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Research Paper

Fixed points for four maps related to generalized weakly contractive condition in partial metric spaces

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Abstract

We prove a unique fixed point theorem for a function depending from four self maps satisfying $(\phi - \psi)$-contractive condition in partial metric spaces. Presented results extend and generalize some existing fixed point results in the literature.

Keywords: *Partial metric; Weakly compatible maps; Complete space.*

1. Introduction

The Notion of partial metric space have originally developed by S.G. Matthews ([3]) to provide mechanism generalizing metric space theories. This relatively new field has been shown to have vast application potentials [6] in the study of computer domains and semantics [7]. The partial metric spaces play an important role in constructing models in the theory of computation see [1, 3, 6, 8].

S.G Matthews ([3])., Sandra Oltra and Oscar Valero [8], Salvador Romaguerra [9], I. Altun, Ferhan Sola [1] and K.P.R Rao and G.N.V. Kishore [5] proved fixed point theorems in partial metric spaces for a single map.

In this paper, we prove a unique fixed point theorem for four self mappings for a generalized operator depending from $(\psi - \varphi)$ contractive condition in partial metric spaces.

First, let us recall some definitions and lemmas of partial metric spaces that we will use in the sequel.

2. Preliminaries

Definition 2.1 *(([3]).). A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:*

(p_1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y)$, $p(y, y) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space (X, p) is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark 2.2 *(([3]).). It is clair that*

a) $|p(x,y) - p(y,z)| \le p(x,z), \forall x,y,z \in X$.

b) $p(x,y) = 0 \implies x = y$.

c) *If* $x = y$, $p(x,y)$ *may not be zero. We consider the following counter-example, the pair* (R^+, p), *where* $p(x,y) = \max\{x,y\}$ *for all* $x,y \in R^+$.

d) *If* p *is a partial metric on* X, *then the function* $p^s : X \times X \to R^+$ *given by* $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ *is a metric on* X.

Each partial metric p *on* X *generates a* T_0 *topology* τ_p *on* X *which has a base the family of open* $p-$*balls* $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, *where* $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ *for all* $x \in X$ *and* $\varepsilon > 0$.

Definition 2.3 *(([3])) Let* (X,p) *be a partial metric space.*

(i) *A sequence* $\{x_n\}$ *in* (X,p) *is said to converge to a point* $x \in X$ *if, and only if* $p(x,x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) *A sequence* $\{x_n\}$ *in* (X,p) *is said to be Cauchy sequence if the limit:* $\lim_{m,n\to\infty} p(x_n, x_m)$ *exists and is finite.*

(iii) (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges, with respect to* τ_p, *to a point* $x \in X$, *such that*

$$p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$$

Lemma 2.4 *(([3])) Let* (X,p) *be a partial metric space. Then:*

(a) $\{x_n\}$ *is a Cauchy sequence in* (X,p) *if, and only if it is a Cauchy sequence in the metric space* (X, p^s).

(b) (X,p) *is complete if, and only if the metric space* (X,p^s) *is complete. Furthermore,* $\lim_{n\to\infty} p^s(x, x_n) = 0$ *if, and only if*

$$p(x,x) = \lim_{n,m\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Matthews ([3]) obtained the following Banach fixed point theorem on complete partial metric spaces.

Theorem 2.5 *([3]). Let* f *be a mapping of a complete partial metric space* (X,p) *into itself such that there is a real number* c *with* $0 \le c < 1$, *satisfying the following condition:*

for all $x,y \in X$: $\quad p(fx, fy) \le cp(x,y)$,

then f *has a fixed point.*

In 2010, I. Altun, F.Sola and H. Simsek [1], proved the following result, that generalizes Theorem 1 of Matthews.

Theorem 2.6 *[1]. Let* (X,p) *be a complete partial metric space and let* $T : X \to X$ *be a map such that:*

$$p(Tx,Ty) \le \varphi\Big(\max\Big\{ p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}\left[p(x,Ty) + p(y,Tx)\right] \Big\}\Big)$$

for all $x,y \in X$, *where* $\varphi : R^+ \to R^+$ *is continuous non-decreasing function such that* $\varphi(t) < t$ *and the series* $\sum_{n\ge1} \varphi(t)$ *converges for all* $t0$. *Then* T *has a unique fixed point.*

Very recently, Ljubomir Ciric, B. Samet, H. Aydi and C. Vetro [4], have proved a common fixed point theorem for four mappings satisfying a generalized nonlinear contraction type condition on partial metric spaces and they have given some application related to the homotopy for some operators on a set endowed with a partial metric. The following theorem [4] extended and generalized the results obtained in [1].

Theorem 2.7 *[4]. Suppose that A, B, S and T are self-maps of a complete partial metric space (X, p) such that $Ax \subset TX,\ BX \subset SX$ and*

$$p(Ax, By) \leq \varphi\left(M\left(x, y\right)\right),$$

for all $x, y \in X$, where $\varphi \in \Phi$ and

$$M\left(x, y\right) = \max\left\{p(Sx, Ty), p(Ax, Sx), p(By, Ty), \frac{1}{2}\left[p(Sx, By) + p(Ax, Ty)\right]\right\}$$

If one of the ranges AX, BX, TX and SX is a closed subset of (X, p), then

(i) A and S have a coincidence point.

(ii) B and T have a coincidence point.

In [5] K.P.R Rao and G.N.V. Kishore have obtained a unique fixed point theorem for self maps satisfying $\psi - \varphi$ contractive condition in partial metric spaces. They generalized and improved some results of Altun et al.[1].

Theorem 2.8 *[5]. Let (X, p) be a complete partial metric space and let*

$$S, T, f, g : X \to X$$

be such that

$$\psi\left(p(Sx, Ty)\right) \leq \psi\left(M\left(x, y\right)\right) - \varphi\left(M\left(x, y\right)\right) \text{ for all } x, y \in X,$$

where $\varphi,\ \psi : [0, \infty[\to [0, \infty[$. ψ is continuous, nondecreasing and φ is lower semi-continuous with $\varphi(t) < t$ and

$$M\left(x, y\right) = \max\left\{p(fx, gy, p(fx, Sx, p(gy, Ty, \frac{1}{2}\left[p(fx, Ty) + p(gy, Sx)\right]\right\}$$

(i) T and F have a coincidence point.

(ii) g and S have a coincidence point.

Before stating our main results, we recall the following definitions.

Definition 2.9 *Let X be a non-empty set and $T_1, T_2 : X \to X$ are given self-maps on X. The pair (T_1, T_2) is said to be weakly compatible if $T_1T_2t = T_2T_1t$, whenever $T_1t = T_2t$ for some t in X.*

Our main results are the following:

3. Main Results

Theorem 3.1 *Let (X, p) be a complete partial metric space and let $A, B, S, T : X \to X$ be such that*

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X) \tag{1}$$

$$\psi\left(p\left(Ax, By\right)\right) \leq \psi\left(\theta\left(x, y\right)\right) - \varphi\left(\theta\left(x, y\right)\right) \quad \text{for all } x, y \in X \tag{2}$$

where

$$\theta\left(x, y\right) = \lambda p\left(Ax, Sx\right) + \mu p\left(By, Ty\right) + \delta p\left(Sx, Ty\right) + \gamma\left[p\left(Ax, Ty\right) + p\left(Sx, By\right)\right] \tag{3}$$

$$\mu, \delta, \gamma, \lambda \in]0, 1[\quad \text{and} \quad \mu + \delta + 2\gamma + \lambda < 1. \tag{4}$$

and $\varphi,\ \psi : [0, \infty[\to [0, \infty[$. ψ is continuous, nondecreasing and φ is lower semi-continuous, $\varphi\left(t\right) = \psi\left(t\right) = 0 \Longleftrightarrow t = 0$. If either $T\left(X\right)$ or $S\left(X\right)$ is a complete subspace of X and the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be any element in X. Using (1), we construct sequences $(x_n), (y_n)$ in X such that

$$\{Ax_{2n} = Tx_{2n+1} = y_{2n} Bx_{2n+1} = Sx_{2n+2} = y_{2n+1} \quad \text{for all } n \geq 1. \tag{5}$$

First we prove that, if there exists $n \geq 1$ such that $\theta(x_{2n}, x_{2n-1}) = 0$, then

$$y_{2n} = y_{2n-1} \tag{6}$$

By taking $x = x_{2n}$ and $y = x_{2n-1}$ in (3), we get

$$\begin{aligned} 0 = \theta(x_{2n}, x_{2n-1}) &= \lambda p(Ax_{2n}, Sx_{2n}) + \mu p(Bx_{2n-1}, Tx_{2n-1}) \\ &+\delta p(Sx_{2n}, Tx_{2n-1}) \\ &+\gamma [p(Ax_{2n}, Tx_{2n-1}) + p(Sx_{2n}, Bx_{2n-1})] \\ &= \lambda p(y_{2n}, y_{2n-1}) + \mu p(y_{2n-1}, y_{2n-2}) + \delta p(y_{2n-1}, y_{2n-2}) \\ &+\gamma [p(y_{2n}, y_{2n-2}) + p(y_{2n-1}, y_{2n-1})]. \end{aligned}$$

Thus, since $\lambda > 0$ and $\lambda p(y_{2n}, y_{2n-1}) \leq \theta(x_{2n}, x_{2n-1}) = 0$, it follows that

$$p(y_{2n}, y_{2n-1}) = 0,$$

hence

$$y_{2n} = y_{2n\ -1} \tag{7}$$

Now we claim if (7) is true, then we have

$$y_{2n} = y_{2n\ +1}, \tag{8}$$

$$\left.\begin{aligned} \theta(x_{2n}, x_{2n+1}) &= \lambda p(Ax_{2n}, Sx_{2n}) + \mu p(Bx_{2n+1}, Tx_{2n+1}) \\ &+\delta p(Sx_{2n}, Tx_{2n+1}) \\ &+\gamma [p(Ax_{2n}, Tx_{2n+1}) + p(Sx_{2n}, Bx_{2n+1})] \\ &= \lambda p(y_{2n}, y_{2n-1}) + \mu p(y_{2n+1}, y_{2n}) + \delta p(y_{2n-1}, y_{2n}) \\ &+\gamma [p(y_{2n}, y_{2n}) + p(y_{2n-1}, y_{2n+1})] \end{aligned}\right\} \tag{9}$$

From (9) and by the triangle inequality we get

$$\begin{aligned} \theta(x_{2n}, x_{2n+1}) &\leq (\lambda + \delta) p(y_{2n}, y_{2n+1}) + \mu p(y_{2n+1}, y_{2n}) \\ &+ \gamma [p(y_{2n}, y_{2n+1}) + p(y_{2n}, y_{2n-1}) - p(y_{2n}, y_{2n}) + p(y_{2n}, y_{2n})] \end{aligned}$$

Hence

$$\theta(x_{2n}, x_{2n+1}) \leq (\gamma + \delta + \lambda) p(y_{2n-1}, y_{2n}) + (\mu + \gamma) p(y_{2n}, y_{2n+1}). \tag{10}$$

Since

$$p(y_{2n}, y_{2n-1}) = p(y_{2n}, y_{2n}) \leq p(y_{2n}, y_{2n+1}), \tag{11}$$

then from (10), (11) and (4) we obtain

$$\left.\begin{aligned} \theta(x_{2n}, x_{2n+1}) &\leq (\lambda + \mu + \delta + 2\gamma) p(y_{2n}, y_{2n+1}) \\ &< p(y_{2n}, y_{2n+1}) \end{aligned}\right\} \tag{12}$$

Since ψ is monotone, then

$$\psi(\theta(x_{2n}, x_{2n+1})) \leq \psi(p(y_{2n}, y_{2n+1})), \tag{13}$$

$$p(Ax, By) = p(Ax_{2n}, Bx_{2n+1}) = p(y_{2n}, y_{2n+1}). \tag{14}$$

From (13) , (14) and (2) we get

$$\psi\left(p\left(y_{2n}, y_{2n+1}\right)\right) \leq \psi\left(p\left(y_{2n}, y_{2n+1}\right)\right) - \varphi\left(\theta\left(x_{2n}, x_{2n+1}\right)\right).$$

By the property of φ, we have $\varphi\left(\theta\left(x_{2n}, x_{2n+1}\right)\right) = 0$, this implies that

$$\theta\left(x_{2n}, x_{2n+1}\right) = 0.$$

By the fact that $\lambda > 0$ and $\lambda p\left(y_{2n}, y_{2n+1}\right) \leq \theta\left(x_{2n}, x_{2n+1}\right) = 0$, therefore $y_{2n} = y_{2n+1}$. Continuing in this way, we can conclude that $y_n = y_{n+k}$ for all $k \geq 0$. Thus, the sequence $\{y_n\}$ is a Cauchy sequence. Now we can suppose that

$$\theta\left(x_{2n}, x_{2n+1}\right) = 0 \text{ for all } n \geq 1. \tag{15}$$

Setting $p_{2n} = p(y_{2n}, y_{2n+1})$. We claim that

$$p_{2n+1} \leq p_{2n} \text{ for all } n \geq 1. \tag{16}$$

Suppose (16) is not true, that is, there exists $n \in N$ such that $p_{2n+1} > p_{2n}$, then

$$\begin{aligned}\psi\left(p_{2n}\right) \leq \psi(p_{2n+1}) &= \psi\left(p\left(y_{2n+1}, y_{2n+2}\right)\right) = \psi\left(p\left(Ax_{2n+1}, Bx_{2n+2}\right)\right) \\ &\leq \psi\left(\theta\left(x_{2n+1}, x_{2n+2}\right)\right) - \varphi\left(\theta\left(x_{2n+1}, x_{2n+2}\right)\right),\end{aligned}$$

$$\begin{aligned}\theta\left(x_{2n+2}, x_{2n+1}\right) &= \lambda p\left(Ax_{2n+2}, Sx_{2n+2}\right) + \mu p\left(Bx_{2n+1}, Tx_{2n+1}\right) \\ &\quad + \delta p\left(Sx_{2n+2}, Tx_{2n+1}\right) \\ &\quad + \gamma\left[p\left(Ax_{2n+2}, Tx_{2n+1}\right) + p\left(Sx_{2n+2}, Bx_{2n+1}\right)\right] \\ &= \lambda p(y_{2n+2}, y_{2n+1}) + \mu p(y_{2n+1}, y_{2n}) + \delta p(y_{2n+1}, y_{2n}) \\ &\quad + \gamma[p(y_{2n+2}, y_{2n}) + p(y_{2n+1}, y_{2n+1})].\end{aligned}$$

Then by triangle inequality, we get

$$\theta\left(x_{2n+2}, x_{2n+1}\right) \leq \lambda p_{2n+1} + \mu p_{2n} + \delta p_{2n+1} + \gamma p_{2n+1} + \gamma p_{2n}.$$

Since

$$p(y_{2n+1}, y_{2n+2}) \leq p(y_{2n}, y_{2n+1}),$$

$\lambda + \mu + \delta + 2\gamma < 1$, then from (16) we have

$$\begin{aligned}\theta\left(x_{2n+1}, x_{2n+2}\right) &\leq (\lambda + \mu + \delta + 2\gamma)\, p_{2n} \\ &\leq p_{2n},\end{aligned}$$

ψ is monotone, we have

$$\psi\left(p_{2n}\right) \leq \psi\left(p_{2n}\right) - \varphi\left(\theta\left(x_{2n+1}, x_{2n+2}\right)\right).$$

This implies $\varphi\left(\theta\left(x_{2n}, x_{2n+1}\right)\right) = 0$, by the property of φ, it follows that $\theta\left(x_{2n}, x_{2n+1}\right) = 0$, which is a contradiction with (15). With the same way, we prove

$$p_{2n+2} \leq p_{2n+1} \text{ for all } n \geq 1. \tag{17}$$

Thus from (16) and from (17) we have

$$p_{n+1} \leq p_n \text{ for all } n \geq 1.$$

Hence, the sequence $\{p_n\}$ is a non-decreasing sequence of non negative real numbers and must convergence to a real number denoted by l. Say:

$$\lim_{n \to \infty} p\left(y_n, y_{n+1}\right) = l, \quad l \geq 0.$$

We shall prove that $l = 0$. We suppose that

$$l > 0, \tag{18}$$

then from (9) and (12) we get

$$\begin{aligned}\lim_{n\to\infty} \lambda p\left(Ax_{2n}, Sx_{2n}\right) = \lim_{n\to\infty} \lambda p\left(y_{2n}, y_{2n-1}\right) \leq \ \limsup_{n\to\infty} \theta\left(x_{2n}, x_{2n+1}\right) \\ \leq (\lambda + \mu + \delta + 2\gamma) \lim_{n\to\infty} p\left(y_{2n}, y_{2n+1}\right),\end{aligned}$$

this implies, by using (4), that

$$0 < \lambda l \leq \ \limsup_{n\to\infty} \theta\left(x_{2n}, x_{2n+1}\right) \leq l,$$

so, there exists $l_1 > 0$ and a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ such that

$$\lim_{n\to\infty} \theta\big(x_{2n_k}, x_{2n_k+1}\big) = l_1 \leq l.$$

Hence, by the lower semicontinuity of φ, we have

$$\varphi\left(l_1\right) \leq \ \liminf_{k\to\infty} \varphi\big(\theta\big(x_{2n_k}, x_{2n_k+1}\big)\big) \tag{19}$$

From (2), we get

$$\psi\big(p(y_{2n_k}, y_{2n_k+1})\big) \leq \psi\big(\theta(x_{2n_k}, x_{2n_k+1})\big) - \varphi\big(\theta(x_{2n_k}, x_{2n_k+1})\big) \tag{20}$$

Taking the upper limit as $k \to \infty$ in (20), we obtain

$$\begin{aligned}\psi(l) &\leq \psi(l_1) - \ \liminf_{k\to\infty} \varphi\big(\theta(x_{2n_k}, x_{2n_k+1})\big) \\ &\leq \psi\left(l_1\right) - \varphi\left(l_1\right) \\ &\leq \psi\left(l\right) - \varphi\left(l_1\right).\end{aligned}$$

This implies that $\varphi\left(l_1\right) = 0$. Thus, by the property of φ, we have $l_1 = 0$, which is a contradiction with (18). Therefore $l = 0$ and so

$$\lim_{n\to\infty} p\left(y_n,\, y_{n+1}\right) = 0, \tag{21}$$

and from (p_2), we have also

$$\lim_{n\to\infty} p\left(y_n,\, y_n\right) = 0, \tag{22}$$

from (21) and (22), we have

$$\lim_{n\to\infty} p^s\left(y_n,\, y_{n+1}\right) = 0\,. \tag{23}$$

Now, we prove that $\{y_{2n}\}$ is a Cauchy sequence in $\big(X, p^s\big)$. On contrary, suppose that $\{y_{2n}\}$ is not a Cauchy sequence in $\big(X, p^s\big)$. There exists an $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{2m_k\}$ and $\{2n_k\}$ such that $n_k < m_k$ and

$$p^s\big(y_{2m_k},\, y_{2n_k}\big) \geq \varepsilon \tag{24}$$

and

$$p^s\big(y_{2m_k},\, y_{2n_k-2}\big) < \varepsilon \tag{25}$$

From (24) and (25) we get

$$\begin{aligned}\varepsilon &\leq p^s\big(y_{2m_k},\, y_{2n_k}\big) \\ &\leq p^s\big(y_{2m_k},\, y_{2n_k-2}\big) + p^s\big(y_{2n_k-2},\, y_{2n_k-1}\big) + p^s\big(y_{2n_k-1},\, y_{2n_k}\big) \\ &< \varepsilon + p^s\big(y_{2n_k-2},\, y_{2n_k-1}\big) + p^s\big(y_{2n_k-1},\, y_{2n_k}\big).\end{aligned}$$

Letting $k \to \infty$ and using (23), we have

$$\lim_{k\to\infty} p^s\big(y_{2m_k}, y_{2n_k}\big) = \varepsilon. \tag{26}$$

Hence from the definition of p^s and from (22), we have

$$\lim_{k\to\infty} p\big(y_{2m_k}, y_{2n_k}\big) = \frac{\varepsilon}{2}. \tag{27}$$

Letting $k \to \infty$ and using (26), (24) in

$$\big|p^s\big(y_{2m_k}, y_{2n_k+1}\big) - p^s\big(y_{2m_k}, y_{2n_k}\big)\big| \le p^s\big(y_{2n_k+1}, y_{2n_k}\big),$$

we obtain

$$\lim_{k\to\infty} p^s\big(y_{2m_k}, y_{2n_k+1}\big) = \varepsilon. \tag{28}$$

Hence, we have

$$\lim_{k\to\infty} p\big(y_{2m_k}, y_{2n_k+1}\big) = \frac{\varepsilon}{2}. \tag{29}$$

Letting $k \to \infty$ and using (26), (24) in

$$\big|p^s\big(y_{2m_k-1}, y_{2n_k+1}\big) - p^s\big(y_{2m_k}, y_{2n_k}\big)\big| \le p^s\big(y_{2m_k-1}, y_{2m_k}\big),$$

we get:

$$\lim_{k\to\infty} p^s\big(y_{2m_k-1}, y_{2n_k}\big) = \varepsilon. \tag{30}$$

Hence, we have

$$\lim_{k\to\infty} p\big(y_{2m_k-1}, y_{2n_k}\big) = \frac{\varepsilon}{2}. \tag{31}$$

Letting $k \to \infty$ and using (30), (24) in

$$\big|p^s(y_{2m_k-1}, y_{2n_k+1}) - p^s(y_{2m_k-1}, y2n_k)\big| \le p^s(y_{2n_k+1}, y_{2n_k}).$$

we get:

$$\lim_{k\to\infty} p^s\big(y_{2m_k-1}, y_{2n_k+1}\big) = \varepsilon. \tag{32}$$

Hence, we have

$$\lim_{k\to\infty} p\big(y_{2m_k-1}, y_{2n_k+1}\big) = \frac{\varepsilon}{2}. \tag{33}$$

Now, by (2) and (3) we have

$$\begin{aligned}\psi\big(p\big(Ax_{2m_k}, Bx_{2n_k+1}\big)\big) &= \psi\big(p\big(y_{2m_k}, y_{2n_k+1}\big)\big)\\ &\le \psi\big(\theta\big(x_{2m_k}, x_{2n_k+1}\big)\big) - \varphi\big(\theta\big(x_{2m_k}, x_{2n_k+1}\big)\big),\end{aligned}$$

$$0.2cm \left.\begin{array}{lll}\theta\big(x_{2m_k}, x_{2n_k+1}\big) & = & \lambda p\big(y_{2m_k}, y_{2m_k-1}\big) + \mu p\big(y_{2n_k+1}, y_{2n_k}\big)\\ & & +\delta p\big(y_{2m_k-1}, y_{2n_k}\big)\\ & & +\gamma\big[p\big(y_{2m_k}, y_{2n_k}\big) + p\big(y_{2m_k-1}, y_{2n_{k+1}}\big)\big]\end{array}\right\} \tag{34}$$

Letting $k \to \infty$ and using (21), (27) , (31), (34) and since $\gamma + \frac{\delta}{2} \le \frac{1}{2}$, we obtain

$$0.3cm \begin{array}{lll}\psi\left(\frac{\varepsilon}{2}\right) & \le & \psi\left(\left(\gamma + \frac{\delta}{2}\right)\varepsilon\right) - \varphi\left(\left(\gamma + \frac{\delta}{2}\right)\varepsilon\right)\\ & \le & \psi\left(\frac{\varepsilon}{2}\right) - \varphi\left(\left(\gamma + \frac{\delta}{2}\right)\varepsilon\right),\end{array}$$

this implies that $\varphi\left(\left(\gamma+\frac{\delta}{2}\right)\varepsilon\right)=0$, then $\varepsilon=0$; which is a contradiction. Hence $\{y_{2n}\}$ is a Cauchy sequence. Letting $n\to\infty$ and $m\to\infty$ in

$$|p^s(y_{2n+1},\,y_{2m+1})-p^s(y_{2m},\,y_{2n})|\le p^s\left(y_{2n+1},\,y_{2n}\right)+p^s\left(y_{2m+1},\,y_{2m}\right),$$

we get $\lim\limits_{n.m\to\infty}p^s\left(y_{2n+1},\,y_{2m+1}\right)=0$. Hence $\{y_{2n+1}\}$ is a Cauchy sequence. Thus $\{y_n\}$ is a Cauchy sequence in (X,p^s). We have $\lim\limits_{n.m\to\infty}p^s(y_n,\,y_m)=0$. Now, from the definition of p^s and from (22), we have

$$\lim_{n.m\to\infty}p(y_n,\,y_m)=0. \tag{35}$$

Suppose $S(X)$ is complete. Since $\{y_{2n+1}\}\subset S(X)$ is a Cauchy sequence in the complete metric space $(S(X),\ p^s)$, therefore there exists $t\in X$ such that $v=S(t)\in S(X)$. Since $\{y_n\}$ is a Cauchy sequence in $(X,\ p^s)$ and $y_{2n+1}\to v$, it follows that $y_{2n}\to v$. From Lemma 1 (b), we have

$$p(v,v)=\lim_{n\to\infty}p(y_{2n+1},v)=\lim_{n\to\infty}p(y_{2n},v)=\lim_{n.m\to\infty}p(y_n,\,y_m) \tag{36}$$

From (35) and (36) ,we have

$$p(v,v)=\lim_{n\to\infty}p(y_{2n+1},v)=\lim_{n\to\infty}p(y_{2n},v)=0 \tag{37}$$

We shall prove that $\lim\limits_{n\to\infty}p(At,y_{2n})=p(At,v)$. Letting $n\to\infty$ in

$$p^s(At,y_{2n})=2p(At,y_{2n})-p(At,At)-p(y_{2n},y_{2n})$$

we get by using (22)

$$p^s(At,v)=2\lim_{n\to\infty}p(At,y_{2n})-p(At,At)-0$$

$$2p(At,v)-p(At,At)-p(v,v)=2\lim_{n\to\infty}p(At,y_{2n})-p(At,At)$$

By (37), we have

$$p(At,v)=\lim_{n\to\infty}p(At,y_{2n})$$

Let $At\neq v$

$$\begin{aligned}p(At,v)&\le p(At,Bx_{2n+1})+p(Bx_{2n+1},v)-p(Bx_{2n+1},Bx_{2n+1})\\&\le p(At,Bx_{2n+1})+p(y_{2n+1},v)\end{aligned}$$

$$\psi\left(p(At,v)\right)\le\psi\left(p(At,Bx_{2n+1})+p(y_{2n+1},v)\right) \tag{38}$$

Hence letting $n\to\infty$ in (38), we obtain

$$\begin{aligned}\psi\left(p(At,v)\right)&\le\psi\left(\lim_{n\to\infty}p(At,Bx_{2n+1})+0\right)\\&=\lim_{n\to\infty}\psi\left(p(At,Bx_{2n+1})\right)\\&\le\lim_{n\to\infty}\left[\psi\left(\theta\left(t,x_{2n+1}\right)\right)-\varphi\left(\theta\left(t,x_{2n+1}\right)\right)\right]\end{aligned}$$

$$\begin{aligned}\theta\left(\left(t,x_{2n+1}\right)\right)&=\lambda p(At,Sx_{2n+1})+\mu p(Bx_{2n+1},Tx_{2n+1})\\&\quad+\delta p(St,Tx_{2n+1})+\gamma\left[p(At,Tx_{2n+1})+p(St,Bx_{2n+1})\right]\end{aligned} \tag{39}$$

Then

$$\begin{aligned}\theta\left(\left(t,x_{2n+1}\right)\right)&=\lambda p(At,y_{2n})+\mu p(y_{2n+1},y_{2n})\\&\quad+\delta p(v,y_{2n})+\gamma\left[p(At,y_{2n})+p(v,y_{2n+1})\right]\end{aligned} \tag{40}$$

Letting $n \to \infty$ in (40) and using (21), (37) and the fact that $\lambda + \gamma < 1$, we obtain

$$\lim_{n\to\infty} \theta\left((t, x_{2n+1})\right) = (\lambda + \gamma)\, p(v, At) \leq p(v, At)$$

Thus

$$\psi\left(p(At, v)\right) \leq \psi\left(p(At, v)\right) - \varphi\left((\lambda + \gamma)\, p(v, At)\right)$$

It follows $\varphi\left((\lambda + \gamma)\, p(v, At)\right) = 0$, from the property of φ we have $p(v, At) = 0$ hence $v = At = St$. Since the pair (A, S) are compatible, We have $Av = Sv$. Suppose

$$Sv \neq v$$

As in above, using the metric p^s and (22) , (37), we can show that

$$p(Av, v) = \lim_{n\to\infty} p(Av, y_{2n})$$
$$\begin{aligned} p(Av, v) &\leq \left[p(Av, Bx_{2n+1}) + p(Bx_{2n+1}, v) - p(Bx_{2n+1}, Bx_{2n+1})\right] \\ &\leq p(Av, Bx_{2n+1}) + p(y_{2n+1}, v) \end{aligned}$$

Then

$$\psi\left(p(Av, v)\right) \leq \psi\left(p(Av, Bx_{2n+1}) + p(y_{2n+1}, v)\right) \tag{41}$$

Letting $n \to \infty$, we get

$$\begin{aligned} \psi\left(p(Av, v)\right) &\leq \psi\left(\lim_{n\to\infty} p(Av, Bx_{2n+1})\right) + 0 \\ &= \lim_{n\to\infty} \psi\left(p(Av, Bx_{2n+1})\right) \\ &\leq \lim_{n\to\infty} \left[\psi\left(\theta\left(v, x_{2n+1}\right)\right) - \varphi\left(\theta\left(v, x_{2n+1}\right)\right)\right] \end{aligned}$$

Since

$$\begin{aligned} \theta\left(v, x_{2n+1}\right) = &\ \lambda p\left(Av, Av\right) + \mu p(y_{2n}, y_{2n+1}) + \delta p(Av, y_{2n}) \\ &+ \gamma\left[p(y_{2n}, Av) + p(Av, y_{2n+1})\right] \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \theta\left(v, x_{2n+1}\right) = &\ \lambda p\left(Av, Av,\right) + 0 + \delta p(Av, v) \\ &+ 2\gamma p(Av, v) \end{aligned}$$

Since $\lambda + \delta + 2\gamma < 1$,We obtain

$$\lim_{n\to\infty} \theta\left(v, x_{2n+1}\right) = (\lambda + \delta + 2\gamma)\, p(Av, v) \leq p(Av, v)$$

Thus

$$\psi\left(p(Av, v)\right) \leq \psi\left(p(Av, v)\right) - \varphi\left((\lambda + \mu + 2\gamma)\, p(Av, v)\right)$$

Hence $\varphi\left((\lambda + \mu + 2\gamma)\, p(Av, v)\right) = 0$, by the property of φ, we have

$$Av = v = Sv \tag{42}$$

Since $A(X) \subset T(X)$, there exists $w \in X$ such that $v = Sv = Tw$. Suppose $v \neq Bw$

$$\psi\left(p(v, Bw)\right) = \psi\left(p(Av, Bw)\right) \leq \psi\left(\theta\left(v, w\right)\right) - \varphi\left(\theta\left(v, w\right)\right)$$

$$\begin{aligned} \theta\left(v, w\right) &= \lambda p\left(Av, Sv\right) + \mu p(Bw, Tw) + \delta p(Sv, Tw) \\ &\quad + \gamma\left[p(Av, Tw) + p(Sv, Bw)\right] \\ &= \lambda p\left(v, v\right) + \mu p(Bw, v) + \delta p(v, v) \\ &\quad + \gamma\left[p(v, v) + p(v, Bw)\right] \\ &= (\lambda + \delta + \gamma) p(v, v) + (\mu + \gamma) p(v, Bw) \\ &= 0 + (\mu + \gamma) p(v, Bw) \\ &\leq p(v, Bw). \end{aligned}$$

Hence

$$\psi\left(p(v,Bw)\right) \leq \psi\left(p(v,Bw)\right) - \varphi\left((\mu+\gamma)p(v,Bw)\right).$$

Thus, $\varphi\left((\mu+\gamma)p(v,Bw)\right) = 0$. By the property of φ, we have $v = Bw$. Thus $Tw = Bw = v$. Since (T, B) is weakly compatible, we have $Tv = Bv$. Suppose $Bv \neq v$.

$$\psi\left(p(v,Bv)\right) = \psi\left(p(Av,Bv)\right) \leq \psi\left(\theta\left(v,v\right)\right) - \varphi\left(\theta\left(v,v\right)\right),$$

$$\begin{aligned}\theta\left(v,v\right) &= \lambda p\left(Av,Sv\right) + \mu p(Bv,Tv) + \delta p(Sv,Tv) \\ &\quad +\gamma\left[p(Av,Tv) + p(Sv,Bv)\right] \\ &= \lambda p\left(v,v\right) + \mu p(Bv,Bv) + \delta p(v,Bv) \\ &\quad +\gamma\left[p(v,Bv) + p(v,Bv)\right] \\ &= \mu p(Bv,Bv) + \left(\delta+2\gamma\right)p(v,Bv) \quad (From p_2) \\ &= \left(\mu+\delta+2\gamma\right)p(v,Bv) \\ &\leq p(v,Bv),\end{aligned}$$

hence

$$\psi\left(p(v,Bv)\right) \leq \psi\left(p(v,Bv)\right) - \varphi\left(\left(\mu+\delta+2\gamma\right)p(v,Bv)\right).$$

It follows $\varphi\left(\left(\mu+\delta+2\gamma\right)p(v,Bv)\right) = 0$, then from the property of φ, we have $p(v,Bv) = 0$, thus $v = Bv$. We have.

$$Tv = Bv = v. \tag{43}$$

From (42) and (43), v is a common fixed point of A, B, T and S. Now we prove the uniqueness of the common fixed point. Let z be another common fixed point of A, B, T and S. Suppose $v \neq z$,

$$\psi\left(p(v,z)\right) = \psi\left(p(Av,Bz)\right) \leq \psi\left(\theta\left(v,z\right)\right) - \varphi\left(\theta\left(v,z\right)\right),$$

$$\begin{aligned}\theta\left(v,z\right) &= \lambda p\left(Av,Sv\right) + \mu p(Bz,Tz) + \delta p(Sv,Tz) \\ &\quad +\gamma\left[p(Av,Tz) + p(Sv,Bz)\right] \\ &= \lambda p\left(v,v\right) + \mu p(z,z) + \delta p(v,z) \\ &\quad +\gamma\left[p(v,z) + p(v,z)\right] \\ &= \left(\delta+\mu+2\gamma\right)p(v,z) \text{ From } p_2 \\ &\leq p(v,z).\end{aligned}$$

Hence

$$\psi\left(p(v,z)\right) \leq \psi\left(p(v,z)\right) - \varphi\left(\left(\delta+\mu+2\gamma\right)p(v,z)\right).$$

It follows that $\varphi\left(\left(\delta+\mu+2\gamma\right)p(v,z)\right) = 0$ and by the property of φ, we have $v = z$. Thus v is the unique common fixed point of A, B, T and S.

Corollary 3.2 *Let $(X,\ p)$ be a partial metric space and let $A, B, S, T : X \to X$ be such that*

$$A(X) \subset T(X) \ \textit{and} \ \ B(X) \subset S(X).$$

If

$$p\left(Ax,Bx\right) \leq \theta\left(x,y\right) \quad \textit{for all} \ \ x,y \in X,$$

where

$$\begin{aligned}\theta\left(x,y\right) &= \lambda p\left(Ax,Sx\right) + \mu p\left(By,Ty\right) + \delta p\left(Sx,Ty\right) \\ &\quad +\gamma\left[p\left(Ax,Ty\right) + p\left(Sx,By\right)\right],\end{aligned}$$

$$\mu,\delta,\gamma,\lambda \in [0.1[\quad \textit{and} \quad \lambda 0,\ \mu+\delta+2\gamma+\lambda < 1,$$

if either $T\left(X\right)$ or $S\left(X\right)$ is a complete subspace of X and the pairs (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Taking $\psi(t) = t$ and $\varphi = 0$ in theorem 3.1.

Corollary 3.3 *Let $(X,\ p)$ be a partial metric space and let $A, T : X \to X$ be such that*

$A(X) \subset T(X)$.

If

$p(Ax, Ax) \leq \theta(x, y)$ *for all* $x, y \in X$,

where

$$\begin{aligned}\theta(x,y) &= \lambda p(Ax,Tx) + \mu p(Ay,Ty) + \delta p(Tx,Ty) \\ &\quad + \gamma\left[p(Ax,Ty) + p(Tx,Ay)\right],\end{aligned}$$

$\mu, \delta, \gamma, \lambda \in [0.1[$ *and* $\lambda 0,\ \mu + \delta + 2\gamma + \lambda < 1$,

if $T(X)$ is a complete subspace of X and the pairs (A,T) are weakly compatible, then A and T have a unique common fixed point in X.

Proof. Taking $\psi(t) = t$ and $\varphi = 0$ and $A = B$ and $T = S$ in theorem 5.

Example 3.4 *Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$ for $x, y \in X$. Let A, B, S and $T : X \to X$ and*

- $S(x) = \frac{x}{2}$, $T(x) = \frac{x}{3}$, $A(x) = \frac{x}{4}$, $B(x) = \frac{x}{6}$,
- $\psi : [0,\infty[\to [0,\infty[$ *defined by:* $\psi(t) = t$,
- $\varphi : [0,\infty[\to [0,\infty[$ *defind by* $\varphi(t) = \frac{t}{2}$,
- $\lambda = \beta = \gamma = \delta = \frac{1}{6}$.

Then all conditions of theorem 3.1 are satisfied and 0 *is the unique fixed point of A, B, S and T.*

Example 3.5 *Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$ for $x, y \in X$. Let A, B, S and $T : X \to X$ and*

- $S(x) = \frac{x}{x+1}$, $T(x) = \frac{x}{x+2}$, $A(x) = \frac{x^2}{2x+2}$ *and* $B(x) = \frac{x^2}{2x+4}$,
  $\psi : [0,\infty[\to [0,\infty[$ *defined by* $\psi(t) = t$,
  $\varphi : [0,\infty[\to [0,\infty[$ *by* $\varphi(t) = \frac{t}{2}$,
- $\lambda = \beta = \gamma = \delta = \frac{1}{6}$.

Then all conditions of theorem 3.1 are satisfied and 0 *is the unique fixed point of A, B, S and T.*

4. Applications

In this section, we give an application of the previous section.

Set $Y = \{\chi : [0,\infty[\to [0,\infty[,\ \chi$ is a Lebesgue integrable mapping which is summable and nonegative and satisfies $\int_0^{\varepsilon} \chi(t)\,\mathrm{d}t > 0$ for each $\varepsilon > 0\}$.

Theorem 4.1 *Let (X, p) be a complete partial metric space and let $A, B, S, T : X \to X$ be such that*

$A(X) \subset T(X)$ *and* $B(X) \subset S(X)$

and for all $x, y \in X$:

$$\int_0^{\psi(p(Ax,By))} \chi(t)\,dt \leq \int_0^{\psi(\theta(x,y))} \chi(t)\,dt - \int_0^{\varphi(\theta(x,y))} \chi(t)\,dt, \quad \chi \in Y,$$

where

$$0.2cm \quad \left.\begin{aligned}\theta(x,y) &= \lambda p(Ax,Sx) + \mu p(By,Ty) + \delta p(Sx,Ty) \\ &\quad + \gamma\left[p(Ax,Ty) + p(Sx,By)\right] \\ &\mu, \delta, \gamma, \lambda \in]0.1[\text{ and } \mu + \delta + 2\gamma + \lambda < 1,\end{aligned}\right\} \quad (44)$$

and $\varphi,\ \psi : [0,\infty\ [\to [0,\infty\ [$. ψ is continuous, nondecreasing and φ is lower semi-continuous, $\varphi(t) = \psi(t) = 0 \Longleftrightarrow t = 0$. If either $T(X)$ or $S(X)$ is a complete subspace of X and the pairs (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof. Define $\Lambda : R_+ \to R_+$ by $\Lambda(x) = \int_0^x \chi(t)\, dt$. then Λ is continuous and nondecreasing with $\Lambda(0) = 0$. Then we obtain

$$\Lambda(\psi(p(Ax, By)) \leq \psi(p(Ax, By)) - \varphi(p(Ax, By))$$

Which further can be written as

$$\Psi_1(p(Ax, By) \leq \Psi_1(\theta(x, y)) - \Phi_1(\theta(x, y))$$

where $\Psi_1 = \Lambda \circ \psi$ and $\Phi_1 = \Lambda \circ \varphi$. Hence by theorem 3.1 we have the desired results.

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