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*Research Paper*

# On the number of paths of lengths 3 and 4 in a graph

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**Abstract**

In this paper, we obtain explicit formulae for the total number of paths of lengths 3 and 4 in a simple graph G. We also determine some formulae for the number of paths of lengths 3 and 4 each of which starts from an specific vertex $v_i$ and for the number of $v_i - v_j$ paths of lengths 3 and 4 in a simple graph G, in terms of the adjacency matrix and with the helps of combinatorics.

***Keywords***: *Adjacency Matrix, Cycle, Graph Theory, Path, Subgraph, Walk .*

## 1. Introduction

In a simple graph G, a walk is a sequence of vertices and edges of the form $v_0, e_1, v_1, ..., e_k, v_k$ such that the edge $e_i$ has ends $v_{i-1}$ and $v_i$. A walk is called closed if $v_0 = v_k$. If the vertices of a walk are distinct then that walk is called a path and a cycle is a non-trivial closed path.
It is known that if a graph G has adjacency matrix A=$[a_{ij}]$, then for k = 0, 1, ... , the ij-entry of $A^k$ is the number of $v_i - v_j$ walks of length k in G. It is also known that tr($A^n$) is the sum of the diagonal entries of $A^n$ and $d_i$ is the degree of the vertex $v_i$.
In 1971, Frank Harary and Bennet Manvel [1], gave a formula for the number of triangles in simple graphs as given by the following theorem:

**Theorem 1.1** *If G is a simple graph with adjacency matrix A, then the number of 3-cycles in G is $\frac{1}{6}$ tr($A^3$). (It is known that* $tr(A^3) = \sum_{i=1}^{n} a_{ii}^{(3)} = \sum_{i \neq j} a_{ij}^{(2)} a_{ij}$*).*

They also gave formulae for the number of cycles of lengths 4 and 5 in simple graphs. Their proofs are based on the following fact:
The number of n-cycles (n= $3, 4, 5$) in a graph G is equal to $\frac{1}{2n}$(tr($A^n$)− x) where x is the number of closed walks of length n, which are not n-cycles.
In 1986, Tomescu [2], gave some formulae for the number of paths of length s having k edges in common with a fixed s-path of a complete graph. In 1994, Bax [3], gave an algorithm to count number of all paths and $v_i - v_j$ paths in a graph. His algorithm was about counting number of all paths in a graph and it can not count number of paths of an specific size.

In 1996, Eric Bax and Joel Franklin [5], gave an algorithm to count paths and cycles of a given length in a directed graph. In [4], [6], [7], [8], [10], [11] and [13], we have also some bounds to estimate the total time complexity for finding or counting paths and cycles in a graph.
In the previous works there is no formula to count the exact number of paths of an specific size in a graph.
In this paper we give some formulae to count the exact number of paths of lengths 3 and 4 in a simple graph G, in terms of the adjacency matrix of G and with the helps of combinatorics.
We state the following propositions which are useful to prove our theorems:

**Proposition 1.2** *In a simple graph G with n vertices and the adjacency matrix A= $[a_{ij}]$, the number of paths of length n is* $\sum_{i\neq j} a_{ij}^{(n)} - x$*, where x is the number of non-closed walks of length n in G, which are not paths.*

**Proposition 1.3** *In a simple graph G with n vertices and the adjacency matrix A= $[a_{ij}]$, the number of paths of length n, each of which begins with an specific vertex $v_i$ is* $\sum_{j=1, i\neq j}^{n} a_{ij}^{(n)} - x$*, where x is the number of non-closed walks of length n in G, starting from the vertex $v_i$, which are not paths.*

**Proposition 1.4** *In a simple graph G with n vertices and the adjacency matrix A= $[a_{ij}]$, the number of $v_i - v_j$ $(i \neq j)$ paths of length n is* $a_{ij}^{(n)} - x$*, where x is the number of non-closed $v_i - v_j$ walks of length n in G, which are not paths.*

## 2. Number of paths of length 2

Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$, then the following results are known by [1] :

1. The number of paths of length 2 in G is $\sum_{i\neq j} a_{ij}^{(2)}$, which is also equal to $\sum_{i\neq j} a_{ij}(d_j - 1)$.
2. The number of paths of length 2 in G, each of which starts from the vertex $v_i$ is $\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)}$, which is also equal to $\sum_{j=1, i\neq j}^{n} a_{ij}(d_j - 1)$.
3. The number of $v_i - v_j$ $(i \neq j)$ paths of length 2 in G is $a_{ij}^{(2)}$.

## 3. Number of paths of length 3

In this section, we give formulae to count the number of paths of length 3 in a simple graph G.

**Theorem 3.1** *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 3 in G is* $\sum_{i\neq j} (a_{ij}^{(3)} - (2d_j - 1)a_{ij})$.

**Proof:** By Proposition 1.2, the number of paths of length 3 in G is equal to $\sum_{i\neq j} a_{ij}^{(3)} - $ x, where x is the number of non-closed walks of length 3, that are not paths. To find x, we have 2 cases as considered below; the cases are based on the configurations-(subgraphs) that generate all non-closed walks of length 3, that are not paths. In each case, N denotes the number of non-closed walks of length 3, that are not paths in the corresponding subgraph, M denotes the number of subgraphs of G of the same configurations, F denotes the total number of non-closed walks of length 3, that are not paths in all possible subgraphs of G of the same configurations. It is clear that F is equal to N× M. To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.
**Case 1:** For the configuration of Fig 1, N= 2, M= $\frac{1}{2}\sum_{i\neq j} a_{ij}$ and F= $\sum_{i\neq j} a_{ij}$.

Fig 1

**Case 2:** For the configuration of Fig 2, N= 4, M= $\frac{1}{2}\sum_{i\neq j} a_{ij}(d_j-1)$ and F= $2\sum_{i\neq j} a_{ij}(d_j-1)$.



Fig 2

Consequently, x= $\sum_{i\neq j} a_{ij} + 2\sum_{i\neq j} a_{ij}(d_j-1)$ and by simplifying, we get the desired result. □

**Example 3.2** *In the graph of Fig 3, $\sum_{i\neq j}(a_{ij}^{(3)}-(2d_j-1)a_{ij})$= 24. So, by Theorem 3.1, the number of paths of length 3 in $K_4$ is 24.*

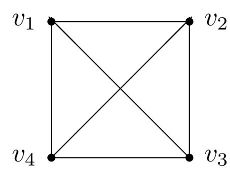


Fig 3

**Theorem 3.3** *Let G be a simple graph with n vertices and the adjacency matrix A= $[a_{ij}]$. The number of paths of length 3 in G, each of which starts from an specific vertex $v_i$ is $\sum_{j=1,i\neq j}^{n}(a_{ij}^{(3)}-(d_i+d_j-1)a_{ij})$.*

**Proof:** By Proposition 1.3, the number of paths of length 3 in G, each of which starts from an specific vertex $v_i$ is $\sum_{j=1,i\neq j}^{n} a_{ij}^{(3)}-$ x, where x is the number of non-closed walks of length 3, that are starting from $v_i$ and are not paths. To find x, we have 3 cases as considered below; the cases are based on the configurations-(subgraphs) that generate all non-closed walks of length 3, each of which starts from the specific vertex $v_i$, that are not paths. In each case, N denotes the number of non-closed walks of length 3, which start from the vertex $v_i$ and are not paths in the corresponding subgraph, M denotes the number of subgraphs of G of the same configurations, F denotes the total number of non-closed walks of length 3, which start from the vertex $v_i$ and are not paths in all possible subgraphs of G of the same configurations. It is clear that F is equal to N× M. To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

**Case 1:** For the configuration of Fig 4, N= 1, M= $\sum_{j=1,i\neq j}^{n} a_{ij}$ and F= $\sum_{j=1,i\neq j}^{n} a_{ij}$.


Fig 4

**Case 2:** For the configuration of Fig 5, N= 1, M= $\sum_{j=1,i\neq j}^{n} a_{ij}(d_j-1)$ and F= $\sum_{j=1,i\neq j}^{n} a_{ij}(d_j-1)$.


Fig 5

**Case 3:** For the configuration of Fig 6, N= 2, M= $\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ij}(d_i-1)$ and F= $\sum_{j=1,i\neq j}^{n} a_{ij}(d_i-1)$.

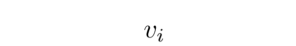


Fig 6

Consequently, x= $\sum_{j=1,i\neq j}^{n} a_{ij} + \sum_{j=1,i\neq j}^{n} a_{ij}(d_j - 1) + \sum_{j=1,i\neq j}^{n} a_{ij}(d_i - 1)$ and by simplifying, we get the desired result. □

**Example 3.4** *In the graph of Fig 3,* $\sum_{j=2}^{4}(a_{1j}^{(3)} - (d_1 + d_j - 1)a_{1j})= 6$*. So, by Theorem 3.3, the number of paths of length 3, each of which starts from the vertex* $v_1$*, is 6.*

**Theorem 3.5** *Let G be a simple graph with n vertices and the adjacency matrix A=* $[a_{ij}]$*. The number of* $v_i - v_j$ $(i \neq j)$ *paths of length* 3 *in G is* $a_{ij}^{(3)} - (d_i + d_j - 1)a_{ij}$*.*

**Proof:** By Proposition 1.4, the number of $v_i - v_j$ $(i \neq j)$ paths of length 3 in G is equal to $a_{ij}^{(3)}$− x, where x is the number of $v_i - v_j$ $(i \neq j)$ walks of length 3, that are not paths. To find x, we have 3 cases as considered below; the cases are based on the configurations-(subgraphs) that generate $v_i - v_j$ $(i \neq j)$ walks of length 3 that are not paths. In each case, N denotes the number of $v_i - v_j$ $(i \neq j)$ walks of length 3 that are not paths in the corresponding subgraph, M denotes the number of subgraphs of G of the same configurations, F denotes the total number of $v_i - v_j$ $(i \neq j)$ walks of length 3 that are not paths in all possible subgraphs of G of the same configurations. It is clear that F is equal to N× M. To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.
**Case 1:** For the configuration of Fig 7, N= 1, M= $a_{ij}$ and F= $a_{ij}$.

Fig 7

**Case 2:** For the configuration of Fig 8, N= 1, M= $a_{ij}(d_j - 1)$ and F= $a_{ij}(d_j - 1)$.

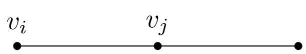

Fig 8

**Case 3:** For the configuration of Fig 9, N= 1, M= $a_{ij}(d_i - 1)$ and F= $a_{ij}(d_i - 1)$.

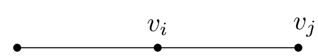

Fig 9

Consequently, x= $a_{ij} + a_{ij}(d_j - 1) + a_{ij}(d_i - 1)$ and by simplifying, we get the desired result. □

**Example 3.6** *In the graph of Fig 3, we have* $a_{12}^{(3)} - (d_1 + d_2 - 1)a_{12} = 2$*. So, by Theorem 3.5, the number of* $v_1 - v_2$ *paths of length* 3 *is 2. Indeed,* $v_1v_4v_3v_2$ *and* $v_1v_3v_4v_2$ *are the two paths.*

Now, we obtain other formulae for the number of paths of length 3.

**Theorem 3.7** *Let G be a simple graph with n vertices and the adjacency matrix A=* $[a_{ij}]$*. The number of paths of length 3 in G is* $\sum_{i\neq j} a_{ij}^{(2)}(d_j - a_{ij} - 1)$*.*

**Proof:** Any path of length 3 in G, is obtainable from a path of length 2 by adding an edge to one of it's end vertices, provided by addition of this edge no triangle is formed. So, if we use $\sum_{i\neq j} a_{ij}^{(2)}(d_j - 1)$ for the number of paths of length 3 in a graph G, it will also contain all the subgraphs of G, that are in the same configurations as the graph of Fig 10 for 6 times and the total number of subgraphs of G of the same configurations as the graph of Fig 10 is $\frac{1}{6}\sum_{i\neq j} a_{ij}^{(2)} a_{ij}$ (see Theorem 1.1). So, the number of paths of length 3 in a graph G is $\sum_{i\neq j} a_{ij}^{(2)}(d_j - 1)$− $6 \times \frac{1}{6}$ $\sum_{i\neq j} a_{ij}^{(2)} a_{ij}$ and by simplifying, we get the desired result. □

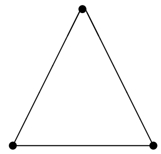

Fig 10

**Example 3.8** *In the graph of Fig 3,* $\sum_{i \neq j} a_{ij}^{(2)}(d_j - a_{ij} - 1) = 24$*. So, by Theorem 3.7, the number of paths of length 3 in* $K_4$ *is 24.*

**Theorem 3.9** *Let G be a simple graph with n vertices and the adjacency matrix* $A = [a_{ij}]$*. The number of paths of length 3 in G, each of which starts from an specific vertex* $v_i$ *is* $\sum_{j=1, i \neq j}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)$*.*

**Proof:** Any path of length 3 that starts from an specific vertex $v_i$ in G, is obtainable from a path of length 2 that starts from the vertex $v_i$ by adding an edge to it's end vertex, provided by addition of this edge no triangle is formed. So, if we use $\sum_{j=1, i \neq j}^{n} a_{ij}^{(2)}(d_j - 1)$ for the number of paths of length 3 that are starting from the vertex $v_i$ in a graph G, it will also contain all the subgraphs of G, that are in the same configurations as the graph of Fig 11 for 2 times and the total number of subgraphs of G of the same configurations as the graph of Fig 11 is $\frac{1}{2} \sum_{j=1, i \neq j} a_{ij}^{(2)} a_{ij}$ (see Theorem 1.1). So, the number of paths of length 3 in a graph G which start from the vertex $v_i$ is $\sum_{j=1, i \neq j} a_{ij}^{(2)}(d_j - 1) - 2 \times \frac{1}{2} \sum_{j=1, i \neq j} a_{ij}^{(2)} a_{ij}$ and by simplifying, we get the desired result. □

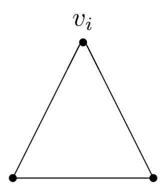


Fig 11

**Example 3.10** *In the graph of Fig 3,* $\sum_{j=2}^{4} a_{1j}^{(2)}(d_j - a_{1j} - 1) = 6$*. So, by Theorem 3.9, the number of paths of length 3, each of which starts from the vertex* $v_1$*, is 6.*

**Theorem 3.11** *Let G be a simple graph with n vertices and the adjacency matrix* $A = [a_{ij}]$*. The number of* $v_i - v_j$ $(i \neq j)$ *paths of length 3 in G is* $\sum_{k=1, k \neq i,j}^{n} (a_{ik}^{(2)} - a_{ij}) a_{jk}$*.*

**Proof:** Any $v_i - v_j$ $(i \neq j)$ path of length 3 in G, is obtainable from a $v_i - v_k$ $(i \neq k$ and $k = 1, 2, ..., n)$ path of length 2 and a $v_k - v_j$ $(k \neq j)$ path of length one. So, if we use $\sum_{k=1, k \neq i,j}^{n} a_{ik}^{(2)} a_{jk}$ for the number of $v_i - v_j$ $(i \neq j)$ paths of length 3 in a graph G, it will also contain all the subgraphs of G that are in the same configurations as the graph of Fig 12 for 1 time and Fig 12 is not the configuration of a path of length 3, that we don't want. The total number of subgraphs of G of the same configurations as the graph of Fig 12 is $\sum_{k=1, k \neq i,j}^{n} a_{ij} a_{jk}$. So, the number of $v_i - v_j$ paths of length 3 in a graph G is $\sum_{k=1, k \neq i,j}^{n} a_{ik}^{(2)} a_{jk} - \sum_{k=1, k \neq i,j}^{n} a_{ij} a_{jk}$ and by simplifying, we get the desired result. □

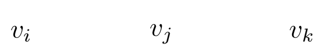


Fig 12

**Example 3.12** *In the graph of Fig 3,* $\sum_{k=3}^{4}(a_{1k}^{(2)} - a_{12})a_{2k} = 2$*. So, by Theorem 3.11, the number of* $v_1 - v_2$ *paths of length* 3 *is 2. Indeed,* $v_1v_4v_3v_2$ *and* $v_1v_3v_4v_2$ *are the two paths.*

# 4. Number of paths of length 4

In this section, we give formulae to count the number of paths of length 4 in a simple graph G.

**Theorem 4.1** *Let G be a simple graph with n vertices and the adjacency matrix* $A = [a_{ij}]$*. The number of paths of length 4 in G is* $\sum_{i\neq j}[a_{ij}^{(4)} - 2a_{ij}^{(2)}(d_j - a_{ij})] - \sum_{i=1}^{n}[(2d_i - 1)a_{ii}^{(3)} + 6\binom{d_i}{3}]$.

**Proof:** By Proposition 1.2, the number of paths of length 4 in a graph G is equal to $\sum_{i\neq j}a_{ij}^{(4)} -$ x, where x is the number of non-closed walks of length 4, that are not paths. To find x, we have 5 cases as considered below; the cases are based on the configurations-(subgraphs) that generate all non-closed walks of length 4, that are not paths. In each case, N denotes the number of non-closed walks of length 4, that are not paths in the corresponding subgraph, M denotes the number of subgraphs of G of the same configurations and F denotes the total number of non-closed walks of length 4, that are not paths in all possible subgraphs of G of the same configurations. It is clear that F is equal to N× M. To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

**Case 1:** For the configuration of Fig 13, N= 4, M= $\frac{1}{2}\sum_{i\neq j}a_{ij}^{(2)}$ and F= $2\sum_{i\neq j}a_{ij}^{(2)}$.



Fig 13

**Case 2:** For the configuration of Fig 14, N= 4, M= $\frac{1}{2}\sum_{i\neq j}a_{ij}^{(2)}(d_j - a_{ij} - 1)$ and F= $2\sum_{i\neq j}a_{ij}^{(2)}(d_j - a_{ij} - 1)$ (see Theorem 3.7).



Fig 14

**Case 3:** For the configuration of Fig 15, N= 6, M=$\sum_{i=1}^{n}\binom{d_i}{3}$ and F= $6\sum_{i=1}^{n}\binom{d_i}{3}$.

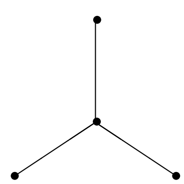

Fig 15

**Case 4:** For the configuration of Fig 16, N= 4, M= $\frac{1}{2}\sum_{i=1}^{n}a_{ii}^{(3)}(d_i - 2)$ and F= $2\sum_{i=1}^{n}a_{ii}^{(3)}(d_i - 2)$ (see Theorem 1.1).

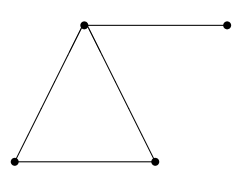

Fig 16

**Case 5:** For the configuration of Fig 17, N= 18, M= $\frac{1}{6}\sum_{i=1}^{n} a_{ii}^{(3)}$ and F= $3\sum_{i=1}^{n} a_{ii}^{(3)}$ (see Theorem 1.1).

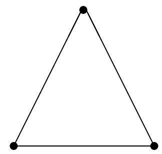

Fig 17

Now we add the values of F arising from the above cases and determine x. By putting the value of x in $\sum_{i\neq j} a_{ij}^{(4)} -$ x and simplifying, we get the desired result. □

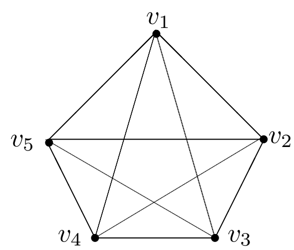


Fig 18

**Example 4.2** *In the graph of Fig 18,* $\sum_{i\neq j}[a_{ij}^{(4)} - 2a_{ij}^{(2)}(d_j - a_{ij})]$*= 660 and* $\sum_{i=1}^{5}[(2d_i - 1)a_{ii}^{(3)} + 6\binom{d_i}{3}]$*= 540. So, by Theorem 4.1, the number of paths of length 4 is 120.*

**Theorem 4.3** *Let G be a simple graph with n vertices and the adjacency matrix A=* $[a_{ij}]$*. The number of paths of length 4 in G, each of which starts from an specific vertex* $v_i$ *is* $\sum_{j=1, i\neq j}^{n} [a_{ij}^{(4)} - (d_i + d_j - 3a_{ij})a_{ij}^{(2)} - (a_{ii}^{(3)} + a_{jj}^{(3)} + 2\binom{d_j - 1}{2})a_{ij}]$.

**Proof:** By Proposition 1.3, the number of paths of length 4 in a graph G, each of which starts from an specific vertex $v_i$ is $\sum_{j=1, i\neq j}^{n} a_{ij}^{(4)} -$ x, where x is the number of non-closed walks of length 4, that begin from $v_i$ and are not paths. To find x, we have 7 cases as considered below; the cases are based on the configurations-(subgraphs) that generate all non-closed walks of length 4, each of which starts from the specific vertex $v_i$, that are not paths. In each case, N denotes the number of non-closed walks of length 4, which start from the vertex $v_i$ and are not paths in the corresponding subgraph, M denotes the number of subgraphs of G of the same configurations, F denotes the total number of non-closed walks of length 4, which start from the vertex $v_i$ and are not paths in all possible subgraphs of G of the same configurations. However, in the cases with more than one figure (cases 3, 6, 7), N, M and F are based on the first graph of the respective figures and P denotes the number of subgraphs of G which don't have the same configurations as the first graph but are counted in M. It is clear that F is equal to N × (M−P). To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.

**Case 1:** For the configuration of Fig 19, N= 2, M= $\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)}$ and F= $2\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)}$.

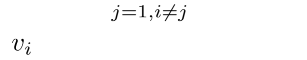


Fig 19

**Case 2:** For the configuration of Fig 20, N= 1, M= $\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)$ and F= $\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)}(d_j - a_{ij} - 1)$ (see Theorem 3.9).

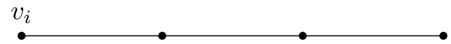


Fig 20

**Case 3:** For the configuration of Fig 21(a), N= 1, M= $\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}(d_i-1)$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 21(b) and are counted in M. Thus P= $2\times\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}a_{ij}$, where $\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 21(b) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}(d_i-a_{ij}-1)$.

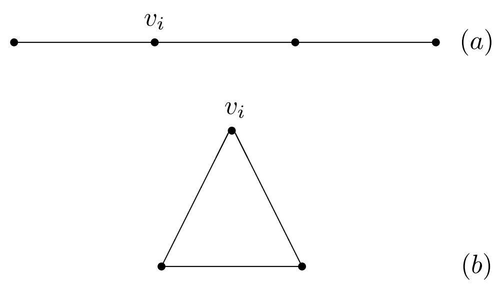


Fig 21

**Case 4:** For the configuration of Fig 22, N= 2, M= $\sum_{j=1,i\neq j}^{n}\binom{d_j-1}{2}a_{ij}$ and F= $2\sum_{j=1,i\neq j}^{n}\binom{d_j-1}{2}a_{ij}$.

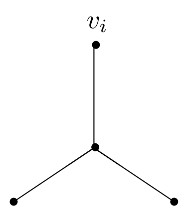


Fig 22

**Case 5:** For the configuration of Fig 23, N= 6, M= $\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}a_{ij}$ and F= $3\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}a_{ij}$.

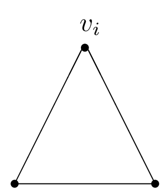


Fig 23

**Case 6:** For the configuration of Fig 24($a$), N= 2, M= $\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ii}^{(3)}a_{ij}$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 24(b) and are counted in M. Thus P= $2\times\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}a_{ij}$, where $\frac{1}{2}\sum_{j=1,i\neq j}^{n} a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 24(b) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{j=1, i\neq j}^{n} a_{ii}^{(3)} a_{ij} - 2\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)} a_{ij}$.

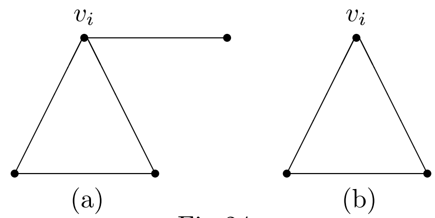

Fig 24

**Case 7:** For the configuration of Fig 25($a$), N= 2, M= $\frac{1}{2}\sum_{j=1, i\neq j}^{n} a_{jj}^{(3)} a_{ij}$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 25(b) and are counted in M. Thus P= $2\times\frac{1}{2}\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)} a_{ij}$, where $\frac{1}{2}\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)} a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 25(b) and 2 is the number of times that this subgraph is counted in M. Consequently, F= $\sum_{j=1, i\neq j}^{n} a_{jj}^{(3)} a_{ij} - 2\sum_{j=1, i\neq j}^{n} a_{ij}^{(2)} a_{ij}$.

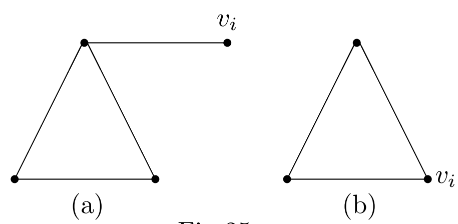

Fig 25

Now we add the values of F arising from the above cases and determine x. By putting the value of x in $\sum_{j=1, i\neq j}^{n} a_{ij}^{(4)}$−x and simplifying, we get the desired result. □

**Example 4.4** *In the graph of Fig 18,* $\sum_{j=2}^{5}[a_{1j}^{(4)} - (d_1 + d_j - 3a_{1j})a_{1j}^{(2)} - (a_{11}^{(3)} + a_{jj}^{(3)} + 2\binom{d_j - 1}{2})a_{1j}] = 24$*. So, by Theorem 4.3, the number of paths of length 4 starting from the vertex* $v_1$ *is 24.*

**Theorem 4.5** *Let G be a simple graph with n vertices and the adjacency matrix A= [*$a_{ij}$*]. The number of* $v_i - v_j$ *paths of length 4 in G is* $a_{ij}^{(4)} - (d_i + d_j - 3a_{ij})a_{ij}^{(2)} - (a_{ii}^{(3)} + a_{jj}^{(3)})a_{ij} - \sum_{k=1, k\neq i,j}^{n} a_{ik}a_{kj}(d_k - 2)$.

**Proof:** By Proposition 1.6, the number of $v_i - v_j$ $(i \neq j)$ paths of length 4 in a graph G is $a_{ij}^{(4)}$− x, where x is the number of $v_i - v_j$ $(i \neq j)$ walks of length 4 , that are not paths. To find x, we have 7 cases as considered below; the cases are based on the configurations-(subgraphs) that generate $v_i - v_j$ $(i \neq j)$ walks of length 4, that are not paths. In each case, N denotes the number of $v_i - v_j$ $(i \neq j)$ walks of length 4 that are not paths in the corresponding subgraph, M denotes the number of subgraphs of G of the same configurations, F denotes the total number of $v_i - v_j$ $(i \neq j)$ walks of length 4 that are not paths in all possible subgraphs of G of the same configurations. However, in the cases with more than one figure (cases 2, 3, 6, 7), N, M and F are based on the first graph of the respective figures and P denotes the number of subgraphs of G which don't have the same configurations as the first graph but are counted in M. It is clear that F is equal to N × (M−P). To find N in each case, we have to include in any walk, all the edges and the vertices of the corresponding subgraphs at least once.
**Case 1:** For the configuration of Fig 26, N= 2, M= $a_{ij}^{(2)}$ and F= $2a_{ij}^{(2)}$.

Fig 26

**Case 2:** For the configuration of Fig 27(a), N= 1, M= $a_{ij}^{(2)}(d_j - 1)$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 27(b) and are counted in M. Thus P= $1 \times a_{ij}^{(2)} a_{ij}$, where $a_{ij}^{(2)} a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 27(b) and 1 is the number of times that this subgraph is counted in M. Consequently, F= $a_{ij}^{(2)}(d_j - a_{ij} - 1)$.

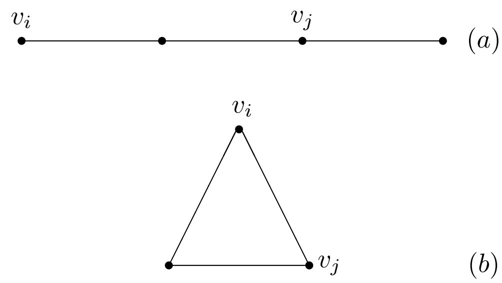


Fig 27

**Case 3:** For the configuration of Fig 28(a), N= 1, M= $a_{ij}^{(2)}(d_i - 1)$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 28(b) and are counted in M. Thus P= $1 \times a_{ij}^{(2)} a_{ij}$, where $a_{ij}^{(2)} a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 28(b) and 1 is the number of times that this subgraph is counted in M. Consequently, F= $a_{ij}^{(2)}(d_i - a_{ij} - 1)$.

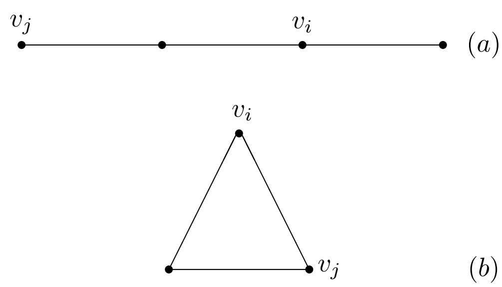


Fig 28

**Case 4:** For the configuration of Fig 29, N= 1, M= $\sum_{k=1, k\neq i,j}^{n} a_{ik} a_{kj}(d_k - 2)$ and F= $\sum_{k=1, k\neq i,j}^{n} a_{ik} a_{kj}(d_k - 2)$.

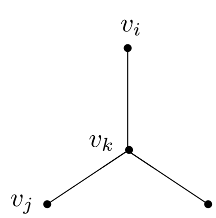


Fig 29

**Case 5:** For the configuration of Fig 30, N= 3, M= $a_{ij}^{(2)} a_{ij}$ and F= $3a_{ij}^{(2)} a_{ij}$.

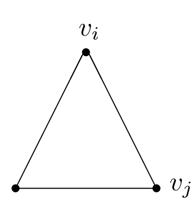


Fig 30

**Case 6:** For the configuration of Fig 31($a$), N= 2, M= $\frac{1}{2}a_{ii}^{(3)}a_{ij}$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 31(b) and are counted in M. Thus P= $1 \times a_{ij}^{(2)}a_{ij}$, where $a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 31(b) and 1 is the number of times that this subgraph is counted in M. Consequently, F= $a_{ii}^{(3)}a_{ij} - 2a_{ij}^{(2)}a_{ij}$.

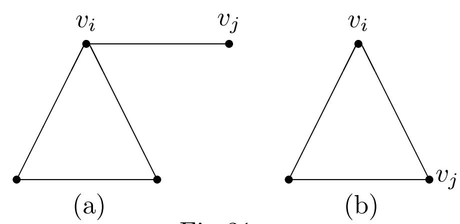

Fig 31

**Case 7:** For the configuration of Fig 32($a$), N= 2, M= $\frac{1}{2}a_{jj}^{(3)}a_{ij}$. Let P denotes the number of all subgraphs of G that have the same configurations as the graph of Fig 32(b) and are counted in M. Thus P= $1 \times a_{ij}^{(2)}a_{ij}$, where $a_{ij}^{(2)}a_{ij}$ is the number of subgraphs of G that have the same configurations as the graph of Fig 32(b) and 1 is the number of times that this subgraph is counted in M. Consequently, F= $a_{jj}^{(3)}a_{ij} - 2a_{ij}^{(2)}a_{ij}$.

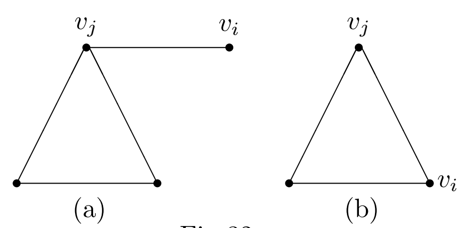

Fig 32

Now we add the values of F arising from the above cases and determine x. By putting the value of x in $a_{ij}^{(4)}-$ x and simplifying, we get the desired result. □

**Example 4.6:** In the graph of Fig 18, $a_{12}^{(4)}$= 51, $(d_1 + d_2 - 3a_{12})a_{12}^{(2)}$= 15, $(a_{11}^{(3)} + a_{22}^{(3)})a_{12}$= 24, $\sum_{k=3}^{5} a_{1k}a_{k2}(d_k - 2)$= 6. So, by Theorem 4.5, the number of $v_1 - v_2$ paths of length 4 is 6.

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