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Research paper

Fractional modeling for prey and predator problem by using optimal homotopy asymptotic method

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Abstract

In this paper, a fractional-ordered prey and predator population model is introduced and applied to obtain an approximate solution with help of optimal homotopy asymptotic method (OHAM). Some plots for populations of the prey and the predator versus time are presented to show the efficiency and the accuracy of the method and confirm that the method is straightforward as well. The fractional derivatives are described in the Caputo sense.

Keywords: *Optimal Homotopy Asymptotic Method; Prey and Predator Problem; Convergence Analysis; Caputo Derivative.*

1. Introduction

Many problems in the fields of mathematics, physics and engineering are modeled by fractional differential equations and integro-differential equations. Fractional differential equations have been solved by some series methods, such as Adomian decomposition [1-3], Variational iteration [4-6], Differential transform [7], Homotopy perturbation (HPM) [12-13], and Homotopy Analysis [8-11], and some others [15-18], [21-26].
In 2008, Marinca et al. [14], introduced a new analytical method known as the OHAM, it can be shown that HPM and HAM are special cases of OHAM. The advantage of OHAM, in comparison with HAM is that, there is no need of h-curves study. This method provides us with a convenient way to control the convergence of the solutions series and allows the adjustment of the convergence region wherever it is needed. In this study, OHAM is applied to approximate a solution of prey and predator problem.
This problem describes the natural habitual existence of rabbits and foxes living together, where foxes eat the rabbits and rabbits eat clover, suppose that there are enough supply of clovers and the rabbits have enough food to eat, When rabbits are abundant, foxes grow and their population increases, Therefore the population of foxes will increase, and they will eat more rabbits, so enter a short period of food and their number decreases, so that the number of foxes has decreased and the rabbits are safe and their number is increasing, then there is an increase and decrease in the number of foxes and rabbits. the number of both animals depends on its primary population, food supply and the parameters which control the model of population. For more details on the mathematical modeling leading to the following system of fractional nonlinear equations governing the problem

$$\begin{cases} D_t^{\alpha} x = x(t)\big(a - b y(t)\big), \\ D_t^{\alpha} y = -y(t)\big(c - d x(t)\big). \end{cases} \tag{1}$$

Where $x(t)$ and $y(t)$ are respectively the populations of rabbits and the foxes at given time. The problem was solved by different methods that we well addressed in [19-20]

2. Basic definitions

In this Section, we recall some definition and basic idea of the method.

Definition 1: *A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $\rho > \mu$, such that $f(t) = t^{\rho} f_1(t)$, where $f_1(t) \in (0, +\infty)$, and it is said to be in the space C_μ^n, if and only if $f^n \in C_\mu$, $n \in \mathbb{N}$.*

Definition 2: *The Rimann-louville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu, \mu > -1$, is defined as*

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds\,, \alpha > 0, \tag{2}$$

$J^0 f(t) = f(t)$.

Some properties of the operator J^{α}, are listed below

For $f \in C_{\mu}, \mu > -1, \alpha, \beta \geq 0$, and $\gamma \geq -1$,

1) $J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t)$,
2) $J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t)$,
3) $J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$.

Definition 3: *Caputo fractional derivative of $f(t)$ is defined as*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} f^{m}(s)ds, \tag{3}$$

For $m-1 < \alpha \leq m, m \in \mathbb{N}, t > 0, f \in C_{-1}^{m}$.

Lemma 1: *If $m-1 < \alpha \leq m, m \in \mathbb{N}, t > 0, f \in C_{-1}^{m}, \mu \geq -1$, then the following two properties hold*

1) $D^{\alpha}J^{\alpha}f(t) = f(t)$, (4)
2) $(J^{\alpha}D^{\alpha})f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^{+})\frac{t^k}{k!}$.

3. Basic idea of OHAM

In this section the basic ideas of the OHAM are introduced. Let's consider the following functional equation;

$$L\big(u(t)\big) + f(t) + N\big(u(t)\big) = 0, B\left(u, \frac{du}{dt}\right) = 0, \tag{5}$$

Where L is a linear operator, $u(t)$ is unknown function, $f(t)$ is known function, N is a non-linear operator, and B is a boundary operator. We construct an optimal convex homotopy $\mathcal{H}(\varphi(t;p),p)\colon [0,1] \to \mathbb{R}$ which satisfies the following equation called the zero-order deformation equation given by

$$(1-p)\big[L\big(\varphi(t;p)\big) + f(t)\big] = H(p)\big[L\big(\varphi(t;p)\big) + f(t) + N\big(\varphi(t;p)\big)\big], B\left(\varphi(t;p), \frac{d\varphi(t;p)}{dt}\right) = 0. \tag{6}$$

Where $p \in [0,1]$, is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$, and $H(0) = 0$, $u_0(t)$ an initial guess of $u(t)$. By substituting $p = 0$ and 1 in (6), respectively obtain

$$\varphi(t;0) = u_0(t), \tag{7}$$

And

$$\varphi(t;1) = u(t). \tag{8}$$

Thus as p increases from 0 to 1, $\varphi(t;p)$ varies continuously from the initial guess $u_0(t)$ to $u(t)$.
By substituting $p = 0$ in Eq. (6), the initial guess $u_0(t)$ is obtained as the solution of the following problem

$$L\big(u_0(t)\big) + f(t) = 0, B\left(u_0(t), \frac{du_0}{dt}\right) = 0. \tag{9}$$

We choose the auxiliary function $H(p)$ in the form

$$H(p) = pc_1 + p^2c_2 + p^3c_3 + \cdots, \tag{10}$$

Wherec_1, c_2, c_3,… are parameters to be determined later.
Expanding $\varphi(t;p,c_1,c_2,\ldots)$ in the powers of p, one gets

$$\varphi(t;p,c_1,c_2,\ldots) = u_0(t) + \sum_{k=1}^{\infty} u_k(t,c_1,c_2,\ldots)p^k, \tag{11}$$

Substituting Eqs, (7), (8) and (10) into (6) and equating the coefficient of p, we obtain

$$L(u_1(t) + f(t)) = c_1N_0\big(u_0(t)\big), B\left(u_1(t), \frac{du_1}{dt}\right) = 0, \tag{12}$$

$$L\big(u_2(t)\big) = L\big(u_1(t)\big) + c_2N_0\big(u_0(t)\big) + c_1\Big(L\big(u_1(t)\big) + N_1\big(u_0(t), u_1(t)\big)\Big), B\left(u_2(t), \frac{du_2}{dt}\right) = 0, \tag{13}$$

And hence the general governing equations for $u_m(t)$ is given by

$$L\big(u_m(t)\big) = L\big(u_{m-1}(t)\big) + c_m N_0\big(u_0(t)\big)$$

$$+\sum_{j=1}^{m-1} c_j\big[L\big(u_{m-1}(t)\big) + N_{m-1}\big(u_0(t), u_1(t), \dots, u_{m-1}(t)\big)\big],\ B\left(u_m(t), \frac{du_m}{dt}\right) = 0, m = 2, 3, \dots \quad (14)$$

Where $N_m\big(u_0(t), u_1(t), \dots, u_m(t)\big)$ is the coefficient of p^m in the expansion of $N(u(t;p))$ about the embedding parameter p and

$$N\big(\varphi(t; p, c_1, c_2, \dots)\big) = N_0\big(u_0(t)\big) + \sum_{m=1}^{\infty} N_m\big(u_0(t), u_1(t), \dots, u_m(t)\big)p^m. \quad (15)$$

Where $u(t; p, c_1, c_2, \dots)$ is given by Eq. (11).
It is observed that the convergence of the series (11) depends upon the auxiliary constants $c_i, i = 1, 2, \dots$
We assume that the parameters $c_1, c_2, c_3, \dots$ are so properly chosen that the series (11), converges at $p = 1$. Hence, we get the approximate solution of Eq. (5) in the following form

$$u(t, c_1, c_2, c_3, \dots) = u_0(t) + \sum_{m=1}^{\infty} u_m(t, c_1, c_2, c_3, \dots). \quad (16)$$

Then the mth order approximation can be denoted in the form

$$\hat{u}(t, c_1, c_2, \dots, c_m) = u_0(t) + \sum_{k=1}^{m} u_i(t, c_1, c_2, , \dots, c_k). \quad (17)$$

Substituting Eq. (17) into Eq. (5) results in the following residual

$$R(t, c_1, c_2, \dots, c_m) = L\big(\hat{u}(t, c_1, c_2, \dots, c_m)\big) + N\big(\hat{u}(t, c_1, c_2, \dots, c_m)\big) + f(t), i = 1, 2, 3, \dots \quad (18)$$

If $R(t, c_1, c_2, \dots, c_m) = 0$ then $\hat{u}(t, c_i)$ happens to be exact solution. Generally such case will not arise for nonlinear problems. Using least square minimization, requires to minimize the function

$$J(c_i) = \int_a^b R^2(t, c_i)dt. \quad (19)$$

Where a and b are two values, depending on the given problem. The unknown constants c_i $(i = 1, 2, .., m)$ can be optimally identified from the conditions

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \frac{\partial J}{\partial c_3} = \dots = \frac{\partial J}{\partial c_m} = 0. \quad (20)$$

The optimal values of c_i $(i = 1, 2, .., m)$ so obtained, when substituting in Eq. (14) give the approximate solution at level m.

4. Convergence analysis of the OHAM

Theorem 4.1: Suppose that $\{u_n\}_{n=0}^{\infty}$ is a sequence of approximate solutions. If $\exists\ 0 < \delta < 1$, the series $\sum_{k=0}^{m-1} u_k(t)$ converges if $\|u_{k+1}\| \le \delta\|u_k\|, \forall k \ge k_0$ for any $k_0 \in N$.
Proof 4.1: Consider the sequence $\{S_n\}_{n=0}^{\infty}$ defined as follows,

$$s_0 = u_0,$$

$$s_1 = u_0 + u_1,$$

$$s_2 = u_0 + u_1 + u_2,$$

$$\vdots$$

$$s_n = u_0 + u_1 + u_2 + \dots + u_n. \quad (21)$$

In order to convert $\{S_n\}_{n=0}^{\infty}$ in to a Cauchy sequence, on the Hilbert space R, the following relationships are considered.

$$\|S_{n+1} - S_n\| = \|u_{n+1}\| \le \delta\|u_n\| \le \delta^2\|u_{n-1}\| \le \dots \le \delta^{n-k_0+1}\|u_{k_0}\|. \quad (22)$$

Now, for every $n, m \in N, n \ge m > k_0$

$$\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \dots + (S_{m+1} - S_m)\| \le \|S_n - S_{n-1}\| + \|S_{n-1} + S_{n-2}\| + \dots + \|S_{m+1} - S_m\| \quad (23)$$

(Triangle inequality)

$$\le \delta^{n-k_0}\|u_{k_0}\| + \delta^{n-k_0-1}\|u_{k_0}\| + \dots + \delta^{m-k_0+1}\|u_{k_0}\| = \left(\frac{1-\delta^{n-m}}{1-\delta}\right)\delta^{m-k_0+1}\|u_{k_0}\|. \quad (24)$$

This implies that $\lim_{n,m\to\infty}\|S_n - S_m\| = 0$ (because $0 < \delta < 1$). Hence, it can be concluded that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence, on the Hilbert space $\mathbb{R}$, and also the solution of the series $\sum_{k=0}^{m-1} u_k(t)$ converges absolutely.

5. Numerical result

Consider the following prey and predator problem as

$$\begin{cases} D_t^{\alpha}x = x(t)\big(a - by(t)\big), \\ D_t^{\alpha}y = -y(t)\big(c - dx(t)\big). \end{cases} \tag{25}$$

Where a, b, c, and d, are known coefficients and x_0, and y_0 are respectively the initial populations of rabbits and foxes.
Choosing the linear operator $L = D_t^{\alpha}$ and applying the algorithm based on OHAM as presented in section 2, the initial guesses $x_0(t)$, and y_0, are obtained by solving the following zero-order problem;

$$\begin{cases} D_t^{\alpha}x_0 = 0, \\ D_t^{\alpha}y_0 = 0 \end{cases} \tag{26}$$

Substituting $m = 1, 2, 3, \ldots$ successively, the various order problems are as follows;
First-order problem;

$$\begin{cases} D_t^{\alpha}x_1 = c_1(-ax_0 + bx_0y_0), \\ D_t^{\alpha}y_1 = k_1(cy_0 - dx_0y_0). \end{cases} \tag{27}$$

Second-order problem;

$$\begin{cases} D_t^{\alpha}x_2 = (1 + c_1)D_t^{\alpha}x_1 + c_1(-ax_1 + b(x_0y_1 + x_1y_0)) + c_2(-ax_0 + bx_0y_0), \\ D_t^{\alpha}y_2 = (1 + k_1)D_t^{\alpha}y_1 + k_1(cy_1 - d(x_0y_1 + x_1y_0)) + k_2(cy_0 - dx_0y_0). \end{cases} \tag{28}$$

$\vdots$

Solving the above problems, one gets the various approximations of the solutions as

$$\begin{cases} x_1(t) = \frac{c_1(-ax_0+bx_0y_0)}{\Gamma(\alpha+1)}t^{\alpha}, \\ y_1(t) = \frac{k_1(cy_0-dx_0y_0)}{\Gamma(\alpha+1)}t^{\alpha}. \end{cases} \tag{29}$$

$$\begin{cases} x_2(t) = \frac{((1+c_1)c_1+c_2)(-ax_0+bx_0y_0)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{c_1((-ac_1+bc_1y_0)(-ax_0+bx_0y_0)+bx_0k_1(cy_0-dx_0y_0))}{\Gamma(2\alpha+1)}t^{2\alpha}, \\ y_2(t) = \frac{((1+k_1)k_1+k_2)(cy_0-dx_0y_0)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{k_1((ck_1-dk_1x_0)(cy_0-dx_0y_0)-dy_0c_1(-ax_0+bx_0y_0))}{\Gamma(2\alpha+1)}t^{2\alpha}. \end{cases} \tag{30}$$

$\vdots$

In this paper three terms approximation of $x(t)$, and $y(t)$, is considered.
For numerical results the following values are used.

case	x_0	y_0	a	b	c	d
1	14	18	1	1	0.1	1
2	16	10	1	1	0.1	1

By using (9, 12, 13) we determine c_1, c_2, k_1, and k_2 as

Case 1:

α	c_1	c_2	k_1	k_2
0.3	0.1472933268	−0.6284038799	−0.03556298506	0.2615307872
0.5	−0.2815391368	0.01558281550	0.04811600960	0.1682116414
1	−1.143689469	0.1885518126	−1.018551744	0.1570187146

Case 2:

α	c_1	c_2	k_1	k_2
0.3	-0.1570187146	0.1570187146	-0.1570187146	0.1570187146
0.5	0.1570187146	0.1570187146	-0.1570187146	0.1570187146
1	−0.3939713344	−0.3099914581	0.3986054111	−1.052748529

Knowing these constants, an approximate solution will be determined for different values of α, as

Case 1:

$$\alpha = 0.5: \begin{cases} x(t) = 14 - 125.7454675\sqrt{t} + 368.1541190t \\ y(t) = 18 - 75.31147225\sqrt{t} + 66.08496023t. \end{cases} \tag{31}$$

$$\alpha = 1 : \begin{cases} x(t) = 14 - 188.2107627t + 605.9178415t^2 \\ y(t) = 18 + 210.8278070t - 691.2232340t^2 \end{cases} \tag{32}$$

Case 2:

$$\alpha = 0.5 : \begin{cases} x(t) = 16 + 128.0159492\sqrt{t} - 445.9314517t, \\ y(t) = 10 - 100.4598951\sqrt{t} + 186.6511705t. \end{cases} \tag{33}$$

$$\alpha = 1 : \begin{cases} x(t) = 16 - 135.7517829t + 300.3320336t^2, \\ y(t) = 10 + 15.36757786t + 313.9083504t^2. \end{cases} \tag{34}$$

According to the following plots, it is clear that with the increase of number of foxes, so rabbits and the source of food for foxes will decrease. Accordingly, because the number of rabbits has decreased, the food required by foxes has decreased and their number has decreased, and so on [3]. Using Maple or Mathematical software, the data generated using Equations (31) through (34) were plotted as shown in Figures 1 and 2.
Relations between number of foxes and the rabbits are shown in following plots

(A) **(B)**

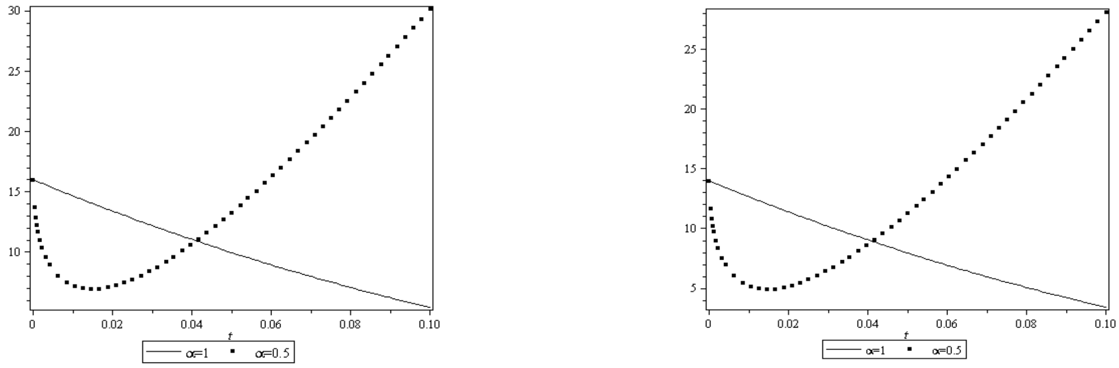

Fig. 1: Number of Rabbits versus Time with Various $A = 1$ and $A = 0.5$, (A) Case 1 and (B) Case 2.

(A) **(B)**

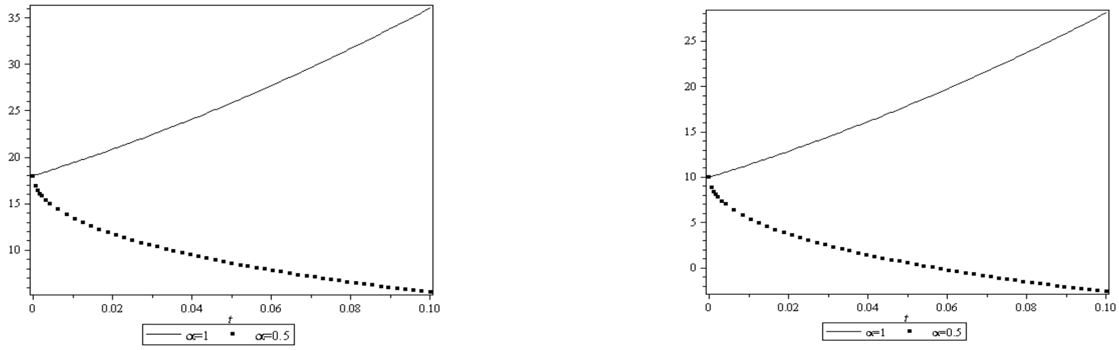

Fig. 2: Number of Foxes versus Time with Various $A = 1$ And $A = 0.5$, (A) Case 1 and (B) Case 2.

6. Conclusion

The convergence of the approximate solution given by OHAM is determined by the auxiliary function $H(p)$, whose parameters c_i are optimally determined. The function $H(p)$ can be chosen under different shapes. OHAM has been successfully applied for the problem of prey and predator. Expected behaviors of the population of the prey and predator are shown in the Figs, 1 and 2, which seems to be realistic.

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