A Note on b-chromatic Number of the Transformation Graph $G^{++}$ and Corona Product of Graphs

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Abstract

In this paper, we find the b-chromatic number of Transformation graph $G^{++}$ for Cycle, Path and Star graph. Also we determine the b-chromatic number of Corona product of Path graph with Cycle and Path graph with Complete graph along with its structural properties.

Keywords: b-chromatic number, b-colouring, chromatic number, Corona product, Transformation graph.

1 Introduction

All graphs in this paper are finite, undirected graphs, loopless graph without multiple edges. A $k$-colouring of a graph $G[I]$ is a labeling $f:V(G)\rightarrow T$, where $|T| = k$ and it is proper if adjacent vertices have different labels. A graph is $k$ colourable if it has a proper colouring. The chromatic number $\chi(G)$ is the least number $k$ such that $G$ is $k$-colourable. The b-chromatic number $\phi(G)$ [2] of a graph $G$ is the largest integer $k$ such that $G$ admits a proper $k$-colouring in which every colour class has a representative adjacent to at least one vertex in each of the other colour classes. Such a colouring is called a b-colouring. The concept of b-chromatic number was introduced in 1999 by Irwing and Manlove[ 3].
For a graph $G$, let $V(G)$ and $E(G)$ [7] denote the point set, line set of graph $G$ respectively. The Transformation graph $G^{++}$[4,8] of $G$ is the graph with point set $V(G) \cup E(G)$ in which the points $X$ and $Y$ are joined by a line if one of the following conditions holds:

- $x,y \in V(G)$ and $x,y$ are adjacent in $G$.
- $x,y \in E(G)$ and $x,y$ are adjacent in $G$.
- one of $x$ and $y$ is in $V(G)$ and the other is in $E(G)$ and they are not incident in $G$.

Corona product [9] or simply corona of graph $G_1$ and $G_2$ is a graph which is the disjoint union of one copy of $G_1$ and $|v_1|$ copies of $G_2$ ($|v_1|$ is number of vertices of $G_1$) in which each vertex copy of $G_1$ is connected to all vertices of separate copy of $G_2$.

2 b-Chromatic Number of $G^{++}$ of Path Graph

Theorem 2.1: The b-Chromatic number of $G^{++}$ of Path graph $P_n$ has $n$ colours.

Proof

Consider a Path graph of length $n-1$ with vertex set $V=\{v_1,v_2,v_3..v_n\}$ and edge set $E=\{e_1,e_2,e_3..e_{n-1}\}$. In Path graph $P_n$, each vertex $v_i$ is adjacent with the vertices $v_{i-1}$ and $v_{i+1}$ for $i=2,3,...n-1$, the vertex $v_1$ is adjacent with $v_2$ and $v_n$ is adjacent with $v_{n-1}$ and the lines $e_i$ and $e_n$ are non-adjacent with $n-3$ lines and remaining $e_i$ for $i=2,3,...n-1$ are non-adjacent with $n-4$ lines.

By the definition of Transformation graph $G^{++}$, the vertex set of $G^{++}(P_n)$ corresponds to both vertex set and edge set of Path graph. The vertex set of $G^{++}(P_n)$ is defined as follows:

i.e.$\{G^{++}(P_n)\}=\{v_i; 1 \leq i \leq n\} \cup \{e_i; 1 \leq i \leq n-1\}$

Consider the colour class $C=\{c_1,c_2,c_3..c_n\}$. Assign the colour $c_i$ to $v_i$ for $i=1,2,3..n$ and assign the colour $c_{n+i}$ to $e_i$ for $i=1,2,3..n-1$. Due to the above mentioned non-adjacency condition the above colouring does not produce a b-chromatic colouring. Thus, to make the above colouring as b-chromatic one, assign the colour $c_i$ to $v_i$ for $1 \leq i \leq n$ and assign the colour $c_l$ to $e_l$ and $c_{l+1}$ to $e_l$ for $i=2,3..n-1$.

Now the vertices $v_i$ for $i=1,2,3$ and the vertices $e_i$ for $3 \leq i \leq n-1$ realizes its own colour, which produces a b-chromatic colouring.

Thus the given colouring is b-chromatic. And by the very construction, it is the maximal colour class.

Hence the proof.
2.1 Structural Properties of $G^{++}(P_n)$

The number of vertices in $G^{++}(P_n)$ i.e. $p[G^{++}(P_n)]= 2n-1$, number of edges in the $G^{++}(P_n)$ i.e. $q[G^{++}(P_n)] = n^2-n-1$. The Maximum and Minimum degree of $G^{++}(P_n)$ is denoted as $\Delta = n$ and $\delta = n-1$ respectively.

3 b-Chromatic Number of $G^{++}$ of Cycle

Theorem 3.1: The b-Chromatic number of $G^{++}$ of the Cycle $C_n$ is $n$.

Proof

Consider a Cycle of length $n$, whose vertices are denoted as $v_1, v_2, v_3, \ldots, v_n$ and edges are denoted as $e_1, e_2, e_3, \ldots, e_n$. We see that every point in Cycle $C_n$ is non-adjacent with $n-2$ lines. Now consider $G^{++}(C_n)$, here there is no non-incident lines. By the definition of Transformation graph $G^{++}$, the vertex set of $G^{++}(C_n)$ corresponds to both vertex set and edge set of Cycle.

i.e. $V[G^{++}(C_n)] = \{v_i; 1 \leq i \leq n\} \cup \{e_i; 1 \leq i \leq n\}$
By observation, $G^{++}(C_n)$ forms an $n$-regular graph. Therefore the b-chromatic number of $G^{++}(C_n) \geq n$. Now we prove for $\varphi[G^{++}(C_n)] \leq n$, for this consider a proper colouring of $G^{++}(C_n)$ as follows.

Consider the colour class $C = \{c_1, c_2, c_3, \ldots, c_n\}$. Assign the colour $c_i$ for $i=1,2,3,\ldots,n$ to the inner cycle of $C_n$. Next if we assign the colour $c_{n+1}$ to any vertices in outer cycle, it does not realize the colour $c_{n+1}$. So we should assign only the existing colours to the vertices in outer cycle. Hence by the colouring procedure, we cannot assign more than $n$ colours to $G^{++}(C_n)$ i.e. $\varphi[G^{++}(C_n)] \leq n$. Therefore $\varphi[G^{++}(C_n)] = n$. Thus by the colouring procedure the above said colouring is maximal and b-chromatic.

Hence the Proof.

**Example**

![Fig. 2: C₅](image1)

![Fig. 3: φ[G^{++}(C₅)] = 5](image2)

**3.1 Structural Properties of $G^{++}(C_n)$**

The number of vertices in $G^{++}(C_n)$ i.e. $p[G^{++}(C_n)] = 2n$, number of edges in the $G^{++}(C_n)$ i.e. $q[G^{++}(C_n)] = n^2$. The Maximum and Minimum degree of $G^{++}(C_n)$ are denoted as $\Delta = n$ and $\delta = n$ respectively. Thus $G^{++}(C_n)$ is an $n$-regular graph.

**4 b-Chromatic Number of $G^{++}$ of Star Graph**

**Theorem 4.1:** If $G$ is $K_{1,n}$, then clearly $\varphi[G^{++}(K_{1,n})] = n+1$

**Proof**

Consider the graph $K_{1,n}$ with pendant vertices $v_1, v_2, v_3, \ldots, v_n$ and $v$ where $v$ is the root vertex with degree $n$. i.e. $V(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{e_i : 1 \leq i \leq n\}$ between the vertices $vv_i$ for $i=1,2,3,\ldots,n$. Here in $K_{1,n}$ we see there is no incident lines.
Consider $G^{++}(K_{1,n})$. By the definition of the Transformation graph $G^{++}$, the vertex set is defined as $V[G^{++}(K_{1,n})] = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Here the vertices $\{e_i : 1 \leq i \leq n\}$ forms a clique of order $n$ (say $K_n$) in $G^{++}(K_{1,n})$. Therefore we say that the b-chromatic number of $G^{++}(K_{1,n}) \geq n$. Consider the colour class $C = \{c_1, c_2, c_3, \ldots, c_{n+1}\}$. Assign a proper colouring to the vertices as follows.

**Case 1**
First assign the proper colouring to the vertex $e_i$. Assign the colour $c_i$ to the vertex $e_i$ for $i=1,2,3,\ldots,n$, and assign the colour $c_{n+1}$ to $v_i$ for $i=1,2,3,\ldots,n$ and assign any colour to root vertex other than the colour $c_{n+1}$. Now the vertices $e_i$ realizes its own colour. Thus, by the colouring procedure the above said colouring produces a maximal and b-chromatic colouring.

**Example**

![Fig 4: $\varphi[G^{++}(K_{1,3})]=4$](image)

**Case 2**
Next assign proper colouring to the vertex $v$ and $v_i$ for $i=1,2,3,\ldots,n$. Assign the colour $c_1$ to the root vertex $v$ and $c_{i+1}$ to $v_i$ for $i=1,2,3,\ldots,n$ and assign the same set of colour to $e_i$, which is already assigned for $v$, because $v_i$ is not adjacent with $e_i$ for $i=1,2,3,\ldots,n$, which produces a b-chromatic colouring. Thus by colouring procedure the above said colouring is maximal and b-chromatic. Hence the proof.

**Example**

![Fig 5: $\varphi[G^{++}(K_{1,3})]=4$](image)
4.1 Structural Properties of $G^{++}(K_{1,n})$

The number of vertices in $G^{++}(K_{1,n})$ i.e. $p[G^{++}(K_{1,n})]= 2n+1$, number of edges in the $G^{++}(K_{1,n})$ i.e. $q[G^{++}(K_{1,n})] = \left\lceil \frac{n(3n-1)}{2} \right\rceil$. The Maximum and Minimum degree of $G^{++}(K_{1,n})$ is denoted as $\Delta = n+1$ and $\delta = n-1$ respectively. The number of vertices having maximum and minimum degree in $G^{++}(K_{1,n})$ is denoted by $n(p_{\Delta})=n$ and $n(p_{\delta})=n+1$.

**Theorem 4.2:** For any Star graph $K_{1,n}$ the number of edges in $G^{++}(K_{1,n})$ is

$$q[G^{++}(K_{1,n})] = \left\lceil \frac{n(3n-1)}{2} \right\rceil$$

**Proof:**

\[
q[G^{++}(K_{1,n})] = \text{Number of edges in } K_{1,n} + \text{Number of edges in } K_n + \text{Number of edges in crown graph } S_n \\
= \binom{n}{1} + \binom{n}{2} + n(n-1) \\
= n + \frac{n(n-1)}{2} + n(n-1) \\
= n + n(n-1) \left(1 + \frac{2+1}{2}\right) \\
= \frac{2n + 3n^2 - 3n}{2} \\
= \frac{3n^2 - n}{2} \\
= \left\lceil \frac{n(3n-1)}{2} \right\rceil
\]

Therefore $q[G^{++}(K_{1,n})] = \left\lceil \frac{n(3n-1)}{2} \right\rceil$

5 b-Chromatic Number of Corona Product of Path Graph with Cycle

**Theorem 5.1:** For any integer $n>3$, $\varphi (P_n \circ C_n) = n$

**Proof:**

Let $G_1 = P_n$ be a Path graph of length $n-1$ with vertices $v_1, v_2, v_3, \ldots, v_n$ and edges $e_1, e_2, e_3, \ldots, e_{n-1}$. Consider $G_2 = C_n$ be a Cycle of length $n$ whose vertices are denoted as $v_1, v_2, v_3, \ldots, v_n$ and edges are denoted by $e_1, e_2, e_3, \ldots, e_n$. 
Consider the Corona product of $G_1$ and $G_2$ i.e. $G = P_n \circ C_n$ is obtained by taking unique copy of $P_n$ with $n$ vertices and $n$ copies of $C_n$ and joining the $i^{th}$ vertex of $P_n$ to every vertex in $i^{th}$ copy of $C_n$.

i.e. $V(G) = V(P_n) \cup V(C_1^n) \cup V(C_2^n) \cup V(C_3^n) \cup \ldots \ldots V(C^n_n)$

where $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $V(C_i^n) = \{u_{i,j}^l : 1 \leq i \leq n, 1 \leq j \leq n\}$

Now assign a proper colouring to these vertices as follows. Consider the colour class $C = \{c_1, c_2, c_3, \ldots, c_n\}$. First assign the colour $c_i$ to vertex $v_i$ for $i=1,2,3,\ldots,n$ and assign the colour to $u_{i,j}^l$ as $c_{i+j}$ when $i+j \leq n$ and $c_{i+j-n}$ when $i+j > n$ for $1 \leq i \leq n$, $l \leq j \leq n-1$. Now the only vertex remaining to be coloured is $u_{i,j}^l$ for $j=n$. Suppose if we assign any new colour to $u_{i,j}^l$ for $i=1,2,3,\ldots,n$, $j=n$ it will not produce a $b$-chromatic colouring, because $u_{i,j}^l$ (i.e. $i=1,2,3,\ldots,n$, $j=n$) is adjacent only with $u_{i,j}^l$ and $u_{i,j}^{l-1}$. So we assign the colour to $u_{i,j}^l$ other than the colour which we assign for $u_{i,j}^l$ and $u_{i,j}^{l-1}$. Now the vertices $\{v_i : 1 \leq i \leq n\}$ realize its own colours, which produces a $b$-chromatic colouring. Thus by the colouring procedure the above said colouring is maximal and $b$-chromatic. Hence the proof.

Example

![Fig. 6: $\phi(P_n \circ C_4) = 4$](image)

### 5.1 Structural Properties of $(P_n \circ C_n)$

The Number of vertices in $P_n \circ C_n (n \geq 3)$ i.e. $p(P_n \circ C_n) = n(n+1)$, number of edges in the $P_n \circ C_n$ i.e. $q(P_n \circ C_n) = 2n^2 + n - 1$. The Maximum and Minimum degree of $P_n \circ C_n$ is denoted as $\Delta = n+2$ and $\delta = n-1$ respectively. The number of vertices having maximum and minimum degree in $P_n \circ C_n$ is denoted by $n(p_\Delta) = n-2$ and $n(p_\delta) = n^2$.

**Corollary 5.1:** For any integer $n < 4$, $\phi (P_n \circ C_n) = n+1$

**Theorem 5.2:** $q(P_n \circ C_n) = 2n^2 + n - 1$
Proof

\[ q(P_n \circ C_n) = \text{Number of edges in largest subgraph} + \text{Number of edges not in any of the largest subgraph} \]
\[ = n \times (2n) + n - 1 \]
\[ = 2n^2 + n - 1 \]

6 b-Chromatic Number of Corona Product of Path with Complete Graph

Theorem 6.1: Let \( P_n \) and \( K_2 \) be the Path graph and Complete graphs with \( n \) vertices respectively. Then

\[ \varphi (P_n \circ K_2) = \begin{cases} 
  n + 1 & \text{for } n = 2 \\
  n & \text{for } n = 3 \text{ and } 4 \\
  n - 1 & \text{for } n = 5 \\
  5 & \text{for every } n > 6 
\end{cases} \]

6.1 Structural Properties of \( P_n \circ K_2 \)

The number of vertices in \( P_n \circ K_2 \) i.e. \( p(P_n \circ K_2) = 3n, \) number of edges in the \( P_n \circ K_2 \) i.e. \( q(P_n \circ K_2) = 3n + (n - 1). \) The Maximum and Minimum degree of \( P_n \circ K_2(n > 3) \) is denoted as \( \Delta = 4 \) and \( \delta = 3 \) respectively. The number of vertices having maximum and minimum degree in \( P_n \circ K_2 \) is denoted by \( n(\Delta) = n - 2 \) and \( n(\delta) = 2. \)

Theorem 6.2: For any integer \( n, \varphi(P_n \circ K_n) = n + 1 \)

Proof

Let \( G_1 = P_n \) be a Path graph of length \( n - 1 \) with \( n \) vertices and \( G_2 = K_n \) be a Complete graph of \( n \) vertices.

Consider the Corona product of \( G_1 \) and \( G_2 \) i.e. \( G = P_n \circ K_n \) is obtained by taking unique copy of \( P_n \) with \( n \) vertices and \( n \) copies of \( K_n \) and joining the \( i^{th} \) vertex of \( P_n \) to every vertex in the \( i^{th} \) copy of \( K_n. \)

i.e. \( V(G) = V(P_n) \cup V(K_n^1) \cup V(K_n^2) \cup V(K_n^3) \cup \ldots \cup V(K_n^n). \)

where \( V(P_n) = \{ v_1, v_2, v_3, \ldots, v_n \} \) and \( V(K_n^i) = \{ u_{j}^{i} : 1 \leq i \leq n, 1 \leq j \leq n \}. \)

By observation, we see that there are \( n \) copies of disjoint subgraph which induces a clique of order \( n + 1 \) (say \( K_{n+1} \)). Therefore we can assign more than or equal to \( n + 1 \) colours to every corona product of path graph with complete graph. Consider the colour class \( C = \{ c_1, c_2, c_3, c_4, \ldots, c_n, c_{n+1} \}. \) Now assign a proper colouring to these vertices as follows. Suppose if we assign more than \( n + 1 \) colours, it contradicts the
definition of b-chromatic colouring. Due to this condition, we cannot assign more than $n+1$ colours. Hence we have $\phi(P_n o K_n) \leq n+1$. Therefore $\phi(P_n o K_n) = n+1$.

Thus by the colouring Procedure the above said colouring is maximal and b-chromatic colouring.

**Example**

![Figure 8: $\phi(P_4 o K_4) = 5$](image)

### 6.2 Structural Properties of $(P_n \circ K_n)$

The number of vertices in $P_n o K_n$ i.e. $p(P_n o K_n) = n(n+1)$, number of edges in the $P_n o K_n$ i.e. $q(P_n o K_n) = \frac{n^3+n^2+2n-2}{2}$. The Maximum and Minimum degree of $P_n o K_n$ is denoted as $\Delta = n+2$ and $\delta = n$ respectively. The number of vertices having maximum and minimum degree in $P_n o C_n$ is denoted by $n(p_{\Delta})=n-2$ and $n(p_{\delta})=n^2$.

**Theorem 6.3:** For any path $P_n$ and complete graph graph $K_n$ the number of edges in corona product of $P_n$ with $K_n$ is

$$q(P_n o K_n) = \left\lfloor \frac{n^3+n^2+2n-2}{2} \right\rfloor$$

**Proof**

\[
q(P_n o K_n) = \text{Number of edges in all } K_{n+1} + \text{Number of edges not in any of the } K_{n+1} = n \times q(K_{n+1}) + \text{Number of edges not in any of the } K_{n+1} = n \times \binom{n+1}{2} + n-1 = n \left( \frac{n(n+1)}{2} \right) + n-1 = n \left( \frac{n^2+n}{2} \right) + n-1 = \left( \frac{n^3+n^2}{2} \right) + n-1 = \left( \frac{n^3+n^2+2n-2}{2} \right)
\]

Therefore $q(P_n o K_n) = \left\lfloor \frac{n^3+n^2+2n-2}{2} \right\rfloor$
7 b-Chromatic Number of Corona Product $K_n$ with Fan Graph

Theorem 7.1:

$$\varphi(F_{1,n}oK_2)=\begin{cases} 
  n+1 & \text{for every } 2 \leq n \leq 4 \\
  5 & \text{for every } n > 5
\end{cases}$$

Theorem 7.2: $\varphi(F_{1,n}oK_n)=n+1$ for every $n>2$.

Proof
The Proof of the theorem is similar to theorem (6.1).

8 b-Chromatic Number of Corona Product $K_{I,n}$ with $K_2$

Theorem 8.1: If $K_{I,n}$ and $K_2$ are Star graph and Complete graphs respectively, then

$$\varphi(K_{I,n}oK_2)=\begin{cases} 
  n+1 & \text{for every } n \leq 3 \\
  4 & \text{for every } n \geq 4
\end{cases}$$

Theorem 8.2: $\varphi(K_{I,n}oK_n)=n+1$ for every $n>2$

Proof
The Proof of the theorem is similar to theorem (6.1).

9 Conclusion

In this paper, we discussed about b-chromatic number of Transformation graph $G^{++}$ of Cycle, Path and Star graph and the Corona product of Path graph with Cycle and Complete graph.

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References


