Positive solutions for one-dimensional p-Laplacian boundary value problems with nonlinear parameter

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Abstract

In this paper, we establish existence of positive solutions of the nonlinear problems of one-dimensional p-Laplacian with nonlinear parameter

$$\phi_p(u'(t))' + a(t)f(\lambda, u) = 0, \quad t \in (0, 1), \quad u(0) = u(1) = 0.$$ 

where \(a : \Omega \to \mathbb{R}\) is continuous and may change sign, \(\lambda > 0\) is a parameter, \(f(\lambda, 0) > 0\) for all \(\lambda > 0\). By applying Leray-Schauder fixed point theorem we obtain the existence of positive solutions.

Keywords: p-Laplacian, Positive solutions, Leray-Schauder fixed point theorem, nonlinear parameter.

1. Introduction

The boundary value problem for one-dimensional p-Laplacian

$$\begin{align*}
\phi_p(u'(t))' + \lambda a(t)f(u) &= 0, \quad t \in (0, 1), \\
u(0) &= u(1) = 0,
\end{align*}$$

(1)

where \(\phi_p(u(t)) = |u|^{p-2} u\), \(p > 1\), has been studied extensively. For details, see for example, Refs [1,2,5], in the case \(p=2\) see [6], and for case \(\lambda = 1\), see [7,8,9].

In a recent paper [4], Hai considered the boundary value problem

$$\begin{align*}
\Delta u + \lambda a(t)f(u) &= 0, \quad t \in \Omega, \\
u &= 0, \quad t \in \partial\Omega,
\end{align*}$$

(2)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(a : \Omega \to \mathbb{R}\) is continuous and changes its sign, \(f(0) > 0\), and \(\lambda > 0\) is sufficiently small, under the following assumptions

(A1) \(f : [0, \infty) \to \mathbb{R}\) is continuous and \(f(0) > 0\).

(A2) \(a : \Omega \to \mathbb{R}\) is continuous, \(a \neq 0\), and there exists a number \(k > 1\) such that
Throughout the paper, we assume that $a^+$ (resp. $a^-$) is the positive (resp. negative) part of $a$, and $G(t, s)$ is the Green’s function of $-\Delta$ with Dirichlet boundary conditions.

They obtained the following interesting result:

**Theorem A.** Let (A1), (A2) hold. Then there exists a positive number $\lambda^*$, such that (2) has a positive solution for $\lambda < \lambda^*$. In another recent paper [3], Ma et al investigated the boundary value problem

\[
\begin{aligned}
\Delta u + a(t) f(\lambda, u) &= 0, & t &\in \Omega, \\
u &= 0, & t &\in \partial\Omega,
\end{aligned}
\]

By applying Leray-Schauder fixed point theorem they obtained that the problem (3) has a positive solution for $\lambda < \lambda^*$. Motivated by the results mentioned in [3,4] above, in this paper we study the existence of positive solutions of the nonlinear one-dimensional p-Laplacian

\[
\begin{aligned}
\varphi_p(u'(t))' + a(t) f(\lambda, u) &= 0, & t &\in (0, 1), \\
u(0) = u(1) &= 0,
\end{aligned}
\]

where $\varphi_p(u(t)) = |u|^p, u, p > 1$, and hence $\varphi_p(u'(t))'$ is the one-dimensional p-Laplacian, and $a : \Omega \to R$ is continuous and changes its sign, $\lambda > 0$ is a parameter, $f(\lambda, 0) > 0$ for all $\lambda > 0$.

The following hypotheses are adopted throughout this paper:

(H1) $f \in C([0, \infty) \times [0, \infty), R)$ and $f(\lambda, 0) > 0$ for all $\lambda > 0, f(0, u) = 0$ for $u \in [0, \infty).

(H2) $a : \Omega \to R$ is continuous, $a \not\equiv 0$, and there exists a number $k > 1$ such that

\[
\int_0^1 \varphi_p^{-1} (h(a^+) + \int_0^s a^+(\tau)d\tau) ds \geq k \int_0^1 \varphi_p^{-1} (h(a^-) + \int_0^s a^-(\tau)d\tau) ds,
\]

where $h : L^1(0, 1) \to R$ is continuous function satisfying

\[
\int_0^1 \varphi_p^{-1} (h(a) + \int_0^s a(\tau)d\tau) ds = 0.
\]

The main result of this paper is as follows

**Theorem 1.1.** Let (H1), (H2) hold. Then there exists a positive number $\lambda^*$, such that (4) has a positive solution for $\lambda < \lambda^*$.

The proof of theorem 1.1 is based on the Leray - Schauder theorem see [10], for more details.

**Remark 1.1.** If we let $f(\lambda, u) := \lambda f(u)$ and $\varphi_p(u) = \Delta u$, in (4), then (4) reduces to (2), (H1) reduces to (A1). Therefore, [4, Theorem 1.1], (see Theorem A above), is the direct consequence of Theorem 1.1.

Clearly, Theorem 1.1 is an extension and improvement of the existence results in [4, Hai], [3, Ma].

The rest of this paper is arranged as follows. In Section 2, we will give some notations and preliminary results, in Section 3, we prove Theorem 1.1 via the Leray - Schauder fixed point theorem.

2. Preliminaries

Throughout the paper, we assume that $f(\lambda, u) = f(\lambda, 0)$ for $u \leq 0$ and given $\lambda > 0$.

For $u \in C_0^1[0, 1]$, define the operator $T$ by

\[
Tu(t) = \int_0^t \varphi_p^{-1} (h(a^+) f(\lambda, u(s)) + \int_0^s a^+(\tau)f(\lambda, u(s)) d\tau) ds.
\]

It’s not difficult to see that $T : C_0^1[0, 1] \to C_0^1[0, 1]$ is completely continuous.

**Lemma 2.1.** Let $0 < \delta < 1$. Then there exists a positive number $\bar{\lambda}$ such that, for $0 < \lambda < \bar{\lambda}$, the equation

\[
\varphi_p(u'(t))' = a^+(t)f(\lambda, u), \quad 0 < t < 1, \quad u(0) = u(1) = 0
\]

has a positive solution $\bar{u}_\lambda$ with $\|\bar{u}_\lambda\|_0 \to 0$ as $\lambda \to 0$, and
\[ \tilde{u}_\lambda(t) \geq \delta f(\lambda, 0)p(t), \quad t \in \Omega, \]

where \( p(t) = \int_0^t \varphi_p^{-1} (h(a^+) + \int_0^s a^-(\tau)d\tau) \, ds. \)

**Proof.** We shall apply the Leray-Schauder fixed point theorem to prove that \( T \) has a fixed point for \( \lambda \) small. Let \( \varepsilon > 0 \) be such that

\[ f(\lambda, u) \geq \delta f(\lambda, 0), \quad \text{for } 0 \leq u \leq \varepsilon. \]  

(5)

From \( f(0, u) \equiv 0, \forall u \geq 0 \), we can suppose that \( 0 < \lambda < \varepsilon \lambda/2 \| p \|_0 \dot{f}(\lambda, 1) \), for given \( \lambda > 0 \), where \( \dot{f}(\lambda, t) = \max_{0 \leq s \leq t} f(\lambda, s) \). Then there exists \( A_\lambda \in (0, \varepsilon) \) such that

\[ \frac{\dot{f}(\lambda, A_\lambda)}{\lambda^{A_\lambda}} = \frac{1}{2\lambda \| p \|_0}. \]  

(6)

Let \( u \in C_0^1[0, 1] \) and \( \theta \in (0, 1) \) be such that \( u = \theta Tu \). Then we have

\[ \| u \|_0 \leq \lambda \| p \|_0 \frac{\dot{f}(\lambda, \| u \|_0)}{\lambda}. \]

or

\[ \frac{\dot{f}(\lambda, \| u \|_0)}{\lambda \| u \|_0} \geq \frac{1}{\lambda \| p \|_0}, \]

which implies that \( \| u \|_0 \neq A_\lambda \). Note that \( A_\lambda \to 0 \) as \( \lambda \to 0 \). By the Leray-Schauder fixed point theorem, \( T \) has a fixed point \( \tilde{u}_\lambda \) with \( \| \tilde{u}_\lambda \|_0 \leq A_\lambda < \varepsilon \). Consequently, \( \tilde{u}_\lambda(t) \geq \lambda \delta f(\lambda, 0)/\lambda p(t), t \in [0, 1] \), and the proof is completed.

### 3. Proof of the Theorem 1.1

Let \( q(t) = \int_0^t \varphi_p^{-1} (h(a^-) + \int_0^s a^-(\tau)d\tau) \, ds. \) By (H2), there exist positive numbers \( \alpha, \gamma \in (0, 1) \) such that

\[ q(t) |f(\lambda, s)| \leq \gamma p(t) f(\lambda, 0) \]  

(7)

for \( s \in [0, \alpha], t \in [0, 1] \). Fix \( \delta \in (\gamma, 1) \) and let \( \lambda^* > 0 \) be such that

\[ \| \tilde{u}_\lambda \|_0 + \lambda \delta f(\lambda, 0)/\lambda \| p \|_0 \leq \alpha \]  

(8)

for \( 0 < \lambda < \lambda^* \), where \( \tilde{u}_\lambda \) is given by Lemma 2.1, and

\[ |f(\lambda, x) - f(\lambda, y)| \leq f(\lambda, 0) \left( \frac{\delta - \gamma}{2} \right) \]  

(9)

for \( x, y \in [-\alpha, \alpha] \) with \( |x - y| \leq \lambda^* \delta L(\lambda, 0)/\lambda \| p \|_0 \).

Let \( 0 < \lambda < \lambda^* \). We look for a solution \( u_\lambda \) of (4) of the form \( \tilde{u}_\lambda + v_\lambda \). Thus \( v_\lambda \) satisfies

\[ v_\lambda(t) = \int_0^t \varphi_p^{-1} \left( h(af(\lambda, \tilde{u}_\lambda + v_\lambda)) + \int_0^s a(\tau) f(\lambda, \tilde{u}_\lambda + v_\lambda) d\tau \right) ds \]

- \[ \int_0^t \varphi_p^{-1} \left( h(a f(\lambda, \tilde{u}_\lambda)) + \int_0^s a^+(\tau) f(\lambda, \tilde{u}_\lambda) d\tau \right) ds, \quad 0 < t < 1. \]

For each \( w \in C_0^1[0, 1] \), let \( v = Tw \) be the solution of

\[ v(t) = \int_0^t \varphi_p^{-1} \left( h(af(\lambda, \tilde{u}_\lambda + \omega)) + \int_0^s a(\tau) f(\lambda, \tilde{u}_\lambda + \omega) d\tau \right) ds \]

- \[ \int_0^t \varphi_p^{-1} \left( h(a f(\lambda, \tilde{u}_\lambda)) + \int_0^s a^+(\tau) f(\lambda, \tilde{u}_\lambda) d\tau \right) ds, \quad 0 < t < 1. \]
Then \( T : C^1_0[0, 1] \rightarrow C^1_0[0, 1] \) is completely continuous. Let \( v \in C^1_0[0, 1] \) and \( \theta \in (0, 1) \) be such that \( v = \theta Tv \). Then we have

\[
v = \theta \int_0^t \varphi_p^{-1} \left( h(af(\lambda, \tilde{u}_\lambda + v)) + \int_0^s a(\tau)f(\lambda, \tilde{u}_\lambda + v)d\tau \right) ds
- \theta \int_0^t \varphi_p^{-1} \left( h(af(\lambda, \tilde{u}_\lambda)) + \int_0^s a(\tau)f(\lambda, \tilde{u}_\lambda)d\tau \right) ds.
\]

We claim that \( \|v\|_0 \neq \delta f(\lambda, 0)\|p\|_0 \). Suppose on the contrary that \( \|v\|_0 = \delta f(\lambda, 0)\|p\|_0 \). Then, by (8) and (9), we obtain

\[
\|\tilde{u}_\lambda + v\|_0 \leq \|\tilde{u}_\lambda\|_0 + \|v\|_0 \leq \alpha
\]

and

\[
|f(\lambda, \tilde{u}_\lambda + v) - f(\lambda, \tilde{u}_\lambda)| \leq f(\lambda, 0)\frac{\delta - \gamma}{2},
\]

which together with (7) implies that

\[
|v(t)| \leq \lambda \frac{\delta - \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t) + \lambda \gamma \frac{f(\lambda, 0)}{\lambda} p(t)
= \lambda \frac{\delta + \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t),
\]

(10)

In particular

\[
\|v\|_0 \leq \lambda \frac{\delta - \gamma}{2} \frac{f(\lambda, 0)}{\lambda} \|p\|_0 < \lambda \delta \frac{f(\lambda, 0)}{\lambda} \|p\|_0.
\]

a contradiction, and the claim is proved. By the Leray -Schauder fixed point theorem, \( T \) has a fixed point \( v_\lambda \) with \( \|v_\lambda\|_0 \leq \delta f(\lambda, 0)\|p\|_0 \). Hence \( v_\lambda \) satisfies (10) and, using Lemma 2.1, we obtain

\[
u_\lambda(t) \geq \tilde{u}_\lambda(t) - v_\lambda(t)
\geq \lambda \frac{\delta - \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t) - \lambda \frac{\delta + \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t)
= \lambda \frac{\delta - \gamma}{2} \frac{f(\lambda, 0)}{\lambda} p(t),
\]

i.e., \( u_\lambda \) is a positive solution of (4). This completes the proof of Theorem 1.1.

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