^2+\|\zeta\|_{L^2(Q_{T,A};H_k^1(0,1))}^2\le\frac{1}{2}\|y(v_1,.)-y(v_2,.)\|_Q^2$$

returning to $\zeta=e^{-rt}\bar{\xi}$

$$\frac{1}{2}\|\bar{\xi}(.,0,.)\|_{Q_{T,1}}^2+\|\bar{\xi}\|_{L^2(Q_{T,A};H_k^1(0,1))}^2\le e^{rT}\|y(v_1,.)-y(v_2,.)\|_Q^2$$

and consequently

$$\|\bar{\xi}\|_{L^2(Q_{T,A};H_k^1(0,1))}^2\le e^{rT}\|y(v_1,.)-y(v_2,.)\|_Q^2$$

Using Proposition 2.2, we get that $v\mapsto\xi(v,.)$ is continuous from $L^2(Q_\omega)$ to $L^2(Q_{T,A};H_k^1(0,1))$

3. Position of the problem

In this section, we are interested with the No-regret control solution of the problem (3). The feature of nonlinearity of the mapping $g\to y(v;g)$ from $L^2(0;A)$ onto $L^2(Q_{T,A};H_k^1(0,1))$ and the regularity results of y proven in Propositions 2.2 and 2.3 involve to replace the cost function defined in (2) by a new one:

$$J_1(v,g)=J(v,0)+\frac{\partial J}{\partial g}(v,0)(g) \tag{13}$$

Thus, the optimization problem (3) becomes:

$$\inf_{v\in L^2(Q_\omega)}\sup_{g\in L^2(0,A)} \quad J_1(v,g)-J_1(0,g) \tag{14}$$

Let $y(v,0)\in L^2(Q_{T,A};H_k^1(0,1))$ be the solution of:

$$\left\{\begin{array}{llll}\dfrac{\partial y(v,0)}{\partial t}+\dfrac{\partial y(v,0)}{\partial a}-(k(x)y_x(v,0))_x+\mu y(v,0) & = & f+v\chi_\omega & \text{in} \quad Q\\ y(t,a,1)(v,0)=y(t,a,0)(v,0) & = & 0 & \text{on} \quad U\\ y(0,a,x)(v,0) & = & y^0(a,x) & \text{in} \quad Q_{A,1}\\ y(t,0,x)(v,0) & = & 0 & \text{in} \quad Q_{T,1}\end{array}\right. \tag{15}$$

In the sequel, we establish the following results:

Proposition 3.1. *For any $(v,g)\in L^2(Q_\omega)\times L^2(0,A)$, the following equality holds:*

$$J_1(v,g)=J(v,0)+2\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,\text{dtdadx} \tag{16}$$

where J is the cost function defined in (2) and $\dfrac{\partial J}{\partial g}(v,0)(g)=\lim\limits_{t\to 0}\dfrac{J(v,tg)-J(v,0)}{t}$

Proof 5. *We have:*

$$J(v,tg)=\|y(v,tg)-z_d\|_{L^2(Q)}^2+N\|v\|_{L^2(Q_\omega)}^2$$
$$=\|y(v,tg)-y(v,0)\|_{L^2(Q)}^2+\|y(v,0)-z_d\|_{L^2(Q)}^2+N\|v\|_{L^2(Q_\omega)}^2+2\int_Q(y(v,tg)-y(v,0))(y(v,0)-z_d)\,\text{dtdadx}$$
$$=J(v,0)+\|y(v,tg)-y(v,0)\|_{L^2(Q)}^2+2\int_Q(y(v,tg)-y(v,0))(y(v,0)-z_d)\,\text{dtdadx}$$

Then, passing to the limit in the expression $\dfrac{J(v,tg)-J(v,0)}{t}$ when $t\to 0$, we obtain:

$$\frac{\partial J}{\partial g}(v,0)(g)=2\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,\text{dtdadx}$$

Consequently,

$$\frac{\partial J}{\partial g}(v,0)(g)+J(v,0)=J(v,0)+2\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,\mathrm{dtdadx}$$

Finally,

$$J_1(v,0)=J(v,0)+2\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,\mathrm{dtdadx}$$

Proposition 3.2. *For any $(v,g)\in L^2(Q_\omega)\times L^2(0,A)$, the following equality holds:*

$$J_1(v,g)-J_1(0,g)=J(v,0)-J(0,0)+2\int_0^A S(a,v)g(a)\mathrm{da} \tag{17}$$

where for any $a\in(0,A)$,

$$S(a,v)=\int_{Q_{T,1}}[y(t,a,x;v,0)\xi(v)(t,0,x)-y(t,a,x;0,0)\xi(0)(t,0,x)]\,\mathrm{dtdx} \tag{18}$$

with $\xi(v)$ solution of (11).

Proof 6. *On the one hand side, for any $(v,g)\in L^2(Q_\omega)\times L^2(0,A)$, we have:*

$$J_1(v,g)-J_1(0,g)=J(v,0)-J(0,0)+2\int_Q\left(\frac{\partial y}{\partial g}(v,0)(g)\right)(y(v,0)-z_d)\,\mathrm{dtdadx}-2\int_Q\left(\frac{\partial y}{\partial g}(0,0)(g)\right)(y(0,0)-z_d)\,\mathrm{dtdadx}$$

And on the other hand side and according to Proposition 2.4, $\bar{y}(g)=\dfrac{\partial y}{\partial g}(v,0)(g)$ is solution of:

$$\begin{cases}\dfrac{\partial\bar{y}(g)}{\partial t}+\dfrac{\partial\bar{y}(g)}{\partial a}-(k(x)\bar{y}_x(g))_x+\mu\bar{y}(g)=f+v\chi_\omega & \text{in}\quad Q\\ \bar{y}(g)(t,a,1)=\bar{y}(g)(t,a,0)=0 & \text{on}\quad Q_{T,A}\\ \bar{y}(g)(0,a,x)=0 & \text{in}\quad Q_{A,1}\\ \bar{y}(g)(t,0,x)=\int_0^A g(a)y(t,a,x;v,0)\mathrm{da} & \text{in}\quad Q_{T,1}\end{cases} \tag{19}$$

Multiply the first equation of (19) by $\xi(v)$ and integrate by parts over Q, it comes:

$$\int_Q\xi(v)\left(\frac{\partial\bar{y}(g)}{\partial t}+\frac{\partial\bar{y}(g)}{\partial a}-(k(x)\bar{y}_x(g))_x+\mu\bar{y}(g)\right)\mathrm{dtdadx}=0$$

Then,

$$\begin{aligned}0=&\int_Q\bar{y}(g)\left(-\frac{\partial\xi(v)}{\partial t}-\frac{\partial\xi(v)}{\partial a}-(k(x)\xi_x(v))_x+\mu\xi(v)\right)\mathrm{dtdadx}+\int_{Q_{T,1}}\xi(v)(.,A,.)\bar{y}(g)(.,A,.)\mathrm{dtdx}-\int_{Q_{T,1}}\xi(v)(.,0,.)\bar{y}(g)(.,0,.)\mathrm{dtdx}\\&+\int_{Q_{A,1}}\xi(v)(T,.,.)\bar{y}(g)(T,.,.)\mathrm{dadx}-\int_{Q_{A,1}}\xi(v)(0,.,.)\bar{y}(g)(0,.,.))\mathrm{dadx}+\int_{Q_{T,A}}[k(x)\xi(v)\bar{y}_x(g)]_0^1\mathrm{dtda}-\int_{Q_{T,A}}[k(x)\xi_x(v)\bar{y}(g)]_0^1\mathrm{dtda}\end{aligned}$$

by taking into account the conditions at the boundary / limits, it comes:

$$\int_Q\bar{y}(g)\left(-\frac{\partial\xi(v)}{\partial t}-\frac{\partial\xi(v)}{\partial a}-(k(x)\xi_x(v))_x+\mu\xi(v)\right)\mathrm{dtdadx}=\int_{Q_{T,1}}\xi(v)(.,0,.)\int_0^A y(t,a,x;v,0)g(a)\mathrm{dtdadx}$$

Next, we obtain:

$$\int_Q\bar{y}(g)(y(v,0)-z_d)\mathrm{dtdadx}=\int_0^A\int_{Q_{T,1}}\xi(v)(.,0,.)y(t,a,x;v,0)g(a)\mathrm{dtdadx}$$

We can get also:

$$\int_Q\bar{y}(g)(y(0,0)-z_d)\mathrm{dtdadx}=\int_0^A\int_{Q_{T,1}}\xi(0)(.,0,.)y(t,a,x;v,0)g(a)\mathrm{dtdadx}$$

by substituting with what below, it comes:

$$J_1(v,g)-J_1(0,g)=J(v,0)-J(0,0)+2\int_Q g(a)\,[y(t,a,x;v,0)\xi(v)(t,0,x)-y(t,a,x;0,0)\xi(0)(t,0,x)]\,\mathrm{dtdadx}$$

by setting,

$$S(a,v)=\int_{Q_{T,1}}[y(t,a,x;v,0)\xi(v)(t,0,x)-y(t,a,x;0,0)\xi(0)(t,0,x)]\mathrm{dadx}$$

we obtain:

$$J_1(v,g)-J_1(0,g)=J(v,0)-J(0,0)+2\int_0^A g(a)S(a,v)\mathrm{da}$$

Proposition 3.3. *Let $S(.,v)$ be the function defined in relation (18). Then the application $v \to S(.,v)$ is continuous from $L^2(Q_\omega)$ onto $L^2(0,A)$.*

Proof 7. *Let $v_1, v_2 \in L^2(Q_\omega)$, we have:*

$$\begin{aligned} S(a,v_1) - S(a,v_2) &= \int_{Q_{T,1}} y(t,a,x;v_1,0)\xi(v_1)(t,0,x)\mathrm{dtdx} - \int_{Q_{T,1}} y(t,a,x;v_2,0)\xi(v_2)(t,0,x)\mathrm{dtdx} \\ &= \int_{Q_{T,1}} [y(t,a,x;v_1,0) - y(t,a,x;v_2,0)]\xi(v_1)(t,0,x)]\mathrm{dtdx} - \int_{Q_{T,1}} y(t,a,x;v_2,0)[\xi(v_1)(t,0,x) - \xi(v_2)(t,0,x)]\mathrm{dtdx} \end{aligned}$$

Passing to the norm

$$\begin{aligned} \|S(a,v_1) - S(a,v_2)\|^2_{L^2(Q\omega)} &\leq \frac{1}{2}\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)} + \frac{1}{2}\|\xi(v_1)\|^2_{L^2(Q_T)} + \frac{1}{2}\|y(v_2,.)\|^2_{L^2(Q)} + \frac{1}{2}\|\xi(v_1) - \xi(v_2)\|^2_{L^2(Q_T)} \\ &\leq \frac{1}{2}\|y(v_1,.) - y(v_2,.)\|^2_{L^2(Q)} + \frac{1}{2}\|\xi(v_1) - \xi(v_2)\|^2_{L^2(Q_T)} \end{aligned}$$

As the Proposition 2.2 is an argument on continuity of the application $v \to y(v,.)$ and using an argument of the proof of Proposition 2.5 on the continuity of application $v \to \xi(v)$, we conclude that $S(a,v_1) \to S(a,v_2)$ when $v_1 \to v_2$.

Lemma 2. *Let $S(.,v)$ be the function defined in relation (18) for any $a \in L^2(0,A)$. For any $\gamma > 0$, we consider the sequences $y_\gamma = y(t,a,x;u_\gamma,0)$ and $\xi(u_\gamma)$ respectively solutions of (15) and (11) with $v = u_\gamma$. Assume that there exists constant $C > 0$ independent of γ such that*

$$\|S(.,u_\gamma)\|_{L^2(0,A)} \leq C \tag{20}$$

Assume also that $\hat{u} \in L^2(Q_\omega)$, $\hat{\xi}(.,0,.) \in L^2(Q_{T,1})$ and $\hat{y} = y(t,a,x;\hat{u},0) \in L^2(Q_{T,A};H^1_k(0,1))$ solution of (15) such that

$$u_\gamma \rightharpoonup \hat{u} \text{ weakly in } L^2(Q_{T,A} \times \omega) \tag{21}$$

$$y_\gamma \rightharpoonup \hat{y} = y(t,a,x;\hat{u},0) \text{ weakly in } L^2(Q_{T,A};H^1_k(0,1)) \tag{22}$$

$$\xi(u_\gamma)(.,0,.) \rightharpoonup \hat{\xi}(.,0,.) \text{ weakly in } L^2(Q_{T,1}) \tag{23}$$

Then we have:

$$S(.,u_\gamma) \rightharpoonup S(.,\hat{u}) \text{ weakly in } \mathscr{D}'(A) \tag{24}$$

Proof 8. *Let $\mathscr{D}(A)$ be the set of C^∞ functions with compact support on $(0,A)$ and $\mathscr{D}'(A)$ its dual. For any $\phi \in \mathscr{D}(A)$,*

$$\eta_\gamma(t,x) = \int_0^A \phi(a)y(t,a,x;u_\gamma,0)\mathrm{da}, \quad \forall (t,x) \in Q_{T,1} \tag{25}$$

In view of (22), there exists a constant C such that

$$\|\eta_\gamma\|_{L^2(Q_{T,1})} \leq \|y_\gamma\|_{L^2(Q)}.\|\phi\|_{L^2(0,A)} \leq C$$

Then, there exists $\eta \in L^2(Q_{T,1})$ such that

$$\eta_\gamma \rightharpoonup \eta \text{ weakly in } L^2(Q_{T,1})$$

More, we have

$$\eta(t,x) = \int_0^A \phi(a)y(t,a,x;\hat{u},0)\mathrm{da}, \quad \forall (t,x) \in Q_{T,1} \tag{26}$$

As $y_\gamma = y(t,a,x;u_\gamma,0)$ solution of (18) with $v = u_\gamma$, we have that η_γ is solution of

$$\left\{ \begin{array}{llll} \dfrac{\partial \eta_\gamma}{\partial t} - (k(x)\eta_{\gamma x})_x & = & v_\gamma & \text{in} \quad Q_{T,1} \\ \eta_\gamma(t,1) = \eta_\gamma(t,0) & = & 0 & \text{on} \quad (0,T) \\ \eta_\gamma(0,x) & = & \int_0^A \phi(a)y^0(a,x)\mathrm{da} & \text{in} \quad (0,1) \end{array} \right.$$

where

$$v_\gamma(t,x) = \int_0^A \phi(a)(f + u_\gamma\chi_\omega)\mathrm{da} - \int_0^A \phi(a)y_\gamma\mu(a)\mathrm{da} - \int_0^A \phi(a)\frac{\partial y_\gamma}{\partial a}\mathrm{da}$$

Using (21) and (22), we can improve a majoration of v_γ:

$$\|v_\gamma\|_{L^2(Q_{T,1})} \leq \left(2\|f\|^2_{L^2(Q)} + 2\|u_\gamma\|^2_{L^2(Q_\omega)} + \|\mu\|^2_{L^\infty(0,A)}.\|y_\gamma\|^2_{L^2(Q)}\right)^{\frac{1}{2}} \|\phi\|_{L^2(0,A)} + \|y_\gamma\|_{L^2(Q)}.\|\frac{\partial \phi}{\partial a}\|_{L^2(0,A)} \leq C$$

Then there exits $C > 0$ independent of γ such that

$$\left\{ \begin{array}{l} \|\eta_\gamma\|_{L^2(0,T);H^1_0(0,1)} \leq C \\ \|\dfrac{\partial \eta_\gamma}{\partial t}\|_{L^2(0,T);H^{-1}(0,1)} \leq C \end{array} \right.$$

Consequently, it follows from Aubin-Lions lemma that

$$\eta_\gamma \to \eta \text{ strongly in } L^2(Q_{T,1}) \tag{27}$$

with

$$\eta(t,x) = \int_0^A \phi(a) y(t,a,x;\hat{u},0)\mathrm{da}, \ \ \forall (t,x) \in Q_{T,1}$$

Now, back to

$$S(a,u_\gamma) = \int_{Q_{T,1}} y(t,a,x;u_\gamma,0)\xi(u_\gamma)(t,0,x)\mathrm{dtdx} - \int_{Q_{T,1}} y(t,a,x;0,0)\xi(0)(t,0,x)\mathrm{dtdx}$$

For any $\phi \in \mathscr{D}(A)$,

$$\begin{aligned}\int_0^A S(a,u_\gamma)\phi(a)\mathrm{da} &= \int_0^A \int_{Q_{T,1}} y(t,a,x;u_\gamma,0)\phi(a)\xi(u_\gamma)(t,0,x)\mathrm{dtdxda} - \int_0^A \int_{Q_{T,1}} y(t,a,x;0,0)\phi(a)\xi(0)(t,0,x)\mathrm{dtdxda} \\ &= \int_{Q_{T,1}} \int_0^A y(t,a,x;u_\gamma,0)\phi(a)\xi(u_\gamma)(t,0,x)\mathrm{dtdxda} - \int_{Q_{T,1}} \int_0^A y(t,a,x;0,0)\phi(a)\xi(0)(t,0,x)\mathrm{dtdxda} \\ &= \int_{Q_{T,1}} \eta_\gamma(t,x)\xi(u_\gamma)(t,0,x)\mathrm{dtdxda} - \int_{Q_{T,1}} \int_0^A y(t,a,x;0,0)\phi(a)\xi(0)(t,0,x)\mathrm{dtdxda}\end{aligned}$$

Passing to the while using (26) and (21) in this latter equality

$$\int_0^A S(a,u_\gamma)\phi(a)\mathrm{da} \longrightarrow \int_{Q_{T,1}} \eta(t,x)\xi(\hat{u})(t,0,x)\mathrm{dtdxda} - \int_{Q_{T,1}} \int_0^A y(t,a,x;0,0)\phi(a)\xi(0)(t,0,x)\mathrm{dtdxda}$$

According to (25),

$$\int_0^A S(a,u_\gamma)\phi(a)\mathrm{da} \longrightarrow \int_{Q_{T,1}} \int_0^A y(t,a,x;\hat{u},0)\phi(a)\xi(\hat{u})(t,0,x)\mathrm{dtdxda} - \int_{Q_T} \int_0^A y(t,a,x;0,0)\phi(a)\xi(0)(t,0,x)\mathrm{dtdxda}$$

We deduce from which below, $S(a,u_\gamma) \rightharpoonup S(a,\hat{u})$ weakly in $\mathscr{D}(A)$.

4. Existence of the No-regret control and the Low-regret control

In view of (17), the optimization problem (14) is equivalent to the following problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} \tilde{\mathscr{J}}(v). \tag{28}$$

with

$$\tilde{\mathscr{J}}(v) = J(v,0) - J(0,0) + 2\int_0^A g(a)S(a,v)\mathrm{da}. \tag{29}$$

The quantity $\int_0^A g(a)S(a,v)\mathrm{da}$ may be equal to 0 or $+\infty$. We define the set:

$$\mathscr{U}_{ad} = \left\{ v \in L^2(Q_\omega);\ \int_0^A g(a)S(a,v)\mathrm{da} = 0,\ \ \forall g \in L^2(0,A) \right\} \tag{30}$$

Further, we give the results of existence of the No-regret control and the Low-regret control.

Lemma 3. *There exists a solution $\tilde{u}$ of (28) in $\mathscr{U}_{ad}$.*

Proof 9. *Firstly, the continuity of the application $v \to J(v,0) - J(0,0)$ is a consequence of the Proposition 2.2. Moreover, the application $v \to J(v,0) - J(0,0)$ is positive (or 0-coercive) and bounded below by $-J(0,0)$. Secondly, due to the fact that $J(v,0) - J(0,0) > -J(0,0)$, we can define a real d such that $d = \inf_{v\in L^2(Q_\omega)} \tilde{\mathscr{J}}(v)$. Let (v_n) be a minimizing sequence such that $d = \lim_{n\to+\infty} \tilde{\mathscr{J}}(v_n)$. We have: $-J(0,0) \leqslant J(v_n,0) - J(0,0) + 2\int_0^A g(a)S(a,v_n)\mathrm{da} \leqslant d+1$. We deduce that there exists a constant C independent of n such that:*

$$\|y(v_n,0) - z_d\|_{L^2(Q)} \leqslant C \tag{31}$$

$$\|v_n\|_{L^2(Q_\omega)} \leqslant C \tag{32}$$

$$\|S(.;v_n)\|_{L^2(0,A)} \leqslant C \tag{33}$$

According to (32), the sequence (v_n) is bounded. Thus we can extract a subsequence still denoted (v_n) such that

$$v_n \rightharpoonup v \text{ weakly in } L^2(Q_\omega) \tag{34}$$

On the one hand side, the continuity of the application $v \to y(v,.)$ (Proposition 2.2) and the relation (32) and on the other hand side, the continuity of the application $v \to S(.,v)$ (Proposition 3.3) and the relation (32) permit us to deduce:

$$y(v_n,.) \rightharpoonup y(v,.) \text{ weakly in } L^2(Q) \tag{35}$$

$$S(.,v_n) \rightharpoonup S(.,v) \text{ weakly in } L^2(0,A) \tag{36}$$

This implies with what above:

$$J(v,0)-J(0,0)+2\int_0^A g(a)S(a,v)\mathrm{da} \leqslant \liminf_{n\to+\infty} J(v_n,0)-J(0,0)+2\int_0^A g(a)S(a,v_n)\mathrm{da} \leqslant d$$

In other words, $\tilde{u}$ is the No-regret control solution of the problem (28).

As such a control $\tilde{u}$ is not easy to characterize, we consider for any $\gamma > 0$, the relaxed optimization problem:

$$\inf_{v\in L^2(Q_\omega)} \sup_{g\in L^2(0,A)} [J(v,0)-J(0,0)+2\int_0^A g(a)S(a,v)\mathrm{da}-\gamma\|g\|^2_{L^2(0,A)}] \tag{37}$$

which means of Fenchel-Legendre transform is equivalent to

$$\inf_{v\in L^2(Q_\omega)} \mathscr{J}_\gamma(v) \tag{38}$$

with

$$\mathscr{J}_\gamma(v) = J(v,0)-J(0,0)+\frac{1}{\gamma}\|S(.;v)\|^2_{L^2(0,A)} \tag{39}$$

Proposition 4.1. *For any $\gamma > 0$, there exists at least a Low-regret control u_γ solution of the problem (38).*

Proof 10. *One can proceed as for Proof 9. Analogously, we can show easily that the application $v \to J(v,0)-J(0,0)$ is coercive in $L^2(Q_\omega)$, bounded below by $-J(0,0)$ and continuous because of Proposition 2.2.*
Furthermore, the continuity of the application $v \to S(.,v)$ (Proposition 3.3) and the use of a minimizing sequence permit us to deduce that there exists at least a Low-regret control u_γ.
Then the strict convexity of the application $v \to S(.,v)$ is not established. Hence the uniqueness of the Low-regret control u_γ is not ensured. Therefore, there is not assurance that the Low-regret u_γ will converge toward the No-regret $\tilde{u} \in \mathscr{U}_{ad}$. Thus in order to obtain this convergence, we adapt the cost function $\tilde{\mathscr{J}}$ to a given No-regret control.

We now turn to the characterization of the adapted Low-regret control $\tilde{u}_\gamma$ in the next section.

5. Existence and characterization of the adapted Low-regret control

For any $\gamma > 0$, the optimization problem (38) becomes:

$$\inf_{v\in L^2(Q_\omega)} \tilde{\mathscr{J}}_\gamma(v) \tag{40}$$

where

$$\tilde{\mathscr{J}}_\gamma(v) = J(v,0)-J(0,0)+\|v-\tilde{u}\|_{L^2(Q_\omega)}+\frac{1}{\gamma}\|S(.;v)\|^2_{L^2(0,A)} \tag{41}$$

Proposition 5.1. *For any $\gamma > 0$, there exists at least an adapted Low-regret control $\tilde{u}_\gamma$ in $L^2(Q_\omega)$ solution of problem (40).*

Proof 11. *As for the previous proofs, we use the fact that the application $v \to J(v,0)-J(0,0)$ is coercive in $L^2(Q_\omega)$, bounded below by $-J(0,0)$ and continuous (Proposition 2.2).*
Then the continuity of the application $v \to S(.,v)$ (Proposition 3.3) and the lower semi-continuity of the function $v \to \tilde{\mathscr{J}}_\gamma$ permit with means of a minimizing sequence to deduce that there exists at least an adapted Low-regret control $\tilde{u}_\gamma$ in $L^2(Q_\omega)$.

Proposition 5.2. *Let be $\tilde{u}_\gamma \in L^2(Q_\omega)$ solution of problem (40). Then there exists $\tilde{p}_\gamma = p(\tilde{u}_\gamma) \in L^2(Q_{T,A};H^1_k(0,1))$ and $\tilde{q}_\gamma = q(\tilde{u}_\gamma) \in L^2(Q_{T,A};H^1_k(0,1))$ such as the quadruplet $\{\tilde{y}_\gamma, \tilde{\xi}_\gamma, \tilde{p}_\gamma, \tilde{q}_\gamma\}$ be solution of systems:*

$$\left\{\begin{array}{llll} \frac{\partial \tilde{y}_\gamma}{\partial t}+\frac{\partial \tilde{y}_\gamma}{\partial a}-(k(x)\tilde{y}_{\gamma x})_x+\mu\tilde{y}_\gamma & = & f+\tilde{u}_\gamma\chi_\omega & \text{in} \quad Q \\ \tilde{y}_\gamma(t,a,1)=\tilde{y}_\gamma(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \tilde{y}_\gamma(0,a,x) & = & y^0(a,x) & \text{in} \quad Q_{A,1} \\ \tilde{y}_\gamma(t,0,x) & = & 0 & \text{in} \quad Q_{T,1} \end{array}\right. \tag{42}$$

$$\left\{\begin{array}{llll} -\frac{\partial \tilde{\xi}_\gamma}{\partial t}-\frac{\partial \tilde{\xi}_\gamma}{\partial a}-(k(x)\tilde{\xi}_{\gamma x})_x+\mu\tilde{\xi}_\gamma & = & \tilde{y}_\gamma-z_d & \text{in} \quad Q \\ \tilde{\xi}_\gamma(t,a,1)=\tilde{\xi}_\gamma(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \tilde{\xi}_\gamma(0,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ \tilde{\xi}_\gamma(t,0,x) & = & 0 & \text{in} \quad Q_{T,1} \end{array}\right. \tag{43}$$

$$\left\{\begin{array}{llll} \frac{\partial \tilde{p}_\gamma}{\partial t}+\frac{\partial \tilde{p}_\gamma}{\partial a}-(k(x)\tilde{p}_{\gamma x})_x+\mu\tilde{p}_\gamma & = & 0 & \text{in} \quad Q \\ \tilde{p}_\gamma(t,a,1)=\tilde{p}_\gamma(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \tilde{p}_\gamma(0,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ \tilde{p}_\gamma(t,0,x) & = & \frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\tilde{u}_\gamma,0)S(a,\tilde{u}_\gamma)\mathrm{da} & \text{in} \quad Q_{T,1} \end{array}\right. \tag{44}$$

$$\left\{\begin{array}{llll} -\frac{\partial \tilde{q}_\gamma}{\partial t}-\frac{\partial \tilde{q}_\gamma}{\partial a}-(k(x)\tilde{q}_{\gamma x})_x+\mu\tilde{q}_\gamma & = & y(\tilde{u}_\gamma,0)-z_d+\zeta^\gamma & \text{in} \quad Q \\ \tilde{q}_\gamma(t,a,1)=\tilde{q}_\gamma(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \tilde{q}_\gamma(0,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ \tilde{q}_\gamma(t,0,x) & = & 0 & \text{in} \quad Q_{T,1} \end{array}\right. \tag{45}$$

and

$$(N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma = 0 \quad in \quad Q_\omega \tag{46}$$

where

$$\zeta^\gamma = \frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma + \frac{1}{\gamma}S(a,\tilde{u}_\gamma)\xi(\tilde{u}_\gamma)(t,0,x)$$

Proof 12. *The optimality condition of Euler-Lagrange which characterizes the adapted Low-regret control $\tilde{u}_\gamma$ is given by:*

$$\lim_{\lambda\to 0}\frac{\tilde{\mathscr{J}}^\gamma(\tilde{u}_\gamma+\lambda w) - \tilde{\mathscr{J}}^\gamma(\tilde{u}_\gamma)}{\lambda} = 0, \qquad \forall w \in L^2(Q_\omega).$$

We use Proposition 2.3 and Proposition 3.2 to transform the latter optimality condition. After some calculations, we obtain, $\forall w \in L^2(Q_\omega)$, the equality:

$$\int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d + \frac{1}{\gamma}S(a,\tilde{u}_\gamma)\bar{\xi}(t,0,x))\mathrm{dtdadx} + \int_{Q_\omega}(\tilde{u}_\gamma - \tilde{u})w\mathrm{dtdadx} + \int_{Q_\omega} N\tilde{u}_\gamma w\mathrm{dtdadx} + \int_Q \frac{1}{\gamma}\tilde{y}_\gamma S(a,\tilde{u}_\gamma)\bar{\xi}(t,0,x)\mathrm{dtdadx} = 0, \tag{47}$$

where $\bar{y}(w) = \frac{\partial y}{\partial v}(\tilde{u}_\gamma,0)(w)$ and $\bar{\xi} = \frac{\partial \xi}{\partial v}(\tilde{u}_\gamma)$ are respective solutions of:

$$\left\{\begin{array}{llll} \frac{\partial \bar{y}(w)}{\partial t} + \frac{\partial \bar{y}(w)}{\partial a} - (k(x)\bar{y}_x(w))_x + \mu y(w) & = & w\chi_\omega & \text{in} \quad Q \\ \bar{y}(w)(t,a,1) = \bar{y}(w)(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \bar{y}(w)(0,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ \bar{y}(w)(t,0,x) & = & 0 & \text{in} \quad Q_{T,1} \end{array}\right. \tag{48}$$

and

$$\left\{\begin{array}{llll} -\frac{\partial \bar{\xi}_\gamma}{\partial t} - \frac{\partial \bar{\xi}_\gamma}{\partial a} - (k(x)\bar{\xi}_x)_x + \mu\bar{\xi}_\gamma & = & \bar{y}(w) & \text{in} \quad Q \\ \bar{\xi}(t,a,1) = \bar{\xi}(t,a,0) & = & 0 & \text{on} \quad Q_{T,A} \\ \bar{\xi}(T,a,x) & = & 0 & \text{in} \quad Q_{A,1} \\ \bar{\xi}(t,A,x) & = & 0 & \text{in} \quad Q_{T,1} \end{array}\right. \tag{49}$$

We introduce adjoint state $\tilde{p}_\gamma$ and $\tilde{q}_\gamma$ solutions of (44) and (45) to interpret (47). Then we multiply the first equation of (48) by $\tilde{q}_\gamma$ and we integrate by parts on Q. We obtain:

$$\int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d + \frac{1}{\sqrt{\gamma}}p_\gamma + \frac{1}{\gamma}S(a,\tilde{u}_\gamma)\bar{\xi}(t,0,x))\mathrm{dtdadx} = \int_{Q_\omega}\tilde{q}_\gamma w\mathrm{dtdadx}, \quad \forall w \in L^2(Q_\omega) \tag{50}$$

And we multiply the first equation of (49) by $\frac{1}{\sqrt{\gamma}}p_\gamma$ and integrate by parts on Q. We get:

$$\frac{1}{\gamma}\int_Q \bar{\xi}(t,0,x)S(a,\tilde{u}_\gamma)\tilde{y}_\gamma\mathrm{dtdadx} = \frac{1}{\sqrt{\gamma}}\int_Q \bar{y}(w)p_\gamma\mathrm{dtdadx}, \quad \forall w \in L^2(Q_\omega) \tag{51}$$

By reducing the relation (47) using relations (50) and (51), we get:

$$\int_{Q_\omega} w((N+1)\tilde{u}_\gamma - \tilde{u} + \tilde{q}_\gamma)\mathrm{dtdadx} = 0, \quad \forall w \in L^2(Q_\omega)$$

Finally we deduce the relation (46).

Proposition 5.3. *Let $\tilde{u}_\gamma \in L^2(Q_\omega)$ be a solution of problem (40). Let also $\{\tilde{y}_\gamma, \tilde{\xi}_\gamma, \tilde{p}_\gamma, \tilde{q}_\gamma\}$ be solution of systems (42)-(46). Then we have following estimations:*

$$\|\tilde{u}_\gamma\|_{L^2(Q_\omega)} \leq C \tag{52}$$

$$\frac{1}{\sqrt{\gamma}}\|S(a,\tilde{u}_\gamma)\|_{L^2(0,A)} \leq C \tag{53}$$

$$\|\tilde{y}_\gamma\|_{L^2(Q_{T,A};H_k^1(0,1))} \leq C \tag{54}$$

$$\|\tilde{\xi}_\gamma\|_{L^2(Q_{T,A};H_k^1(0,1))} \leq C \tag{55}$$

$$\|\tilde{\xi}_\gamma(.,0,.)\|_{L^2(Q_{T,1})} \leq C \tag{56}$$

$$\|\tilde{p}_\gamma\|_{L^2(Q_{T,A};H_k^1(0,1))} \leq C \tag{57}$$

$$\|\tilde{p}_\gamma(.,0,.)\|_{L^2(Q_{T,1})} \leq C \tag{58}$$

$$\|\tilde{q}_\gamma\|_{L^2(Q_{T,A};H_k^1(0,1))} \leq C \tag{59}$$

where C is a positive constant and none the same in each inequality.

Proof 13. *We proceed by steps:*
Step 1: *We prove the estimations (52)-(55).*
According to Proposition 5.1, $\tilde{u}_\gamma$ is a solution of (40). So,

$$\tilde{\mathscr{J}}_\gamma(\tilde{u}_\gamma) \leq \tilde{\mathscr{J}}^\gamma(0) = \|\tilde{u}\|_{L^2(Q_\omega)}$$

Then, from defnition of $\tilde{\mathscr{J}}^\gamma$, J and by relations (41) and (2) that,

$$\|\tilde{y}_\gamma - z_d\|^2_{L^2(Q)} + N\|\tilde{u}_\gamma\|^2_{L^2(Q_\omega)} + \|\tilde{u}_\gamma - \tilde{u}\|^2_{L^2(Q_\omega)} + \frac{1}{\gamma}\|S(.,\tilde{u}_\gamma)\|^2_{L^2(0,A)} \leq \|\tilde{u}_\gamma\|^2_{L^2(Q_\omega)} + \|y(0,0) - z_d\|^2_{L^2(Q)}$$
$$= C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_{A,1})}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right)$$

as the state $y(v,g)$ depends of the initial term y^0, the control $\tilde{u}$ and the source term f. In addition, we can set $C = C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_{A,1})}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right)$.
Hence, we deduce the relations (52), (53) and

$$\|\tilde{y}_\gamma - z_d\|_{L^2(Q)} \leq C \tag{60}$$

For the proof of (54)-(56), we proceed as for Proof 1. As $\tilde{y}_\gamma$ and $\tilde{\xi}_\gamma$ are respectively solutions of (48) and (49), we obtain the following estimations:

$$\|\tilde{y}_\gamma\|_{L^2(Q_{T,A};H^1_k(0,1))} \leq \frac{1}{\sqrt{2}}e^{rT}\left(\|\tilde{u}_\gamma\|_{L^2(Q_\omega)} + \|y^0\|_{L^2(Q_{A,1})} + \|f\|_{L^2(Q)}\right)$$

and

$$\|\tilde{\xi}_\gamma\|_{L^2(Q_{T,A};H^1_k(0,1))} + \|\tilde{\xi}_\gamma(.,0,.)\|_{L^2(Q_T)} \leq \frac{\sqrt{2}}{2}e^{rT}\|\tilde{y}_\gamma - z_d\|_{L^2(Q)}.$$

Step 2: *Now, we will prove the estimations (57) and (58).*
The proof of (57) is similar to the latter one. As $\tilde{p}_\gamma$ is the solution of (44), we get:

$$\|\tilde{p}_\gamma\|_{L^2(Q_{T,A};H^1_k(0,1))} \leq C$$

The proof of (58) uses the inequality:

$$\int_{Q_{T,1}}\left|\frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\tilde{u}_\gamma,0)S(a,\tilde{u}_\gamma)\mathrm{da}\right|^2 \mathrm{dtdx} \leq \frac{1}{\gamma}\|S(.,\tilde{u}_\gamma)\|^2_{L^2(0,A)}\|\tilde{y}_\gamma\|^2_{L^2(Q)} \tag{61}$$

and the relations (53), (54). We deduce:

$$\|\tilde{p}_\gamma(.,0,.)\|_{L^2(Q_{T,1})} \leq C$$

Step 3: *At last, we will prove the estimation (59).*
We decompose the solution of (45) $\tilde{q}_\gamma$ as $\tilde{q}_\gamma = \tilde{q}^1_\gamma + \tilde{q}^2_\gamma$ where $\tilde{q}^1_\gamma$ is the solution of:

$$\begin{cases} -\frac{\partial \tilde{q}^1_\gamma}{\partial t} - \frac{\partial \tilde{q}^1_\gamma}{\partial a} - (k(x)\tilde{q}^1_{\gamma x})_x + \mu\tilde{q}^1_\gamma & = \tilde{y}_\gamma - z_d \quad \text{in } Q \\ \tilde{q}^1_\gamma(t,a,1) = \tilde{q}^1_\gamma(t,a,0) & = 0 \quad \text{on } Q_{T,A} \\ \tilde{q}^1_\gamma(T,a,x) & = 0 \quad \text{in } Q_{A,1} \\ \tilde{q}^1_\gamma(t,A,x) & = 0 \quad \text{in } Q_{T,1} \end{cases} \tag{62}$$

and $\tilde{q}^2_\gamma$ is the solution of:

$$\begin{cases} -\frac{\partial \tilde{q}^2_\gamma}{\partial t} - \frac{\partial \tilde{q}^2_\gamma}{\partial a} - (k(x)\tilde{q}^2_{\gamma x})_x + \mu\tilde{q}^2_\gamma & = \frac{1}{\sqrt{\gamma}}p_\gamma + \frac{1}{\gamma}\tilde{\xi}(0)S(a,\tilde{u}_\gamma) \quad \text{in } Q \\ \tilde{q}^2_\gamma(t,a,1) = \tilde{q}^2_\gamma(t,a,0) & = 0 \quad \text{on } Q_{T,A} \\ \tilde{q}^2_\gamma(T,a,x) & = 0 \quad \text{in } Q_{A,1} \\ \tilde{q}^2_\gamma(t,A,x) & = 0 \quad \text{in } Q_{T,1} \end{cases} \tag{63}$$

The same reasoning as in Proof 1 and the estimation (60) involve:

$$\|\tilde{q}^1_\gamma\|_{L^2(Q_{T,A};H^1_k(0,1))} \leq C \tag{64}$$

where $C = C\left(\|\tilde{u}\|_{L^2(Q_\omega)}, \|y^0\|_{L^2(Q_{A,1})}, \|f\|_{L^2(Q)}, \|z_d\|_{L^2(Q)}\right)$.
The relations (47) and (51) permit to obtain:

$$0 = \int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d)\mathrm{dtdadx} + \int_{Q_\omega} N\tilde{u}_\gamma w\mathrm{dtdadx} + \int_{Q_\omega}(\tilde{u}_\gamma - \tilde{u})w\mathrm{dtdadx} + \int_Q \bar{y}(w)(\frac{1}{\sqrt{\gamma}}p_\gamma + \frac{1}{\gamma}\tilde{\xi}(0)S(a,\tilde{u}_\gamma))\mathrm{dtdadx}, \quad \forall w \in L^2(Q_\omega) \tag{65}$$

We set

$$\mathscr{O} = \{\bar{y}(w), w \in L^2(Q_\omega)\}.$$

Then $\mathscr{O} \subset L^2(Q)$. We define on $\mathscr{O} \times \mathscr{O}$ the inner product

$$\langle \bar{y}(v), \bar{y}(w)\rangle_{\mathscr{O}} = \int_{Q_\omega} vw\mathrm{dtdadx} + \int_Q \bar{y}(v)\bar{y}(w)\mathrm{dtdadx}, \quad \forall (\bar{y}(v), \bar{y}(w)) \in \mathscr{O} \times \mathscr{O}$$

The set $\mathscr{O}$ endowed with norm $\|\bar{y}(w)\|^2_{\mathscr{O}} = \|w\|^2_{L^2(Q_\omega)} + \|\bar{y}(w)\|^2_{L^2(Q)}$ is a Hilbert space.
By setting

$$T_\gamma(\tilde{u}_\gamma) = \frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma}\tilde{\xi}(0)S(a,\tilde{u}_\gamma)$$

For any $w \in L^2(Q_\omega)$ and taking account (65), we get :

$$\int_Q T_\gamma(\tilde{u}_\gamma)\bar{y}(w)\text{dtdadx} = -\int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d)\text{dtdadx} - \int_{Q_\omega} N\tilde{u}_\gamma w\text{dtdadx} - \int_{Q_\omega}(\tilde{u}_\gamma - \tilde{u})w\text{dtdadx} \tag{66}$$

The inequality of Cauchy-Schwarz permits to write:

$$\begin{aligned}\left|-\int_Q \bar{y}(w)(\tilde{y}_\gamma - z_d)\text{dtdadx} - \int_{Q_\omega}((N+1)\tilde{u}_\gamma - \tilde{u})w\text{dtdadx}\right| &\leq \|\tilde{y}_\gamma - z_d\|_{L^2(Q)}\|\bar{y}(w)\|_{L^2(Q)} + (N+1)\|\tilde{u}_\gamma\|_{L^2(Q_\omega)}\|w\|_{L^2(Q_\omega)} + \|\tilde{u}\|_{L^2(Q_\omega)}\|w\|_{L^2(Q_\omega)} \\ &\leq \|\tilde{y}_\gamma - z_d\|_{L^2(Q)} + (N+1)\|\tilde{u}_\gamma\|_{L^2(Q_\omega)} + \|\tilde{u}\|_{L^2(Q_\omega)}\|\bar{y}(w)\|_{\mathscr{O}} \\ &\leq C\|\bar{y}(w)\|_{\mathscr{O}}, \quad \forall w \in L^2(Q_\omega),\end{aligned}$$

where $C = C\left(N, \|\tilde{u}\|_{L^2(Q_\omega)}, \|z_d\|_{L^2(Q)}, \|y^0\|_{L^2(0,A)}, \|f\|_{L^2(Q)}\right) > 0$
Using (65), we have:

$$\left|\int_Q T_\gamma(\tilde{u}_\gamma)\bar{y}(w)\text{dtdadx}\right| \leq C\|\bar{y}(w)\|_{\mathscr{O}}, \quad \forall w \in L^2(Q_\omega)$$

Whence

$$\|T_\gamma(\tilde{u}_\gamma)\|_{\mathscr{O}} = \|\frac{1}{\sqrt{\gamma}} p_\gamma + \frac{1}{\gamma}\tilde{\xi}(0)S(a,\tilde{u}_\gamma)\|_{\mathscr{O}} \leq C \tag{67}$$

Finally, we use the reasoning of Proof 1 and the inequality (67) to obtain:

$$\|\tilde{q}_2^\gamma\|_{L^2(Q_{T,A};H^1_k(0,1))} \leq C \tag{68}$$

Thus, from (63) and (67), we deduce (59).

6. Characterization of the No-regret control

Proposition 6.1. *The adapted Low-regret control $\tilde{u}_\gamma$ converges toward the No-regret control $\tilde{u}$ in $L^2(Q_\omega)$.*

Proof 14. *Using (52)-(56), there exists a subsequence of $(\tilde{u}_\gamma, \tilde{y}_\gamma, \tilde{\xi}_\gamma, S(.,\tilde{u}_\gamma))$ still denoted $(\tilde{u}_\gamma, \tilde{y}_\gamma, \tilde{\xi}_\gamma, S(.,\tilde{u}_\gamma))$ and $\tilde{u} \in L^2(Q_\omega)$, $\tilde{y} \in L^2(U;H^1_k(0,1))$, $\tilde{\xi} \in L^2(U;H^1_k(0,1))$, $\lambda \in L^2(0,A)$, $\zeta \in L^2(Q_{T,1})$ such that*

$$\tilde{u}_\gamma \rightharpoonup \tilde{u} \text{ weakly in } L^2(Q_\omega) \tag{69}$$

$$\frac{1}{\sqrt{\gamma}}S(a,\tilde{u}_\gamma) \rightharpoonup \lambda \text{ weakly in } L^2(0,A) \tag{70}$$

$$S(.,\tilde{u}_\gamma) \to 0 \text{ strongly in } L^2(0,A) \tag{71}$$

$$\tilde{y}_\gamma \rightharpoonup \tilde{y} \text{ weakly in } L^2(Q_{T,A};H^1_k(0,1)) \tag{72}$$

$$\tilde{\xi}_\gamma \rightharpoonup \tilde{\xi} \text{ weakly in } L^2(Q_{T,A};H^1_k(0,1)) \tag{73}$$

$$\tilde{\xi}(.,0,.) \rightharpoonup \zeta \text{ weakly in } L^2(Q_{T,1}) \tag{74}$$

We multiply the first equation of (42) by $\varphi \in C^\infty(Q)$ and the first equation of (43) by $\psi \in C^\infty(Q)$ and integrate by parts over Q:

$$\int_Q \varphi\left(\frac{\partial\tilde{y}_\gamma}{\partial t} + \frac{\partial\tilde{y}_\gamma}{\partial a} - (k(x)\tilde{y}_{\gamma x})_x + \mu\tilde{y}_\gamma\right)\text{dtdadx} = \int_Q \varphi(f + \tilde{u}_\gamma\chi_\omega)\text{dtdadx}$$

and

$$\int_Q \psi\left(-\frac{\partial\tilde{\xi}_\gamma}{\partial t} - \frac{\partial\tilde{\xi}_\gamma}{\partial a} - (k(x)\tilde{\xi}_{\gamma x})_x + \mu\tilde{\xi}_\gamma\right)\text{dtdadx} = \int_Q \psi(\tilde{y}_\gamma - z_d)\text{dtdadx}.$$

We obtain respectively:

$$\int_Q \tilde{y}_\gamma\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right)\text{dtdadx} = \int_Q \varphi(f + \tilde{u}_\gamma\chi_\omega)\text{dtdadx}$$

and

$$\int_Q \tilde{\xi}_\gamma\left(\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right)\text{dtdadx} = \int_Q \psi(\tilde{y}_\gamma - z_d)\text{dtdadx}$$

The limit of these two latter identities while $\gamma \to 0$ and using (69),(72) and (73) leads respectively to:

$$\int_Q \tilde{y}\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right)\mathrm{dtdadx} = \int_Q \varphi(f + \bar{u}\chi_\omega)\mathrm{dtdadx}$$

and

$$\int_Q \tilde{\xi}\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right)\mathrm{dtdadx} = \int_Q \psi(\tilde{y} - z_d)\mathrm{dtdadx}$$

We integrate these two latter identities by parts over Q and we get:

$$\int_Q \varphi\left(\frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{y}}{\partial a} - (k(x)\tilde{y}_x)_x + \mu\tilde{y}\right)\mathrm{dtdadx} = \int_Q \varphi(f + \bar{u}\chi_\omega)\mathrm{dtdadx}$$

and

$$\int_Q \psi\left(-\frac{\partial \tilde{\xi}}{\partial t} - \frac{\partial \tilde{\xi}}{\partial a} - (k(x)\tilde{\xi}_x)_x + \mu\tilde{\xi}\right)\mathrm{dtdadx} = \int_Q \psi(\tilde{y} - z_d)\mathrm{dtdadx}$$

Thus

$$\frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{y}}{\partial a} - (k(x)\tilde{y}_x)_x + \mu\tilde{y} = f + \bar{u}\chi_\omega \quad \textit{a.e. in } Q \tag{75}$$

and

$$-\frac{\partial \tilde{\xi}}{\partial t} - \frac{\partial \tilde{\xi}}{\partial a} - (k(x)\tilde{\xi}_x)_x + \mu\tilde{\xi} = \tilde{y} - z_d \quad \textit{a.e. in } Q \tag{76}$$

Then $\tilde{y}, \tilde{\xi} \in L^2(Q_{T,A}; H_k^1(0,1))$ implie $\tilde{y}|_{Q_{T,A}}$, $\tilde{\xi}|_{Q_{T,A}}$ exist a.e.$(t,a) \in Q_{T,A}$ and belong to $L^2(Q_{T,A})$. Using (75) and (76) on the other hand, we deduce that $\tilde{y}, \tilde{\xi} \in L^2(Q_{T,A}; H_k^1(0,1))$ involve $\frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{y}}{\partial a} \in L^2(Q_{T,A}; H_k^{-1}(0,1))$ and $-\frac{\partial \tilde{\xi}}{\partial t} - \frac{\partial \tilde{\xi}}{\partial a} \in L^2(Q_{T,A}; H_k^{-1}(0,1))$. Which implies $\tilde{y} \in \mathscr{W}(T,A)$ and $\tilde{\xi} \in \mathscr{W}(T,A)$. Thus using Lemma 1, the traces $\tilde{y}(.,0,.)$, $\tilde{y}(0,.,.)$ exist and belong respectively to $L^2(Q_{T,1})$ and to $L^2(Q_{A,1})$. Too the traces $\tilde{\xi}(.,A,.)$, $\tilde{\xi}(T,.,.)$ exist and belong respectively to $L^2(Q_{T,1})$ and to $L^2(Q_{A,1})$.
Now multiply the first equation of (42) by $\varphi \in C^\infty(Q)$ such that $\varphi = 0$ on $Q_{T,A}$, $\varphi(T,.,.) = 0$ in $Q_{A,1}$ and $\varphi(.,A,.) = 0$ in $Q_{T,1}$ and the first equation of (43) by $\psi \in C^\infty(Q)$ such that $\psi = 0$ on $Q_{T,A}$, $\psi(0,.,.) = 0$ in $Q_{A,1}$ then integrate by parts on Q:

$$-\int_{Q_{A,1}} y^0\varphi(0,a,x)\mathrm{dadx} + \int_Q \tilde{y}_\gamma\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right)\mathrm{dtdadx} = \int_Q \varphi(f + \tilde{u}_\gamma\chi_\omega)\mathrm{dtdadx}$$

and

$$\int_{Q_{T,1}} \tilde{\xi}_\gamma(t,0,x)\psi(t,0,x)\mathrm{dtdx} + \int_Q \tilde{\xi}_\gamma\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right)\mathrm{dtdadx} = \int_Q \psi(\tilde{y}_\gamma - z_d)\mathrm{dtdadx}$$

Passing these two latter identities to the limit while $\gamma \to 0$ using (69),(72),(73) and (74), we get respectively, $\forall\varphi \in C^\infty(\bar{Q})$ such that $\varphi = 0$ on $Q_{T,A}$, $\varphi(T,.,.) = 0$ on $Q_{A,1}$ and $\varphi(.,A,.) = 0$ on $Q_{T,1}$:

$$-\int_{Q_{A,1}} y^0\varphi(0,a,x)\mathrm{dadx} + \int_Q \tilde{y}\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu\varphi\right)\mathrm{dtdadx} = \int_Q \varphi(f + \hat{u}\chi_\omega)\mathrm{dtdadx}$$

and, $\forall\psi \in C^\infty(\bar{Q})$ such that $\psi = 0$ on $Q_{T,A}$, $\psi(0,.,.) = 0$ on $Q_{A,1}$

$$\int_{Q_{T,1}} \zeta\psi(t,0,x)\mathrm{dtdx} + \int_Q \tilde{\xi}\left(\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial a} - (k(x)\psi_x)_x + \mu\psi\right)\mathrm{dtdadx} = \int_Q \psi(\tilde{y} - z_d)\mathrm{dtdadx}$$

which after an integration by parts over Q give respectively

$$\int_Q \varphi(f + \hat{u}\chi_\omega)\mathrm{dtdadx} = \int_{Q_{A,1}} (\tilde{y}(0,a,x) - y^0)\varphi(0,a,x)\mathrm{dadx} + \int_Q \varphi\left(\frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{y}}{\partial a} - (k(x)\tilde{y}_x)_x + \mu\tilde{y}\right)\mathrm{dtdadx} + \int_{Q_{T,1}} (\tilde{y}(t,0,x)\varphi(t,0,x)\mathrm{dtdx}$$

and

$$\begin{aligned}\int_Q \psi(\tilde{y} - z_d)\mathrm{dtdadx} = &\int_{Q_{T,1}} (\zeta - \tilde{\xi}(t,0,x))\psi(t,0,x)\mathrm{dtdx} + \int_Q \psi\left(-\frac{\partial \tilde{\xi}}{\partial t} - \frac{\partial \tilde{\xi}}{\partial a} - (k(x)\tilde{\xi}_x)_x + \mu\tilde{\xi}\right)\mathrm{dtdadx} + \int_{Q_{T,1}} \tilde{\xi}(t,A,x))\psi(t,A,x)\mathrm{dtdx} \\ &+ \int_{Q_{A,1}} \tilde{\xi}(T,a,x))\psi(T,a,x)\mathrm{dadx}\end{aligned}$$

$\forall\varphi \in C^\infty(Q)$ such that $\varphi = 0$ on $Q_{T,A}$, $\varphi(T,.,.) = 0$ in $Q_{A,1}$ and $\varphi(.,A,.) = 0$ in $Q_{T,1}$ and using (75) and (76), it leads to:

$$0 = \int_{Q_{A,1}} (\tilde{y}(0,a,x) - y^0)\varphi(0,a,x)\mathrm{dadx} + \int_{Q_{T,1}} (\tilde{y}(t,0,x)\varphi(t,0,x)\mathrm{dtdx} \tag{77}$$

and forall $\psi \in C^\infty(Q)$ *such that* $\psi = 0$ *on* $Q_{T,A}$ *and* $\psi(0,.,.) = 0$ *in* $Q_{A,1}$*:*

$$0 = \int_{Q_{T,1}} (\zeta - \tilde{\xi}(t,0,x))\psi(t,0,x)\mathrm{dtdx} + \int_{Q_{T,1}} \tilde{\xi}(t,A,x))\psi(t,A,x)\mathrm{dtdx} + \int_{Q_{A,1}} \tilde{\xi}(T,a,x))\psi(T,a,x)\mathrm{dadx} \tag{78}$$

Then, we deduce:

$$\begin{aligned} \tilde{y}(0,a,x) = y^0 \qquad & \text{in } Q_{A,1} \\ \tilde{y}(t,0,x) = 0 \qquad & \text{in } Q_{T,1} \end{aligned} \tag{79}$$

and

$$\begin{aligned} & \tilde{\xi}(T,a,x) = 0 \qquad \text{in } Q_{A,1} \\ \tilde{\xi}(t,0,x) = \zeta, \qquad & \tilde{\xi}(t,A,x) = 0 \qquad \text{in } Q_{T,1} \end{aligned} \tag{80}$$

Then, according to (69),(72), (74), (80) *and* (53)*, we have from Lemma 2 that*

$$S(.,\tilde{u}_\gamma) \rightharpoonup S(.,\bar{u}) \text{ weakly in } \mathscr{D}'(A).$$

Hence, using (71) *and the uniqueness of the limit that*

$$S(.,\tilde{u}_\gamma) \to S(.,\bar{u}) \text{ strongly in } L^2(0,A).$$

Therefore,

$$\int_0^A S(.,\tilde{u}_\gamma)g(a)\mathrm{da} \to \int_0^A S(.,\bar{u})g(a)\mathrm{da} = 0.$$

Thus $\bar{u} \in \mathscr{U}_{ad}$ *and we also have* $\|S(.,\bar{u})\|_{L^2(0,A)} = 0$*. Since* $\tilde{u}$ *is a No-regret control and* $\bar{u} \in \mathscr{U}_{ad}$ *, it follows from* (28)-(29) *that*

$$J(\tilde{u},0) - J(0,0) \le J(\bar{u},0) - J(0,0), \tag{81}$$

Observing that $\tilde{u}_\gamma$ *solves the problem* $\inf\limits_{v \in L^2(Q_\omega)} \tilde{\mathscr{J}}_\gamma(v)$*, we have*

$$\tilde{\mathscr{J}}_\gamma(\tilde{u}_\gamma) \le \tilde{\mathscr{J}}_\gamma(\tilde{u}) = J(\tilde{u},0) - J(0,0) \tag{82}$$

which in view of the definition of $\tilde{\mathscr{J}}_\gamma$ *given by* (41) *implies that*

$$J(\tilde{u}_\gamma,0) - J(0,0) + \|\tilde{u}_\gamma - \tilde{u}\|^2_{L^2(Q_\omega)} \le \tilde{\mathscr{J}}_\gamma(\tilde{u}_\gamma) \le \tilde{\mathscr{J}}_\gamma(\tilde{u}) = J(\tilde{u},0) - J(0,0)$$

Using the convexity and lower semi-continuity of J *on* $L^2(Q_\omega)$*,* (69) *and* (72)*, we obtain:*

$$J(\bar{u},0) - J(0,0) + \|\bar{u} - \tilde{u}\|^2_{L^2(Q_\omega)} \le \liminf_{\gamma \to 0} \tilde{\mathscr{J}}_\gamma(\tilde{u}_\gamma) \le J(\tilde{u},0) - J(0,0). \tag{83}$$

which combining with (81) *gives*

$$\|\bar{u} - \tilde{u}\|^2_{L^2(Q_\omega)} \le 0$$

Hence,

$$\bar{u} = \tilde{u} \text{ in } Q_\omega. \tag{84}$$

Thus the adapted Low-regret control converges in $L^2(Q_\omega)$ *to the No-regret control* $\tilde{u} \in \mathscr{U}_{ad}$*.*
Further, $\tilde{y}, \tilde{\xi} \in L^2(Q_{T,A}; H^1_k(0,1))$ *involve:*

$$\begin{aligned} \tilde{y}(t,a,0) = \tilde{y}(t,a,1) = 0 \\ \tilde{\xi}(t,a,0) = \tilde{\xi}(t,a,1) = 0 \end{aligned} \tag{85}$$

Now using successively (75), (76), (79),(84) *and* (85) *we deduce, on one hand side that* $\tilde{y} = y(\tilde{u},0) \in L^2(Q_{T,A}; H^1_k(0,1))$ *unique solution of:*

$$\begin{cases} \frac{\partial \tilde{y}}{\partial t} + \frac{\partial \tilde{y}}{\partial a} - (k(x)\tilde{y}_x)_x + \mu \tilde{y} & = f + \tilde{u}\chi_\omega & \text{in } Q \\ \tilde{y}(t,a,1) = \tilde{y}(t,a,0) & = 0 & \text{on } Q_{T,A} \\ \tilde{y}(0,a,x) & = y^0 & \text{in } Q_{A,1} \\ \tilde{y}(t,0,x) & = 0 & \text{in } Q_{T,1} \end{cases}$$

and on the other hand side that $\tilde{\xi} = \xi(\tilde{u}) \in L^2(Q_{T,A}; H^1_k(0,1))$ *is the unique solution of:*

$$\begin{cases} -\frac{\partial \tilde{\xi}}{\partial t} - \frac{\partial \tilde{\xi}}{\partial a} - (k(x)\tilde{\xi}_x)_x + \mu \tilde{\xi} & = \tilde{y} - z_d & \text{in } Q \\ \tilde{\xi}(t,a,1) = \tilde{\xi}(t,a,0) & = 0 & \text{on } Q_{T,A} \\ \tilde{\xi}(T,a,x) & = 0 & \text{in } Q_{A,1} \\ \tilde{\xi}(t,A,x) & = 0 & \text{in } Q_{T,1} \end{cases}$$

In what follows, we will try to characterize the unique no regret control $\tilde{u}$.

Proposition 6.2. *The No-regret control $\tilde{u}$ solution of problem (3) is characterized by the quadruplet $\{\tilde{y},\tilde{\xi},\tilde{p},\tilde{q}\}$ such that:*

$$\begin{cases} \frac{\partial \tilde{y}}{\partial t}+\frac{\partial \tilde{y}}{\partial a}-(k(x)\tilde{y}_x)_x+\mu\tilde{y} & =f+\tilde{u}\chi_\omega & \text{in } Q \\ \tilde{y}(t,a,1)=\tilde{y}(t,a,0) & =0 & \text{on } Q_{T,A} \\ \tilde{y}(0,a,x) & =y^0 & \text{in } Q_{A,1} \\ \tilde{y}(t,0,x) & =0 & \text{in } Q_{T,1} \end{cases} \tag{86}$$

$$\begin{cases} -\frac{\partial \tilde{\xi}}{\partial t}-\frac{\partial \tilde{\xi}}{\partial a}-(k(x)\tilde{\xi}_x)_x+\mu\tilde{\xi} & =\tilde{y}-z_d & \text{in } Q \\ \tilde{\xi}(t,a,1)=\tilde{\xi}(t,a,0) & =0 & \text{on } Q_{T,A} \\ \tilde{\xi}(T,a,x) & =0 & \text{in } Q_{A,1} \\ \tilde{\xi}(t,A,x) & =0 & \text{in } Q_{T,1} \end{cases} \tag{87}$$

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t}+\frac{\partial \tilde{p}}{\partial a}-(k(x)\tilde{p}_x)_x+\mu\tilde{p} & = & 0 & \text{in} & Q \\ \tilde{p}(t,a,1)=\tilde{p}(t,a,0) & = & 0 & \text{on} & Q_{T,A} \\ \tilde{p}(0,a,x) & = & 0 & \text{in} & Q_{A,1} \\ \tilde{p}(t,0,x) & = & k_1 & \text{in} & Q_{T,1} \end{cases} \tag{88}$$

$$\begin{cases} -\frac{\partial \tilde{q}}{\partial t}-\frac{\partial \tilde{q}}{\partial a}-(k(x)\tilde{q}_x)_x+\mu\tilde{q} & = & y(\tilde{u},0)-z_d+k_2 & \text{in} & Q \\ \tilde{q}(t,a,1)=\tilde{q}(t,a,0) & = & 0 & \text{on} & Q_{T,A} \\ \tilde{q}(T,a,x) & = & 0 & \text{in} & Q_{A,1} \\ \tilde{q}(t,A,x) & = & 0 & \text{in} & Q_{T,1} \end{cases} \tag{89}$$

and

$$N\tilde{u}+\tilde{q}=0 \quad in \quad Q_\omega \tag{90}$$

where

$$k_1=\lim_{\gamma\to 0}\frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\tilde{u}_\gamma,0)S(a,\tilde{u}_\gamma)\mathrm{da} \quad and \quad k_2=\lim_{\gamma\to 0}\frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma+\frac{1}{\gamma}S(a,\tilde{u}_\gamma)\xi(\tilde{u}_\gamma)(t,0,x)$$

Proof 15. *The results (86) and (87) have been demonstrated during the proof of the previous proposition. It remains the demonstration of (88)-(90). For this, we will proceed by steps:*
Step 1*: show that p is solution of (88).*
Using relations (61), (53), and (54) on the one hand side, we can deduce that there exists a constant $C>0$ such that:

$$\int_{Q_{T,1}}\left|\frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\tilde{u}_\gamma,0)S(a,\tilde{u}_\gamma)\mathrm{da}\right|^2\mathrm{dtdx}\le\frac{1}{\gamma}\|S(.,\tilde{u}_\gamma)\|^2_{L^2(0,A)}\|\tilde{y}_\gamma\|^2_{L^2(Q)}\le C \tag{91}$$

And on the other hand side, noting that $\tilde{p}_\gamma=\tilde{p}(u_\gamma,.)$ satisfies the system (44), we deduce that there exists an other constant $C>0$ such that:

$$\|\tilde{p}_\gamma\|_{L^2(Q)}\le C \tag{92}$$

With regard to estimates (91) and (92), we deduce that there exists $k_1\in L^2(0,A)$ and $\tilde{p}\in L^2(Q)$ such that:

$$\frac{1}{\sqrt{\gamma}}\int_0^A y(t,a,x;\tilde{u}_\gamma,0)S(a,\tilde{u}_\gamma)\mathrm{da}\rightharpoonup k_1 \qquad weakly\ in \quad L^2(0,A) \tag{93}$$

$$\tilde{p}_\gamma\rightharpoonup\tilde{p} \qquad weakly\ in \quad L^2(Q) \tag{94}$$

Now multiply the first equation of (44) by a test function $\varphi\in C^\infty(Q)$ such that $\varphi=0$ on $Q_{T,A}$, $\varphi(T,.,.)=0$ in $Q_{A,1}$ and $\varphi(.,A,.)=0$ in $Q_{T,1}$. Then we integrate by parts on Q:

$$\begin{aligned} 0&=\int_Q\varphi\left(\frac{\partial\tilde{p}_\gamma}{\partial t}+\frac{\partial\tilde{p}_\gamma}{\partial a}-(k(x)\tilde{p}_{\gamma x})_x+\mu\tilde{p}_\gamma\right)\mathrm{dtdadx} \\ &=\int_Q\tilde{p}_\gamma\left(-\frac{\partial\varphi}{\partial t}-\frac{\partial\varphi}{\partial a}-(k(x)\varphi_x)_x+\mu\varphi\right)\mathrm{dtdadx} \end{aligned}$$

passing to the limit when $\gamma\to 0$ in the last equation and using (94), we obtain:

$$0=\int_Q\tilde{p}\left(-\frac{\partial\varphi}{\partial t}-\frac{\partial\varphi}{\partial a}-(k(x)\varphi_x)_x+\mu\varphi\right)\mathrm{dtdadx}$$

We integrate this last equality by parts on Q:

$$0=\int_Q\varphi\left(\frac{\partial\tilde{p}}{\partial t}+\frac{\partial\tilde{p}}{\partial a}-(k(x)\tilde{p}_x)_x+\mu\tilde{p}\right)\mathrm{dtdadx}$$

So we deduce that:

$$\frac{\partial \tilde{p}}{\partial t} - \frac{\partial \tilde{p}}{\partial a} - (k(x)\tilde{p}_x)_x + \mu \tilde{p} = 0 \qquad a.e. \text{ in } \quad Q \tag{95}$$

Then, $\tilde{p} \in L^2(Q_{T,A}; H_k^1(0,1))$ *implies* $\tilde{p}|_{Q_{T,A}}$ *exists* $a.e.(t,a) \in Q_{T,A}$ *and belongs to* $L^2(Q_{T,A})$*. Using* (95) *on the other hand, we deduce that* $\tilde{p} \in L^2(Q_{T,A}; H_k^1(0,1))$ *involves* $\dfrac{\partial \tilde{p}}{\partial t} + \dfrac{\partial \tilde{p}}{\partial a} \in L^2(Q_{T,A}; H_k^{-1}(0,1))$*. Which implies* $\tilde{p} \in \mathscr{W}(T,A)$*. Thus using Lemma 1, the traces* $\tilde{p}(.,0,.)$ *and* $\tilde{p}(0,.,.)$ *exist and belong respectively to* $L^2(Q_{T,1})$ *and to* $L^2(Q_{A,1})$*.*
Consider now $\varphi \in C^\infty(Q)$ *such that* $\varphi = 0$ *on* $Q_{T,A}$*,* $\varphi(T,.,.) = 0$ *in* $Q_{A,1}$ *and* $\varphi(.,A,.) = 0$ *in* $Q_{T,1}$*. Multiply the first equation of* (44) *by* φ *then integrate by parts on* Q*:*

$$0 = \int_Q \tilde{p}_\gamma \left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu \varphi \right) \text{dtdadx} + \int_{Q_{T,1}} \tilde{p}(.,A,.)\varphi(.,A,.)\text{dtdx} - \int_{Q_{T,1}} \tilde{p}_\gamma(.,0,.)\varphi(.,0,.)\text{dtdx}$$
$$+ \int_{Q_{A,1}} \tilde{p}_\gamma(T,.,.)\varphi(T,.,.)\text{dadx} - \int_{Q_{A,1}} \tilde{p}_\gamma(0,.,.)\varphi(0,.,.))\text{dadx} + \int_{Q_{T,A}} [k(x)\tilde{p}_\gamma \varphi_x]_0^1 \text{dtda} - \int_{Q_{T,A}} [k(x)\tilde{p}_{\gamma x}\varphi]_0^1 \text{dtda}$$

which gives, by taking into account the boundary and/or limits conditions:

$$0 = \int_Q \tilde{p}_\gamma \left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu \varphi \right) \text{dtdadx} - \int_{Q_{T,1}} \tilde{p}_\gamma(.,0,.)\varphi(.,0,.)\text{dtdx}$$
$$= \int_Q \tilde{p}_\gamma \left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu \varphi \right) \text{dtdadx} - \int_{Q_{T,1}} \varphi(.,0,.) \left(\frac{1}{\sqrt{\gamma}} \int_0^A y(t,a,x;\tilde{u}_\gamma,0) S(a,\tilde{u}_\gamma)\text{da} \right) \text{dtdx}.$$

Using (91) *and* (93)*, the Lebesgue-dominated convergence theorem permits to deduce:*

$$0 = \int_Q \tilde{p} \left(-\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial a} - (k(x)\varphi_x)_x + \mu \varphi \right) \text{dtdadx} - \int_{Q_{T,1}} k_1 \varphi(.,0,.)\text{dtdx}$$

We integrate this latter equality by parts on Q*:*

$$0 = \int_Q \varphi \left(\frac{\partial \tilde{p}}{\partial t} + \frac{\partial \tilde{p}}{\partial a} - (k(x)\tilde{p}_x)_x + \mu \tilde{p} \right) \text{dtdadx} - \int_{Q_{T,1}} \varphi(.,0,.)k_1 \text{dtdx} - \int_{Q_{T,1}} \tilde{p}(.,A,.)\varphi(.,A,.)\text{dtdx} + \int_{Q_{T,1}} \tilde{p}(.,0,.)\varphi(.,0,.)\text{dtdx}$$
$$- \int_{Q_{A,1}} \tilde{p}(T,.,.)\varphi(T,.,.)\text{dadx} + \int_{Q_{A,1}} \tilde{p}(0,.,.)\varphi(0,.,.))\text{dadx} - \int_{Q_{T,A}} [k(x)\tilde{p}\varphi_x]_0^1 \text{dtda} + \int_{Q_{T,A}} [k(x)\tilde{p}_x \varphi]_0^1 \text{dtda}$$

by taking into account the boundary and/or limits conditions:

$$0 = \int_{Q_{T,1}} \varphi(.,0,.)\left(\tilde{p}(.,0,.) - k_1\right)\text{dtdx} + \int_{Q_{A,1}} \varphi(0,.,.)\tilde{p}(0,.,.)\text{dadx}$$

$\forall \varphi \in C^\infty(Q)$ *such that* $\varphi = 0$ *on* $Q_{T,A}, \varphi(T,.,.) = 0$ *in* $Q_{A,1}$ *and* $\varphi(.,A,.) = 0$ *in* $Q_{T,1}$

we obtain:

$$\tilde{p}(.,0,.) = k_1 \text{ in } Q_{T,1} \tag{96}$$
$$\tilde{p}(0,.,.) = 0 \text{ in } Q_{A,1} \tag{97}$$

By combining (95)*,* (96)*,* (97) *and the fact that* $\tilde{p} \in L^2(Q_{T,A}; H_k^1(0,1))$*, we deduce that* $\tilde{p}$ *is solution of system:*

$$\begin{cases} \frac{\partial \tilde{p}}{\partial t} + \frac{\partial \tilde{p}}{\partial a} - (k(x)\tilde{p}_x)_x + \mu \tilde{p} & = 0 & \text{in } Q \\ \tilde{p}(t,a,1) = \tilde{p}(t,a,0) & = 0 & \text{on } Q_{T,A} \\ \tilde{p}(0,a,x) & = 0 & \text{in } Q_{A,1} \\ \tilde{p}(t,0,x) & = k_1 & \text{in } Q_{T,1} \end{cases}$$

Step 2*: We will show now that* $\tilde{q}_\gamma$ *converges toward* $\tilde{q}$ *solution of* (89)
Let $\tilde{q}_\gamma^1$ *and* $\tilde{q}_\gamma^2$ *be respective solutions of:*

$$\begin{cases} -\frac{\partial \tilde{q}_\gamma^1}{\partial t} - \frac{\partial \tilde{q}_\gamma^1}{\partial a} - (k(x)\tilde{q}_{\gamma x}^1)_x + \mu \tilde{q}_\gamma^1 & = \tilde{y}_\gamma - z_d & \text{in } Q \\ \tilde{q}_\gamma^1(t,a,1) = \tilde{q}_\gamma^1(t,a,0) & = 0 & \text{on } Q_{T,A} \\ \tilde{q}_\gamma^1(T,a,x) & = 0 & \text{in } Q_{A,1} \\ \tilde{q}_\gamma^1(t,A,x) & = 0 & \text{in } Q_{T,1} \end{cases} \tag{98}$$

and

$$\begin{cases} -\frac{\partial \tilde{q}_\gamma^2}{\partial t} - \frac{\partial \tilde{q}_\gamma^2}{\partial a} - (k(x)\tilde{q}_{\gamma x}^2)_x + \mu \tilde{q}_\gamma^2 & = \frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma + \frac{1}{\gamma}\tilde{\xi}(0)S(a,\tilde{u}_\gamma) & \text{in } Q \\ \tilde{q}_\gamma^2(t,a,1) = \tilde{q}_\gamma^2(t,a,0) & = 0 & \text{on } Q_{T,A} \\ \tilde{q}_\gamma^2(T,a,x) & = 0 & \text{in } Q_{A,1} \\ \tilde{q}_\gamma^2(t,A,x) & = 0 & \text{in } Q_{T,1} \end{cases} \tag{99}$$

Then $\tilde{q}_\gamma = \tilde{q}^1_\gamma + \tilde{q}^2_\gamma$ is solution of (45).
We proceed as for Proof 4 and we use the inequality (60). There exists constant $C > 0$ such that:

$$\|\tilde{q}^1_\gamma\|_{L^2(Q_{T,A};H^1_k(0,1))} \leq C \tag{100}$$

So, there exists $\tilde{q}^1 \in L^2(Q_{T,A};H^1_k(0,1))$ such that:

$$\tilde{q}^1_\gamma \rightharpoonup \tilde{q}^1 \quad \text{weakly in} \quad L^2(Q_{T,A};H^1_k(0,1)) \tag{101}$$

Let now $\varphi_1 \in C^\infty(Q)$ be a test function. Multiply the first equation of (98) by φ_1 then integrate by parts on Q:

$$\int_Q \varphi_1 \left(-\frac{\partial \tilde{q}^1_\gamma}{\partial t} - \frac{\partial \tilde{q}^1_\gamma}{\partial a} - (k(x)\tilde{q}^1_{\gamma x})_x + \mu \tilde{q}^1_\gamma\right) \mathrm{dtdadx} = \int_Q \varphi_1 (\tilde{y}_\gamma - z_d) \mathrm{dtdadx}$$

we obtain:

$$\int_Q \tilde{q}^1_\gamma \left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu \varphi_1\right) \mathrm{dtdadx} = \int_Q \varphi_1 (\tilde{y}_\gamma - z_d) \mathrm{dtdadx}$$

passing to the limit when $\gamma \to 0$ in the last equality and using, we get:

$$\int_Q \tilde{q}^1 \left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu \varphi_1\right) \mathrm{dtdadx} = \int_Q \varphi_1 (\tilde{y} - z_d) \mathrm{dtdadx}$$

We integrate the latter equality by parts on Q:

$$\int_Q \varphi_1 \left(-\frac{\partial \tilde{q}^1}{\partial t} - \frac{\partial \tilde{q}^1}{\partial a} - (k(x)\tilde{q}^1_x)_x + \mu \tilde{q}^1\right) \mathrm{dtdadx} = \int_Q \varphi_1 (\tilde{y} - z_d) \mathrm{dtdadx}$$

we deduce that:

$$-\frac{\partial \tilde{q}^1}{\partial t} - \frac{\partial \tilde{q}^1}{\partial a} - (k(x)\tilde{q}^1_x)_x + \mu \tilde{q}^1 = \tilde{y} - z_d \tag{102}$$

As $\tilde{q}^1 \in L^2(Q_{T,A};H^1_k(0,1))$, then $\tilde{q}^1|_{Q_{T,A}}$ exists and belongs to $L^2(Q_{T,A})$ a.e.$(t,a) \in Q_{T,A}$. On the other hand side, relation (102) and the fact that $\tilde{q}^1 \in L^2(Q_{T,A};H^1_k(0,1))$ implie that $-\frac{\partial \tilde{q}^1}{\partial t} - \frac{\partial \tilde{q}^1}{\partial a} \in L^2(Q_{T,A};H_k^{-1}(0,1))$. Whence $\tilde{q}^1 \in \mathscr{W}(T,A)$. Then according to Lemma 1, the traces $\tilde{q}^1(T,.,.)$ and $\tilde{q}^1(.,A,.)$ exist and belong respectively to $L^2(Q_{A,1})$ and to $L^2(Q_{T,1})$.
Consider $\varphi_1 \in C^\infty(Q)$ knowing that $\varphi_1 = 0$ on $Q_{T,A}$, $\varphi_1(.,0,.) = 0$ on $Q_{T,1}$ and $\varphi_1(0,.,.) = 0$ on $Q_{A,1}$. Multiply the first equation of (98) by φ_1 then integrate by parts on Q:

$$\begin{aligned}\int_Q \varphi_1 (\tilde{y}_\gamma - z_d) \mathrm{dtdadx} = & \int_Q \tilde{q}^1_\gamma \left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu \varphi_1\right) \mathrm{dtdadx} - \int_{Q_{T,1}} \tilde{q}^1_\gamma(.,A,.)\varphi_1(.,A,.)\mathrm{dtdx} + \int_{Q_{T,1}} \tilde{q}^1_\gamma(.,0,.)\varphi_1(.,0,.)\mathrm{dtdx} \\ & - \int_{Q_{A,1}} \tilde{q}^1_\gamma(T,.,.)\varphi_1(T,.,.)\mathrm{dadx} + \int_{Q_{A,1}} \tilde{q}^1_\gamma(0,.,.)\varphi_1(0,.,.))\mathrm{dadx} - \int_{Q_{T,A}} [k(x)\tilde{q}^1_\gamma \varphi_{1x}]_0^1 \mathrm{dtda} + \int_{Q_{T,A}} [k(x)\tilde{q}^1_{\gamma x}\varphi_1]_0^1 \mathrm{dtda}\end{aligned}$$

by taking into account the boundary / limits conditions:

$$\int_Q \varphi_1 (\tilde{y}_\gamma - z_d) \mathrm{dtdadx} = \int_Q \tilde{q}^1_\gamma \left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu \varphi_1\right) \mathrm{dtdadx}$$

passing to the limit when $\gamma \to 0$ in the last equality and taking into account (101) and (72), we get:

$$\int_Q \varphi_1 (\tilde{y} - z_d) \mathrm{dtdadx} = \int_Q \tilde{q}^1 \left(\frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial a} - (k(x)\varphi_{1x})_x + \mu \varphi_1\right) \mathrm{dtdadx}$$

If we integrate the latter equality by parts over Q, we obtain:

$$\begin{aligned}\int_Q \varphi_1 (\tilde{y}_\gamma - z_d) \mathrm{dtdadx} = & \int_Q \varphi_1 \left(-\frac{\partial \tilde{q}^1}{\partial t} - \frac{\partial \tilde{q}^1}{\partial a} - (k(x)\tilde{q}^1_x)_x + \mu \tilde{q}^1\right) \mathrm{dtdadx} + \int_{Q_{T,1}} \tilde{q}^1(.,A,.)\varphi_1(.,A,.)\mathrm{dtdx} - \int_{Q_{T,1}} \tilde{q}^1(.,0,.)\varphi_1(.,0,.)\mathrm{dtdx} \\ & + \int_{Q_{A,1}} \tilde{q}^1(T,.,.)\varphi_1(T,.,.)\mathrm{dadx} - \int_{Q_{A,1}} \tilde{q}^1(0,.,.)\varphi_1(0,.,.))\mathrm{dadx} + \int_{Q_{T,A}} [k(x)\tilde{q}^1_\gamma \varphi_{1x}]_0^1 \mathrm{dtda} - \int_{Q_{T,A}} [k(x)\tilde{q}^1_{\gamma x}\varphi_1]_0^1 \mathrm{dtda}\end{aligned}$$

$\forall \varphi_1 \in C^\infty(Q)$ *such that* $\varphi_1 = 0$ *on* $Q_{T,A}$, $\varphi_1(0,.,.) = 0$ *on* $Q_{A,1}$ *and* $\varphi_1(.,0,.) = 0$ *in* $Q_{T,1}$.

by taking into account the boundary and/or limits conditions:

$$\int_Q \varphi_1 (\tilde{y} - z_d) \mathrm{dtdadx} = \int_Q \varphi_1 \left(-\frac{\partial \tilde{q}^1}{\partial t} - \frac{\partial \tilde{q}^1}{\partial a} - (k(x)\tilde{q}^1_x)_x + \mu \tilde{q}^1\right) \mathrm{dtdadx} + \int_{Q_{T,1}} \tilde{q}^1(.,A,.)\varphi_1(.,A,.)\mathrm{dtdx} + \int_{Q_{A,1}} \tilde{q}^1(T,.,.)\varphi_1(T,.,.)\mathrm{dadx}$$

$\forall \varphi_1 \in C^\infty(Q)$ *such that* $\varphi_1 = 0$ *on* $Q_{T,A}$, $\varphi_1(0,.,.) = 0$ *on* $Q_{A,1}$ *and* $\varphi_1(.,0,.) = 0$ *on* $Q_{T,1}$.

We obtain successively:

$$q_1(T,.,.) = 0 \text{ in } Q_{A,1} \tag{103}$$

$$q_1(.,A,.) = 0 \text{ in } Q_{T,1} \tag{104}$$

By combining (102), (103), (104) *and the fact that* $\tilde{q}^1 \in L^2(Q_{T,A}; H^1_k(0,1))$, *we deduce that* $\tilde{q}^1$ *is solution of system*

$$\begin{cases} -\frac{\partial \tilde{q}^1}{\partial t} - \frac{\partial \tilde{q}^1}{\partial a} - (k(x)\tilde{q}^1_x)_x + \mu \tilde{q}^1 & = \tilde{y} - z_d \quad \text{in } Q \\ \tilde{q}^1(t,a,1) = \tilde{q}^1(t,a,0) & = 0 \quad \text{on } Q_{T,A} \\ \tilde{q}^1(T,a,x) & = 0 \quad \text{in } Q_{A,1} \\ \tilde{q}^1(t,A,x) & = 0 \quad \text{in } Q_{T,1} \end{cases} \tag{105}$$

We are now trying to show that $\tilde{q}^2$ *satisfies system:*

$$\begin{cases} -\frac{\partial \tilde{q}^2}{\partial t} - \frac{\partial \tilde{q}^2}{\partial a} - (k(x)\tilde{q}^2_x)_x + \mu \tilde{q}^2 & = k_2 \quad \text{in } Q \\ \tilde{q}^2(t,a,1) = \tilde{q}^2(t,a,0) & = 0 \quad \text{on } Q_{T,A} \\ \tilde{q}^2(T,a,x) & = 0 \quad \text{in } Q_{A,1} \\ \tilde{q}^2(t,A,x) & = 0 \quad \text{in } Q_{T,1} \end{cases} \tag{106}$$

For the sequel, we use the results and the Hilbert space $\mathscr{O}$ *define in Proof 13.*
According to inequalities (67) *and* (68), *there exists* $k_2 \in L^2(Q)$ *and* $\tilde{q}^2 \in L^2(Q_{T,A}; H^1_k(0,1))$ *such that:*

$$\frac{1}{\sqrt{\gamma}}\tilde{p}_\gamma + \frac{1}{\gamma}\tilde{\xi}(0)S(a,\tilde{u}_\gamma) \rightharpoonup \quad k_2 \quad \textit{weakly in} \quad L^2(Q) \tag{107}$$

$$\tilde{q}^2_\gamma \rightharpoonup \quad \tilde{q}^2 \quad \textit{weakly in} \quad L^2(Q_{T,A}; H^1_k(0,1)) \tag{108}$$

Furthermore, using the same reasoning as for $\tilde{q}^1$, *we prove by using* (107) *and* (108) *that* $\tilde{q}^2$ *satisfies* (106).
Now back to equality $\tilde{q}_\gamma = \tilde{q}^1_\gamma + \tilde{q}^2_\gamma$. *Using* (64) *and* (68), *we deduce that*

$$\|\tilde{q}_\gamma\|_{L^2(Q_{T,A}; H^1_k(0,1))} \leq C \tag{109}$$

Then, there exists $\tilde{q} \in L^2(Q_{T,A}; H^1_k(0,1))$ *such that:*

$$\tilde{q}_\gamma \rightharpoonup \tilde{q} \quad \textit{weakly in} \quad L^2(Q_{T,A}; H^1_k(0,1)) \tag{110}$$

By proceeding in the same way as for $\tilde{q}^1$ *and* $\tilde{q}^2$ *and using* (110), *we show that* $\tilde{q}$ *is the solution of* (89).
Finally, passing to the limit in (46) *and using* (69), (84) *and* (110) , *we deduce* (90).

Conclusion

We considered a population dynamics problem with missing data on the birth rate. This nonlinear problem is presented in its divergent form. We used the concepts of No-regret control and Low-regret control of J.L.Lions to solve the problem. After showing the existence of the adapted Low-regret control, we found a singular optimality system to characterize it. Then, we proved the existence of the No-regret control as the limit of a serie of adapted Low-regret controls. At end, we established a singular optimality system to characterize it.

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