



Application of mathematical methods of engineering to transient analysis of r-l-c series circuit with sinusoidal voltage

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Abstract

The paper has made an attempt to obtain transient response of an AC series circuit by mathematical methods of engineering namely classical method and Laplace transform method. The equations developed are differential equations of second order. The solution is obtained by classical method with initialization condition and Laplace transform method. A comparative study is done in analyzing the transient response of an R-L-C series circuit connected to sinusoidal source of voltage by the two methods. The derived equations of transient response by two methods are similar although Laplace transform method is better tool with respect to initialization conditions in a circuit. Proposed work has established similarity of transient response of all the three conditions of under and over damped, as well as critically damped case by two different methods.

Keywords: Classical Method, Differential Equation, Initialization, Laplace Transforms Transient Response.

1. Introduction

Transient response of An AC series circuit is obtained by traditional methods like classical method as well as more generalized methods as Laplace transform method. In case of R-L-C series circuit connected to sinusoidal source of voltage $v = V_m \sin \omega t$, a non-homogeneous differential equation of second order is formed which is solved.

For a differential equation, the complete solution consists of two parts; the complementary function and the particular solution. The complementary function dies out after short interval and is referred to as the transient response or source free response. The particular solution is the steady state response, or the forced response. The first step in finding the complete solution of a circuit is to form a differential equation for the circuit. Differential equation can be solved by several methods to find out the complete solution.

The transient response of R-L-C AC series circuit is important due to the fact that actual transmission line undergoes similar phenomena. The current through the transmission line is affected due to application of voltage and also due to withdrawal of voltage or case of sudden switching. The performance of the transmission line is due to steady state condition and transient condition. Therefore need is to evaluate the characteristic equation of the current due to Forced and Natural response. Also transient response is classified in to over damped, under damped and critically damped case. Thus paper attempts to analyse and establish similarity with 3 phase transmission line performance.

1.1. Problem formulation

Fig 1 shows a circuit consisting of resistance, inductance and capacitance connected in series to sinusoidal source of voltage. Inductance and capacitance are in uncharged state initially.

The current equation consists of two parts namely one due to application of voltage source called as forced response. Fig 2 indicates flow of current.

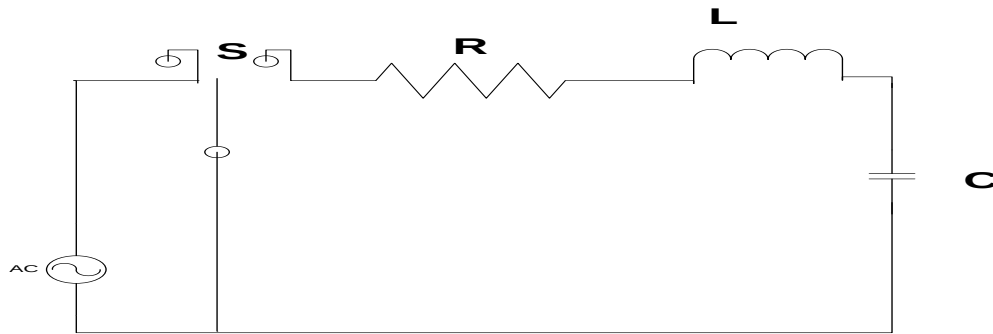


Fig. 1: R-L-C Connected to Sinusoidal Source

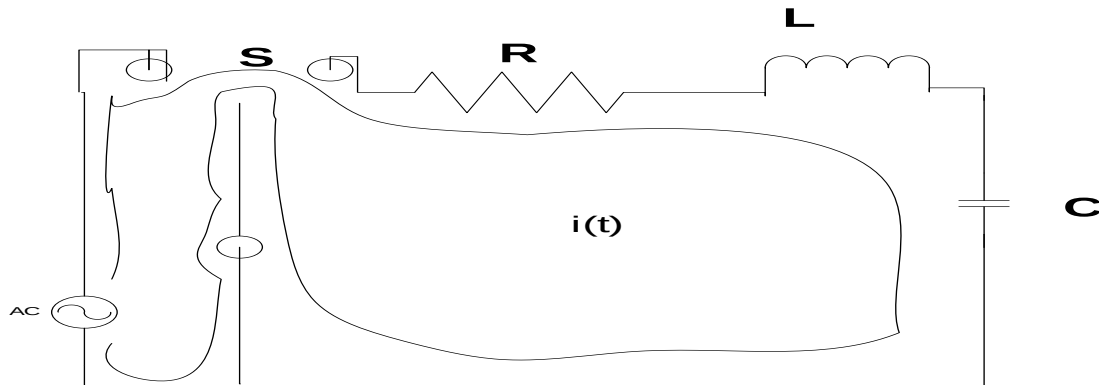


Fig. 2: Current Due to Forced Response

The other part of the current is due to withdrawal of the source called as Natural or transient response. Fig 3 indicates flow of current.

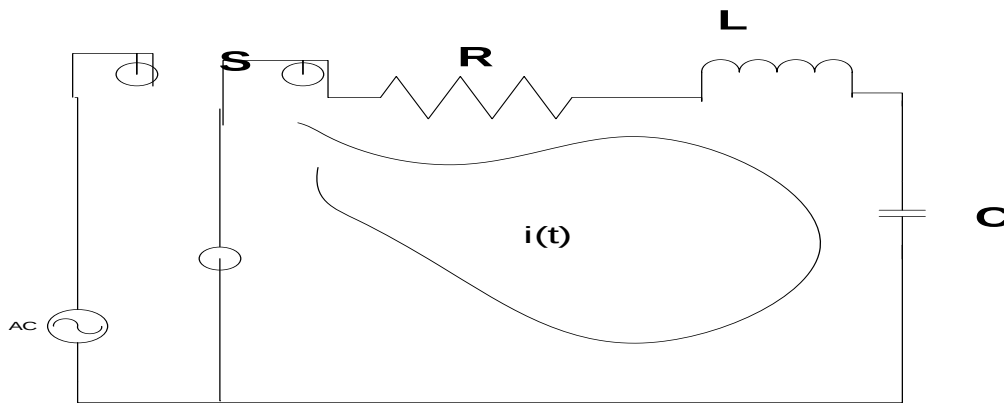


Fig. 3: Current Due to Natural Response

1.2. Solution methodology

The solution of above problem is obtained by Classical method and also by Laplace transform method.

1.2.1. Classical method

Consider the circuit consisting of resistance, inductance and capacitance connected in series to sinusoidal source of voltage.

Current circulates through the circuit when switch S is closed and application of sinusoidal voltage, $V\sin(\omega t + \theta)$, where V is the amplitude of the wave and θ is the phase angle.

Switch S is closed at $t=0$.

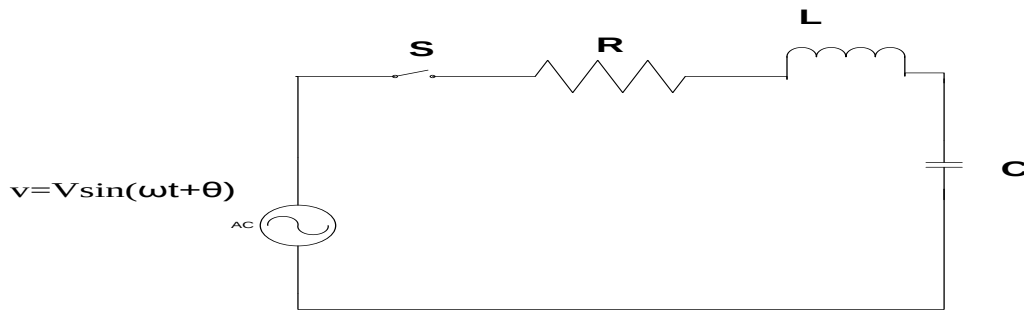


Fig. 4: R-L-C Connected to Sinusoidal Source of Voltage

Applying KVL to the circuit results in the following differential equation.

$$V \sin(\omega t + \theta) = iR + \frac{1}{C} \int i dt + L \frac{di}{dt} \quad (1)$$

Differentiating above equation (1), we get

$$\frac{V \omega \cos(\omega t + \theta)}{L} = L \frac{d^2 i}{dt^2} + \frac{1}{C} i + iR \text{ or}$$

$$\frac{V \omega \cos(\omega t + \theta)}{L} = \frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} \quad (2)$$

$$D^2 + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = \frac{V \omega \cos(\omega t + \theta)}{L} \quad (3)$$

Solution of above equation is $i(t) = i_p + i_h$ where i_p is the particular solution and i_h is the complementary solution. i_p is obtained by solution of non-homogeneous linear differential equation

$$D^2 + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = \frac{V \omega \cos(\omega t + \theta)}{L} \quad (4)$$

And i_c is obtained as homogeneous linear differential equation.

$$D^2 + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0 \quad (5)$$

Complementary solution of eq (2) is $i_c = c e^{\frac{-t}{RC}}$ where value of c can be obtained by applying initial conditions.

For obtaining particular solution, the current solution is estimated as

$$i_p = A \cos(\omega t + \theta) + B \sin(\omega t + \theta) \quad (6)$$

Differentiating eq (6) we get,

$$i_p' = -A \omega \sin(\omega t + \theta) + B \omega \cos(\omega t + \theta) \quad (7)$$

$$i_p'' = -A \omega^2 \cos(\omega t + \theta) - B \omega^2 \sin(\omega t + \theta) \quad (8)$$

Substituting the value of i_p' and i_p'' in eq (2), we get

$$-A \omega^2 \cos(\omega t + \theta) - B \omega^2 \sin(\omega t + \theta) + \frac{R}{L} (-A \omega \sin(\omega t + \theta) + B \omega \cos(\omega t + \theta)) + \frac{1[A \cos(\omega t + \theta) + B \sin(\omega t + \theta)]}{LC} = \frac{V \omega \cos(\omega t + \theta)}{L} \quad (9)$$

Collecting and equating real and imaginary components of (9) on both sides, we get equation

$$A = \frac{V \omega}{L} \times \frac{\omega^2 - \frac{1}{LC}}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \quad (10)$$

and

$$B = -\frac{V \omega^2 R}{L^2} \times \frac{1}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \quad (11)$$

Substituting value of A and B in equation (6) results in to

$$i_p = \frac{V \omega}{L} \times \frac{\omega^2 - \frac{1}{LC}}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \cos(\omega t + \theta) - \frac{V \omega^2 R}{L^2} \times \frac{1}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \sin(\omega t + \theta) \quad (12)$$

Assuming $M \cos \varphi$

$$\varphi = \frac{V \omega^2 R}{L^2} \times \frac{1}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \quad (13)$$

And

$$M \sin \varphi = \frac{V \omega}{L} \times \frac{\omega^2 - \frac{1}{LC}}{\left(\frac{\omega R}{L}\right)^2 + \left(\omega^2 - \frac{1}{LC}\right)^2} \frac{V \omega C}{1 + \omega^2 R^2 C^2} \quad (14)$$

Dividing equation (14) by (13), we get

$$\tan \varphi = \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \text{ or } \varphi = \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \quad \text{and } M$$

With above assumptions of $M \cos \varphi$ and $M \sin \varphi$ Equation (12) reduces to

$$i_p = M \sin \varphi \cos(\omega t + \theta) - M \cos \varphi \sin(\omega t + \theta) \quad (15)$$

$$\therefore i_p = M \sin(\omega t + \theta - \varphi).$$

Substituting the value of M and φ , resulting equation of

$$i_p = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) \quad (16)$$

For obtaining value of complementary function the characteristic equation is $D^2 + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} = 0$.

The roots of the characteristic equation are (i) $D_1 = k_1 + K_2$ and $D_2 = k_1 - k_2$. Case for over damped state

The resulting current equation is

$$i(t) = i_p + i_c = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + [D - (k_1 + k_2)][D - (k_1 - k_2)i] \quad (17)$$

$$i(t) = i_p + i_c = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + [c_1 e^{(k_1 + k_2)t} - c_2 e^{(k_1 - k_2)t}] \quad (18)$$

(ii) $D_1 = k_1 + jk_2$ and $D_2 = k_1 - jk_2$. Case for under damped state.

$$i(t) = i_p + i_c = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + [D - (k_1 + jk_2)][D - (k_1 - jk_2)i] \quad (19)$$

$$i(t) = i_p + i_c = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] \quad (20)$$

(iii) $D_1 = k_1$ and $D_2 = k_1$

$$i(t) = i_p + i_c = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + [D - (k_1)][D - (k_1)i] \quad (21)$$

$$i(t) = i_p + i_c = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + e^{k_1 t} [c_1 + c_2] \quad (22)$$

1.2.2. Laplace transform method

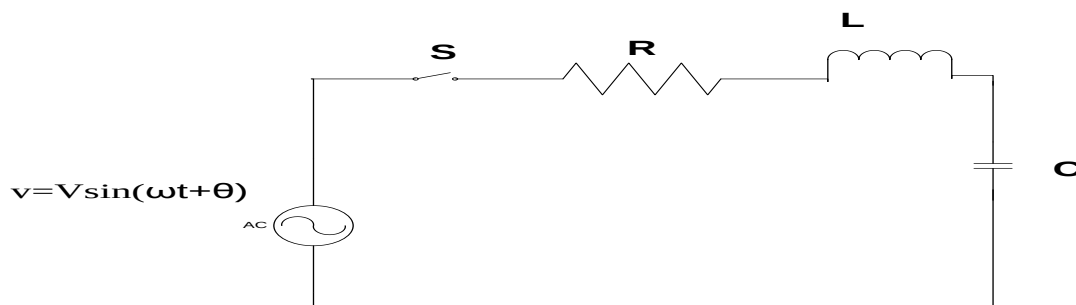


Fig. 5: R-L-C Connected to Sinusoidal Source of Voltage

Consider the circuit consisting of resistance, inductance and capacitor connected in series to sinusoidal source of voltage.

Current circulates through the circuit when switch S is closed and application of sinusoidal Voltage $V \sin(\omega t + \theta)$, where V is the amplitude of the wave and θ is the phase angle.

Switch S is closed at $t=0$.

Applying KVL to the circuit results in the following differential equation.

$$V \sin(\omega t + \theta) = iR + L \frac{di}{dt} + \frac{1}{C} \int i dt \quad (23)$$

Taking Laplace transform of above equation,

$$I(s) \left[Ls + R + \frac{1}{Cs} \right] = \frac{V(s \sin \theta + \omega \cos \theta)}{s^2 + \omega^2} \quad (24)$$

$$I(s) \left[s^2 + \frac{R}{L}s + \frac{1}{LC} \right] = \frac{Vs(s \sin \theta + \omega \cos \theta)}{L(s^2 + \omega^2)} \quad (25)$$

$$I(s) = \frac{Vs}{L} \frac{(s \sin \theta + \omega \cos \theta)}{(s^2 + \omega^2) \left(s^2 + \frac{R}{L}s + \frac{1}{LC} \right)} \quad (26)$$

Where s_1 and s_2 are roots of the characteristic equation? $\left(s^2 + \frac{R}{L}s + \frac{1}{LC} \right) = 0$

$$I(s) = \frac{Vs}{L} \frac{(s \sin \theta + \omega \cos \theta)}{(s + j\omega)(s - j\omega)(s - s_2)(s - s_1)} \quad s_1, s_2 = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L} \right)^2 - \frac{1}{LC}} \quad (27)$$

Expressing RHS of equation (27) we get,

$$\text{Let } \frac{Vs}{L} \frac{(s \sin \theta + \omega \cos \theta)}{(s+j\omega)(s-j\omega)(s-s_2)(s-s_1)} = \frac{k_1}{(s-s_1)} + \frac{k_2}{(s-s_2)} + \frac{k_3}{(s+j\omega)} + \frac{k_4}{s-j\omega} \quad (28)$$

Using Laplace method to determine coefficient on RHS,

Multiplying eq (28) on both sides by $(s - s_1)$, and putting $s = s_1$ we get

$$\frac{V}{L} \frac{s(s \sin \theta + \omega \cos \theta)(s-s_1)}{(s+j\omega)(s-j\omega)(s-s_2)(s-s_1)} = k_1 + 0.$$

$$\therefore k_1 = \frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1+j\omega)(s_1-j\omega)(s_1-s_2)} \quad (29)$$

Multiplying eq (28) on both sides by $(s - s_2)$, and putting $s = s_2$ we get

$$\frac{V}{L} \frac{s(s \sin \theta + \omega \cos \theta)(s-s_2)}{(s+j\omega)(s-j\omega)(s-s_2)(s-s_1)} = k_2 + 0.$$

$$\therefore k_2 = \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2+j\omega)(s_2-j\omega)(s_2-s_1)} \quad (30)$$

Multiplying eq (28) on both sides by $(s + j\omega)$, and putting $s = -j\omega$ we get

$$\frac{V}{L} \frac{s(s \sin \theta + \omega \cos \theta)(s+j\omega)}{(s+j\omega)(s-j\omega)(s-s_2)(s-s_1)} = k_3 + 0,$$

$$\therefore k_3 = \frac{V}{L} \frac{-j\omega(-j\omega \sin \theta + \omega \cos \theta)}{(-j\omega-j\omega)(-j\omega-s_1)(-j\omega-s_2)} \quad (31)$$

$$k_3 = \frac{\omega(\cos \theta - j \sin \theta)V}{2(s_1+j\omega)(s_2+j\omega)L} \quad (32)$$

Multiplying eq (28) on both sides by $(s - j\omega)$, and putting $s = j\omega$ we get

$$\frac{V}{L} \frac{s(s \sin \theta + \omega \cos \theta)(s-j\omega)}{(s+j\omega)(s-j\omega)(s-s_2)(s-s_1)} = k_4 + 0,$$

$$\therefore k_4 = \frac{V}{L} \frac{-j\omega(-j\omega \sin \theta + \omega \cos \theta)}{(-j\omega-j\omega)(-j\omega-s_1)(-j\omega-s_2)} \quad (33)$$

$$k_4 = \frac{\omega(\cos \theta + j \sin \theta)V}{2(s_1-j\omega)(s_2-j\omega)L} \quad (34)$$

Substituting the value of k_1, k_2, k_3 and k_4 in (28), and taking inverse laplace transform, we get

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} + k_3 e^{-j\omega t} + k_4 e^{j\omega t} \quad (35)$$

$$i(t) = \frac{V}{L} [k_1 e^{s_1 t} + k_2 e^{s_2 t}] + \frac{V}{L} [k_3 e^{-j\omega t} + k_4 e^{j\omega t}] \quad (36)$$

Equation (36) consists of two parts. First part is made up of two roots s_1 and s_2 , and second part consists of two transitory components due to $j\omega t$.

$$i(t) = i(t)' + i(t)''$$

$$i(t)' = \frac{V}{L} [k_1 e^{s_1 t} + k_2 e^{s_2 t}], \quad (37)$$

substituting the value of k_1, k_2 we get

$$i(t)' = \frac{V}{L} \left[\frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1+j\omega)(s_1-j\omega)(s_1-s_2)} e^{s_1 t} + \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2+j\omega)(s_2-j\omega)(s_2-s_1)} e^{s_2 t} \right] \quad (38)$$

$$i(t)' = \frac{V}{L(s_1-s_2)} \left[\frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1+j\omega)(s_1-j\omega)} e^{s_1 t} - \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2+j\omega)(s_2-j\omega)} e^{s_2 t} \right] \quad (39)$$

Equation (39) can be written as

$$i(t)' = \frac{V}{L(s_1-s_2)} [c_1 e^{s_1 t} - c_2 e^{s_2 t}] \quad (40)$$

Eq (40) is a generalized case but if we consider,

Case 1 $s_1 = D_1 = k_1 + k_2$ and $D_2 = s_2 = k_1 - k_2$

Case 2 $s_1 = D_1 = k_1 + jk_2$ and $D_2 = s_2 = k_1 - jk_2$

Case 3 $s_1 = D_1 = k_1$ and $D_2 = s_2 = k_1$,

Then three different equations are possible.

Following is the derivation of second part,

$$i''_t = \frac{V}{L} [k_3 e^{-j\omega t} + k_4 e^{j\omega t}] \quad (41)$$

Substituting the value of k_3 and k_4 , we get

$$i(t)'' = \frac{V}{2L} \left[\frac{\omega(\cos \theta - j \sin \theta) e^{-j\omega t}}{(s_1+j\omega)(s_2+j\omega)} + \frac{\omega(\cos \theta + j \sin \theta) e^{j\omega t}}{(s_1-j\omega)(s_2-j\omega)} \right] \quad (42)$$

$$i(t)'' = \frac{V\omega}{2L} \left[\frac{e^{-j\theta} e^{-j\omega t}}{(s_1+j\omega)(s_2+j\omega)} + \frac{e^{j\theta} e^{j\omega t}}{(s_1-j\omega)(s_2-j\omega)} \right] \quad (43)$$

$$= \frac{V\omega}{2L} \left[\frac{[e^{-j(\omega t + \theta)}(s_1-j\omega)(s_2-j\omega)] + e^{j(\omega t + \theta)}(s_1+j\omega)(s_2+j\omega)}{(s_1^2 + \omega^2)(s_2^2 + \omega^2)} \right] \quad (44)$$

$$= \frac{V\omega}{2L} \frac{[e^{-j(\omega t + \theta)}(s_1 s_2 - \omega^2 - j\omega s_1 - j\omega s_2) + e^{j(\omega t + \theta)}(s_1 s_2 - \omega^2 + j\omega s_1 + j\omega s_2)]}{(s_1^2 + \omega^2)(s_2^2 + \omega^2)} \quad (45)$$

$$= \frac{V\omega}{2L} \frac{[e^{-j(\omega t + \theta)} + e^{j(\omega t + \theta)}(s_1 s_2 - \omega^2) - j(\omega s_1 + \omega s_2) e^{-j(\omega t + \theta)} - e^{j(\omega t + \theta)}]}{(s_1^2 + \omega^2)(s_2^2 + \omega^2)} \quad (46)$$

$$= \frac{V\omega}{L} \frac{[\cos(\omega t + \theta) \left(\frac{1}{LC} - \omega^2 \right) - \left(\frac{-\omega R}{L} \right) \sin(\omega t + \theta)]}{(s_1^2 + \omega^2)(s_2^2 + \omega^2)} \quad (47)$$

$$= \frac{V\omega}{L} \frac{\left[\left(\frac{\omega R}{L} \right) \sin(\omega t + \theta) - \cos(\omega t + \theta) \left(\omega^2 - \frac{1}{LC} \right) \right]}{(s_1^2 + \omega^2)(s_2^2 + \omega^2)} \quad (48)$$

Assuming

$$M \cos \varphi = \frac{\omega R}{L} \quad (49)$$

And

$$M \sin \varphi = \left(\omega^2 - \frac{1}{LC} \right) \quad (50)$$

$$\therefore \tan \varphi = \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \quad (51)$$

$$\text{or } \varphi = \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \quad (52)$$

$$M = \frac{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}}{1} \quad (53)$$

Equation (4) reduces to

$$\frac{V \omega}{L} \frac{[M \cos \varphi \sin(\omega t + \theta) - M \sin \varphi \cos(\omega t + \theta)]}{s_1^2 s_2^2 + \omega^2 (s_1^2 + s_2^2) + \omega^4} \quad (54)$$

Substituting the value of $s_1^2 s_2^2 + \omega^2 (s_1^2 + s_2^2) + \omega^4$ in (54), we get

$$= \frac{V \omega [M \sin(\omega t + \theta - \varphi)]}{L(s_1^2 s_2^2 + \omega^2 (s_1^2 + s_2^2) + \omega^4)} \quad (55)$$

or

$$\frac{V \omega^2 L^2 \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2} \left[\sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) \right]}{L^2 \left(R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2 \right) \omega^2} \quad (56)$$

$$i(t)'' = \frac{V \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}} \quad (57)$$

$$i(t) = i(t)' + i(t)''$$

Substituting the value of $i(t)'$ and $i(t)''$ from equation (39) and (57), we get

$$i(t) = \frac{V}{L(s_1 - s_2)} \left[\frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)} e^{s_1 t} - \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)} e^{s_2 t} \right] + \frac{V \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}} \quad (58)$$

Generalizing the two equations without attempting to apply initial conditions at $t=0$, for finding the value of c_1, c_2, s_1 and s_2 , similarity between the two equations derived by classical method and laplace transform method can be drawn in the following way.

Following is the generalized equation by Laplace Transform method.

Classical Method:

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + c_1 e^{(k_1 + k_2)t} - c_2 e^{(k_1 - k_2)t} \quad (59)$$

Laplace Transform Method:

$$i(t) = \frac{V \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}} + \left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (60)$$

1.2.2.1. Case 1 Above equations are same if we substitute the two roots $D1=k1+K2=S1$ and $D2= k1-k2=S2$

And also

$$c_1 = \frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} \quad (61)$$

And

$$c_2 = -\frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} \quad (62)$$

Above table can be re written as for classical method by substituting as above

Classical Method

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}} \sin \left(\omega t + \theta - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right) + \frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \quad (63)$$

Laplace Transform Method:

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + \left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (64)$$

Above equations are same if we substitute the two roots $D_1 = k_1 + K_2 = S_1$ and $D_2 = k_1 - k_2 = S_2$

And also

$$c_1 = \frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} \quad (65)$$

And

$$c_2 = -\frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} \quad (66)$$

Above table can be re written as for Laplace Transform method by substituting as

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + c_1 e^{(k_1 + k_2)t} - c_2 e^{(k_1 - k_2)t} \quad (67)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + [c_1 e^{(k_1 + k_2)t} - c_2 e^{(k_1 - k_2)t}] \quad (68)$$

Equatons shown are same.

1.2.2.2. Generalized equation case 2, if we substitute the two roots $d_1 = k_1 + jk_2 = s_1$ and $d_2 = k_1 - jk_2 = s_2$ classical method

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] \quad (69)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + \left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (70)$$

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] \quad (71)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + \frac{V [k_1' e^{s_1 t} + k_2' e^{s_2 t}]}{L(s_1 - s_2)} \quad (72)$$

If we substitute the two roots $D_1 = k_1 + jk_2 = S_1$ and $D_2 = k_1 - jk_2 = S_2$ in above equation, we get

$$i(t)' = \frac{V}{L(k_1 + jk_2 - (k_1 - jk_2))} [k_1' e^{(k_1 + jk_2)t} - k_2' e^{(k_1 - jk_2)t}] \quad (73)$$

$$i(t)' = \frac{V}{L(2j)k_2} [(k_1' e^{k_1 t} e^{jk_2 t}) - (k_2' e^{k_1 t} e^{-jk_2 t})] \quad (74)$$

Rearranging and substituting value of $e^{jk_2 t}$ and $e^{-jk_2 t}$ in terms of cos and sin components

$$i(t)' = e^{k_1 t} \left[\frac{V[(k_1' - k_2')2 \cos k_2 t + (k_1' + k_2')2 \sin k_2 t]}{L(2j)k_2} \right] \quad (75)$$

$$i(t)' = e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] \quad (76)$$

where $c_1 = \frac{V(k_1' - k_2')2}{2jLk_2}$ and $c_2 = \frac{V(k_1' + k_2')2}{2jLk_2}$

which is same as equation by classical method if we substitute

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] \quad (77)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + [e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t]] \quad (78)$$

Equatons shown are same.

1.2.2.3. Generalised equation case 3, if we substitute the two roots $d_1=k_1=s_1$ and $d_2=k_1=s_2$ classical method

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 + c_2] \quad (79)$$

Laplace transforms method:

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + \left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (80)$$

In the above equation of $i(t)$ by laplace transform method

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + \left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (81)$$

Rearranging the second term with substitution of $D_1=K_1=S_1=D_2=K_1=S_2$ in the following equation

$$\left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (82)$$

$$\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{k_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{k_1 t} \quad (83)$$

$$\left[e^{k_1 t} \left(\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} \right) \right] \quad (84)$$

$$= e^{k_1 t} [c_1 + c_2] \quad (85)$$

$$\left[\text{where } c_1 = \left(\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} \right) \right] \quad (86)$$

And

$$c_2 = \left(\frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} \right) \quad (87)$$

With this substitution, Laplace Transform equation can be rewritten in following way,

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 + c_2] \quad (88)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + e^{k_1 t} [c_1 + c_2] \quad (89)$$

Comparative for all the three different cases:

(i) $D_1=k_1+K_2=S_1$ and $D_2=k_1-k_2=S_2$

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + c_1 e^{(k_1+k_2)t} - c_2 e^{(k_1-k_2)t} \quad (90)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + [c_1 e^{(k_1+k_2)t} - c_2 e^{(k_1-k_2)t}] \quad (91)$$

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + \frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \quad (92)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + \left[\frac{V}{L} \frac{s_1(s_1 \sin \theta + \omega \cos \theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} e^{s_1 t} + \frac{V}{L} \frac{s_2(s_2 \sin \theta + \omega \cos \theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)} e^{s_2 t} \right] \quad (93)$$

(ii) $D_1=k_1+jk_2=S_1$ and $D_2=k_1-jk_2=S_2$

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t] \quad (94)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + [e^{k_1 t} [c_1 \cos k_2 t + c_2 \sin k_2 t]] \quad (95)$$

(iii) $D_1=K_1=S_1=D_2=K_1=S_2$

$$i(t) = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + e^{k_1 t} [c_1 + c_2] \quad (96)$$

$$i(t) = \frac{V \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} + e^{k_1 t} [c_1 + c_2] \quad (97)$$

2. Conclusion

It is proved that equations governing sinusoidal excitation to R-L-C series circuit for transient behavior with storage elements initially in uncharged state are second order differential equations of non-homogeneous type. The mathematical methods of engineering namely classical method and Laplace Transform method have given similar results for the analysis which was carried out in this work. Also cases of over damped, under damped and critically damped condition was evaluated significant for natural response of R-L-C AC series circuit. Laplace transform method is proved to be a better tool with respect to initialization condition in comparison to classical method. Thus analysis of R-L-C AC series circuit with sinusoidal excitation during transient condition can be utilized to explain the phenomena occurring on transmission line.

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