



# Dual series method for solving Helmholtz equation with mixed boundary conditions of the third kind

N. A. Hoshan

Department of Mathematics, Tafila Technical University, Tafila, Jordan  
E-mail: [dr\\_naserh@yahoo.com](mailto:dr_naserh@yahoo.com)

Copyright © 2014 N. A. Hoshan. This is an open access article distributed under the [Creative Commons Attribution License](https://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## Abstract

The paper involves with the application of the dual series equation to the problem of Helmholtz equation of cylindrical coordinates subject to inhomogeneous mixed boundary conditions of the third kind located on the surface of the cylinder of bounded radius and infinite high. By choosing the Hankel integral transform, the dual series equations were reduced to a Fredholm integral equation of the second kind which is solved conveniently by using numerical techniques.

**Keywords:** Helmholtz Equation, Dual Series Equations, Mixed Boundary Conditions.

## 1. Introduction

It is well known that heat and wave equations with different coordinate systems and various initial and boundary conditions can be reduced to Helmholtz equation, for example temperature function  $T(t, P)$  is transformed to  $T(t, P) = e^{-t\gamma} u(P)$ , such transformation simplify the solution of the required partial differential equation. In this paper we presented a dual series equations related to Helmholtz equation in cylindrical coordinates subject to inhomogeneous mixed boundary conditions of the third kind located on a level surface of the cylinder of the bounded radius and infinite high. The radius of the surface cylinder  $R$  is divided into two intervals such that, inside the disk  $0 < r < R_1$  located a third kind boundary conditions which is different with the boundary conditions of the third kind located outside the disk  $R_1 < r < R_2$ . No boundary conditions on the line of discontinuity  $r = R_1$ . Application of separation of variables to the Helmholtz partial differential equation, we obtain

a general solution with unknown coefficients, next step the use of the boundary conditions leads to dual series equations with Bessel function as a kernel for determination unknown coefficients.

Problems of the heat and Helmholtz equations with mixed boundary conditions of the first, the second and of the third kind in axial cylindrical coordinates have previously been treated and founded in the scientific or technical monographs on this subject [1-6], [8]. Much attention was received to investigate mixed problems with partly infinite boundaries can often be reduced to study dual integral equations [1-6], while for the finite regions the problems can often reduce to study dual series equations. Dual series equations related to Laplace equation in cylindrical, spherical and other coordinates involving potential theory, diffraction theory, elasticity theory and other applications can be found in [10-12] and other monographs. In this paper, we treated the dual series equations by defining Hankel integral transform of a function related to the unknown function satisfying the dual series equations, and then the inverse transformation produces ultimately the function sought for. In [9-11] Hankel integral transform method is used for investigation solutions for dual integral equations involving a Bessel function of the first kind of order zero as a kernel. The analysis of the dual series equations led to a Fredholm integral equation of the second kind which is solved numerically.

## 2. Formulation and solution of the problem

Find the solution of Helmholtz mixed boundary value problem in a axial symmetrical solid cylinder

$$\nabla^2 v + k^2 v = 0,$$

(1)

$$av_z - bv = -f_1(r), \quad r \in D \quad (2)$$

$$a_2u_z - b_2u = -f_2(r), \quad r \in \bar{D} \quad (3)$$

Where  $v = v(r, z)$ ,  $\nabla^2 = (\partial^2 / \partial r^2 + \frac{1}{r} \partial / \partial r + \partial^2 / \partial z^2)$ ,  $0 < r < R_2$ ,

$0 < z < \infty$  Are the corresponding cylindrical coordinates,  $k$  is a constant,  $k \neq 0$   
 $D = \{0 < r < R_1, z = 0\}$ ,  $\bar{D} = \{R_1 < r < R_2, z = 0\}$   $R_2$  and is the radius of the cylinder

$a_i, b_i, i = 1, 2$  Are constants,  $f_i(r), i = 1, 2$  known continuous functions, further we assume that as  $\sqrt{r^2 + z^2} \rightarrow 0, v \rightarrow 0$ .

Separation variables in (1), under the assumption that the conditions bounded at infinity and zero, we obtain the general solution of the problem

$$v(r, z) = \sum_{n=1}^{\infty} C_n \exp(-z \sqrt{\lambda_n^2 + k^2}) J_0(\lambda_n r) \quad (4)$$

$C_n$  Unknown coefficients,  $\lambda_n$  is the root of Bessel function of the first kind order zero  $J_0(\lambda_n R_2) = 0$ .

Use mixed conditions (2) and (3) for (4), we obtain the following dual series equations to determine the unknown coefficients  $C_n$

$$\sum_{n=1}^{\infty} C_n \{a_1 \sqrt{\lambda_n^2 + k^2} + b_1\} J_0(\lambda_n \rho) = f_1(r), \quad r \in D \quad (5)$$

$$\sum_{n=1}^{\infty} C_n \{a_2 \sqrt{\lambda_n^2 + k^2} + b_2\} J_0(\lambda_n \rho) = f_2(r), \quad r \in \bar{D} \quad (6)$$

If  $k = 0$ , and some of the coefficients  $a_i, b_i$  is zero, the dual equations (5) and (6) reduced to known solutions [10], [11].  
 Now to solve (5) and (6), let us to introduce the substitution

$$A_n = C_n (a_2 \sqrt{\lambda_n^2 + k^2} + b_2), \quad G_n = \frac{a_1 \sqrt{\lambda_n^2 + k^2} + b_1}{a_2 \sqrt{\lambda_n^2 + k^2} + b_2}$$

Dual equations (5) and (6) accept the form

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) G_n(\lambda_n) = f_1(r), \quad r \in D \quad (7)$$

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = f_2(r), \quad r \in \bar{D} \quad (8)$$

To solve (7), (8), let us to write down the weight function  $G_n(\lambda_n)$  as

$$G_n(\lambda_n) = G_n(\lambda_n) - q + q = g_n(\lambda_n) + q$$

Where  $q$  is constant,  $g_n$  is continuous function

$$g_n = \frac{\sqrt{\lambda_n^2 + k^2} (a_1 - qa_2) + (b_1 - qb_2)}{a_2 \sqrt{\lambda_n^2 + k^2} + b_2} \quad (9)$$

Dual series equations (7), (8) by meaning (9) will be

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) (g_n(\lambda_n) + q) = f_1(r), \quad r \in D \quad (10)$$

Series (8) is written over the interval  $0 < r < R_2$  such that

$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = \begin{cases} y(r), & r \in D, \\ f_2(r), & r \in \bar{D}. \end{cases}$$

$y(r)$  Is continuous unknown function defined over the interval  $0 < r < R_1$  appears is the nature of a mixed boundary conditions. Use an inversion formula for determination of  $A_n$  in (8), we have

$$A_n = \frac{2}{R_2^2 J_1^2(\lambda_n R_2)} \left\{ \int_0^{R_1} y(u) J_0(\lambda_n u) u du + \int_{R_1}^{R_2} f_2(u) J_0(\lambda_n u) u du \right\} \quad (11)$$

Putting (11) into (10), we have

$$\begin{aligned} q \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) g_n(\lambda_n) = \\ q \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) + \frac{2}{R_2^2 J_1^2(\lambda_n R_2)} \sum_{n=1}^{\infty} \left\{ \int_0^{R_1} y(u) J_0(\lambda_n u) u du + \int_{R_1}^{R_2} f_2(u) J_0(\lambda_n u) u du \right\} \\ J_0(\lambda_n r) g_n(\lambda_n) = f_1(r), \quad r \in D \end{aligned} \quad (12)$$

First sum in (12), is equal to  $y(r)$ , simplifying the second sum and integration yields a Fredholm integral equation of the second kind

$$y(r) + \frac{1}{q} \int_0^{R_1} y(u) K(r, u) du = \frac{1}{q} F(r), \quad r \in D \quad (13)$$

With kernel and free term respectively

$$K(r, u) = \frac{2}{R_2^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n u) J_{-1/2}(\lambda_n r)}{J_1^2(\lambda_n R_2)} g_n$$

$$F(r) = f_1(r) + \frac{2}{R_2^2} \int_{R_1}^{R_2} f_2(u) \left( \sum_{n=1}^{\infty} \frac{J_0(\lambda_n u) J_{-1/2}(\lambda_n r)}{J_1^2(\lambda_n R_2)} g_n \right) du$$

The known function and the kernel satisfy the inequalities [12]

$$\int_0^{R_2} |F(r)| dr < \infty, \quad L^2 = \int_0^{R_1} \int_0^{R_1} K^2(r, u) dr du < \infty, \quad \lambda < 1/L, \quad \lambda = 1/q$$

$$\text{And } \lambda < \frac{1}{MR_1}, \quad M = \max K(r, u), \quad 0 < r, u < R_1$$

### 3. Numerical solution for integral equation

It can be seen that the problem are reduced to determine the solution of an integral equation for the unknown function  $y(r)$  as presented in (13). This equation can be approximated by a sum discrete values of  $r$  and  $u$  leading to the system of linear equations [8]

$$y(r_i) + \sum_{j=1}^n B_j K(r_i, u_j) y(u_j) = F(r_i) + E_i \quad (14)$$

$B_j$  Is the weight function depending on the numerical integration? Both  $r_i, u_j$  are equally from 0 to  $R_1$ ,  $K(r_i, u_j)$  is the discretized kernel integral equation,  $y(r_i), F(r_i)$  discretized values for the right-hand side function and auxiliary function respectively,  $E_i$  is the error resulted by replacing integral by series. In expression (14) neglecting the error  $E_i$ , then replacing the integral by a set of simultaneous algebraic equations, we have

$$y_i + \sum_{j=1}^n B_j K_{ij} y_j = F_i, \quad i = 1, \dots, n. \quad (15)$$

Where  $y_i = y(r_i)$ , the values of  $F_i = F(r_i)$ ,  $K_{ij} = K(r_i, y_j)$   $0 \leq r_i, u_j \leq R_1$

Approximation  $\tilde{y}(r)$  of the unknown function  $y(r)$  by using (14) and (15) is

$$\tilde{y}(r) = F(r) - \sum_{j=1}^n A_j K(r, r_j) y_j \quad (16)$$

Where

$$K(r, r_j) = \frac{2}{R_2^2} \sum_{n=1}^{\infty} \frac{J_{-1/2}(\lambda_n r) J_0(\lambda_n r_j)}{J_1^2(\lambda_n R_2)} g_n$$

$$K(u_i, r_j) = \frac{2}{R_2^2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n u_i) J_{-1/2}(\lambda_n r_j)}{J_1^2(\lambda_n R_2)} g_n$$

To solve equation (15), Simpson rule is used for this purpose to set up the simultaneous equations [10], the weight function  $B_j$  is

$$B_1 = B_n = \frac{R_1}{3(n-1)}$$

$$B_2 = B_4 = \dots = \frac{4R_1}{3(n-1)}$$

$$B_3 = B_5 = \dots = \frac{2R_1}{3(n-1)}$$

In view of equation (15), one expands to the linear  $n$  equations in the matrix

$$Ky = F \Rightarrow y = K^{-1}F \quad (17)$$

$$K = \begin{bmatrix} 1 + \lambda B_1 K(r_1, u_1) & B_2 K(r_1, u_2) & \dots & B_n K(r_1, u_n) \\ B_1 K(r_2, u_1) & 1 + \lambda B_2 K(r_2, u_2) & \dots & B_n K(r_2, u_n) \\ \vdots & \vdots & \ddots & \vdots \\ B_1 K(r_n, u_1) & B_1 K(r_n, u_1) & \dots & 1 + \lambda B_n K(r_n, u_n) \end{bmatrix}$$

$$y = \begin{bmatrix} y(r_1) \\ y(r_2) \\ \vdots \\ y(r_n) \end{bmatrix} \quad F = \begin{bmatrix} F(r_1) \\ F(r_2) \\ \vdots \\ F(r_n) \end{bmatrix}, \quad \lambda = 1/q$$

It is noted that for simplification the infinite series  $K(r,u)$  and free term  $F(r)$  appeared in (13) can be treated numerically by approximated  $K(r,u)$  such that

$$\tilde{K}(r,u) \approx \frac{2}{R_2^2} \sum_{n=1}^N \frac{J_0(\lambda_n u) J_{-1/2}(\lambda_n r)}{J_1^2(\lambda_n R_2)} g_n$$

Without loss of generality if we consider  $f_1(r) = \exp(-r^2), f_2(r) = 0, R_1 = 1$  in the equality

(17) Calculation will be more simple moreover, the functional series  $K(r,u)$  converges for any values  $r,u$  and  $r_1$

.Notice that as  $R_2 \rightarrow \infty$ , the dual series (5), (6) reduced to dual integral equations of the form

$$\int_0^\infty A(\lambda) J_0(\lambda r) (a_1 \sqrt{\lambda^2 + k^2} + b_1) d\lambda = f_1(r), \quad r \in (0, R_1) \quad \int_0^\infty A(\lambda) J_0(\lambda r) (a_2 \sqrt{\lambda^2 + k^2} + b_2) d\lambda = f_2(r), \quad r \in (R_1, \infty)$$

The solution of the above dual integral equations was discussed with details in [4].

## 4. Conclusion

The obtained results the conclusion can be drawn that the paper aim an analytical method the mixed boundary value problem with boundary conditions of the third kind will lead to study dual series equations. The use of a Hankel integral transform always reduces to inhomogeneous Fredholm integral equation of the second kind which is treated numerically by using simple numerical method (Simpson's rule of integration). It can be revealed that there is no approximation before the numerical evaluation of the integral equation solution. The present results can be served other investigations mixed problems, in particular for mixed boundary condition of the third kind with various applications and coordinate systems.

## References

- [1] N. Hoshan, The Dual integral equations method to solve heat conduction equation for unbounded plate, *Compl Math and Modeling*, v1, 21, no2, (2010), pp. 226-238.
- [2] N. Hoshan, Exact solution of certain dual integral equations involving heat equation. *F. East Journal of Applied Math*, V 35, issue1, Pupsha, (2009), pp.81- 88.
- [3] N. Hoshan, The dual integral equations method involving heat equation with mixed boundary conditions, *Engineering Mathematics Letters*, 2, (2013), pp. 137-142.
- [4] N. Hoshan The dual integral equations method for solving Hlmholtz mixed boundary value problem, *American Journal of computational and applied mathematics*, 3(2),(2013), pp. 138-142.
- [5] N. Hoshan, Solution of Fredholm integral equations of the first kind, *applied mathematical sciences*, vol. 7, no. 77, (2013), pp. 3847-3852.
- [6] N. Hoshan, Dual integral equations and singular integral equations for Helmholtz equation. *International Journal of Contemp Math. Sciences*, V4, no. 34, Hikari Ltd. (2009), pp. 1695-1699.
- [7] E. Kindal, P. Atkinson, *Numerical Solution of Integral Equations of the Second Kind*, Cambridge University, (1997).
- [8] B. Mandal, N.Mandal, *Advances in Dual Integral Equation*. London, CRC, (1999).
- [9] I. Sneddon, *Mixed boundary value problems*, Amesterdam, (1966).
- [10] Y. Sompornjaroensul, K. Kiattikomo, Dual series equations for static deformation of plate, *Theoret. Appl. Mech.* V34, N3, (2007), pp. 221-248. <http://dx.doi.org/10.2298/TAM0703221S>.
- [11] J. Uflyand, *Dual Equations in Mathematical Physics Equations*, L, Nauka, (1977).
- [12] A. Wazwa, *Linear and nonlinear integral equations, Methods and Applications* Beijing, Springer. (2011). <http://dx.doi.org/10.1007/978-3-642-21449-3>.