

Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers

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Abstract

In this paper, we present identities involving common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers. Binet's formula will employ to obtain the identities.

Keywords: *Generalized Fibonacci numbers, Jacobsthal numbers, Binet's formula.*

1 Introduction

It is well-known that the Fibonacci sequence is most prominent examples of recursive sequence. The Fibonacci sequence is famous for possessing wonderful and amazing properties. Fibonacci numbers are a popular topic for mathematical enrichment and popularization. The Fibonacci appear in numerous mathematical problems. The Fibonacci numbers F_n are terms of the sequence $\{0, 1, 1, 2, 3, 5, \dots\}$ wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

2 Generalized Fibonacci Sequences

Generalized Fibonacci sequence [8], is defined as

$$F_k = pF_{k-1} + qF_{k-2}, \quad k \geq 2 \text{ with } F_0 = a, F_1 = b, \quad (2.1)$$

where p, q, a & b are positive integers.

For different values of p, q, a & b many sequences can be determined.

We focus two cases of sequences $\{V_k\}_{k \geq 0}$ and $\{U_k\}_{k \geq 0}$ which generated in (2.1).

If $p=1, q=a=b=2$, we get

$$V_k = V_{k-1} + 2V_{k-2} \quad \text{for } k \geq 2 \text{ with } V_0 = 2, V_1 = 2 \quad (2.2)$$

The first few terms of $\{V_k\}_{k \geq 0}$ are 2, 2, 6, 10, 22, 42 and so on.

Its Binet forms is defined by

$$V_k = 2 \frac{\mathfrak{R}_1^{k+1} - \mathfrak{R}_2^{k+1}}{\mathfrak{R}_1 - \mathfrak{R}_2} \quad (2.3)$$

If $p=1, q=a=2, b=0$ we get

$$U_k = U_{k-1} + 2U_{k-2} \quad \text{for } k \geq 2 \text{ with } U_0 = 2, U_1 = 0 \quad (2.4)$$

The first few terms of $\{U_k\}_{k \geq 0}$ are 2, 0, 4, 4, 12, 20 and so on.

Its Binet forms is defined

$$U_k = 4 \frac{\mathfrak{R}_1^{k-1} - \mathfrak{R}_2^{k-1}}{\mathfrak{R}_1 - \mathfrak{R}_2} \quad (2.5)$$

The Jacobsthal sequence [1], is defined by the recurrence relation

$$J_k = J_{k-1} + 2j_{k-2}, \quad k \geq 2 \text{ with } J_0 = 0, J_1 = 1 \quad (2.6)$$

Its Binet's formula is defined by

$$J_k = \frac{\mathfrak{R}_1^k - \mathfrak{R}_2^k}{\mathfrak{R}_1 - \mathfrak{R}_2} \quad (2.7)$$

The Jacobsthal-Lucas sequence [1], is defined by the recurrence relation

$$j_k = j_{k-1} + 2j_{k-2}, \quad k \geq 2 \text{ with } j_0 = 2, j_1 = 1 \quad (2.8)$$

Its Binet's formula is defined by

$$j_k = \mathfrak{R}_1^k + \mathfrak{R}_2^k \quad (2.9)$$

where $\mathfrak{R}_1$ & $\mathfrak{R}_2$ are the roots of the characteristic equation $x^2 - x - 2 = 0$.

3 Identities for The Common Factors of Generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas Numbers

There are a lot of identities of Fibonacci and Lucas numbers described in [3]. M. Thongmoon [5], defined various identities of Fibonacci and Lucas numbers. B. Singh, P. Bhadouria and O. Sikhwal [7], present some generalized identities involving common factors of Fibonacci and Lucas numbers. In [8], V. K. Gupta, Y. K. Panwar and O. Sikhwal have defined generalized Fibonacci sequences. In this paper, we present identities of common factors of generalized Fibonacci, Jacobsthal and Jacobsthal-Lucas numbers.

Theorem 3.1 *If V_k and U_k are the generalized Fibonacci numbers and j_k is Jacobsthal-Lucas numbers, then*

$$(i) \quad V_{2k+1}j_{2k+1} = V_{4k+2} - 4^{k+1} \quad (1)$$

$$(ii) \quad V_{2k+2}j_{2k+1} = V_{4k+3} - 4^{k+1} \quad (2)$$

$$(iii) \quad U_{2k+1}j_{2k+1} = U_{4k+4} - 4^{k+1} \quad (3)$$

$$(iv) \quad U_{2k+2}j_{2k+1} = U_{4k+1} \quad (4)$$

Proof (i).

$$\begin{aligned} V_{2k+1}j_{2k+1} &= 2 \left(\frac{\mathfrak{R}_1^{2k+2} - \mathfrak{R}_2^{2k+2}}{\mathfrak{R}_1 - \mathfrak{R}_2} \right) (\mathfrak{R}_1^{2k+1} + \mathfrak{R}_2^{2k+1}) \\ &= 2 \left(\frac{\mathfrak{R}_1^{4k+3} - \mathfrak{R}_2^{4k+3}}{\mathfrak{R}_1 - \mathfrak{R}_2} \right) + \frac{2}{(\mathfrak{R}_1 - \mathfrak{R}_2)} (\mathfrak{R}_1 \mathfrak{R}_2)^{2k+1} (\mathfrak{R}_1 - \mathfrak{R}_2) \\ &= V_{4k+2} - 2(\mathfrak{R}_1 \mathfrak{R}_2)^{2k+1} \\ &= V_{4k+2} - 4^{k+1} \end{aligned}$$

This completes the proof.

Proof (ii). It can be proved same as **Theorem1: (i)**

Proof (iii).

$$\begin{aligned}
 U_{2k+1}j_{2k+1} &= 4 \left(\frac{\mathfrak{R}_1^{2k} - \mathfrak{R}_2^{2k}}{\mathfrak{R}_1 - \mathfrak{R}_2} \right) (\mathfrak{R}_1^{2k+1} + \mathfrak{R}_2^{2k+1}) \\
 &= 4 \left(\frac{\mathfrak{R}_1^{4k+3} - \mathfrak{R}_2^{4k+3}}{\mathfrak{R}_1 - \mathfrak{R}_2} \right) + \frac{4}{(\mathfrak{R}_1 - \mathfrak{R}_2)} (\mathfrak{R}_1 \mathfrak{R}_2)^{2k} (\mathfrak{R}_2 - \mathfrak{R}_1) \\
 &= U_{4k+4} - 4(\mathfrak{R}_1 \mathfrak{R}_2)^{2k} \\
 &= U_{4k+4} - 4^{k+1}
 \end{aligned}$$

This completes the proof.

Proof (iv). It can be proved same as **Theorem1: (iii)**

Corollary 3.2:

$$(i) \quad V_{2k+1}j_{2k+1} = 2J_{4k+3} - 4^{k+1} \quad (5)$$

$$(ii) \quad V_{2k+2}j_{2k+1} = 2J_{4k+4} - 4^{k+1} \quad (6)$$

$$(iii) \quad U_{2k+1}j_{2k+1} = 4(J_{4k+3} - 4^k) \quad (7)$$

$$(iv) \quad U_{2k+2}j_{2k+1} = 4J_{4k} \quad (8)$$

Following theorems can be solved by Binet's formulae (2.3), (2.5), (2.7) and (2.9)

Theorem 3.3: If V_k & U_k are the generalized Fibonacci numbers and J_k & j_k are Jacobsthal and jacobsthal-Lucas numbers, then

$$(i) \quad V_{2k+1}j_{2k+2} = 2J_{4k+4} \quad (9)$$

$$(ii) \quad V_{2k+1}j_{2k} = 2(J_{4k+2} + 4^k) \quad (10)$$

$$(iii) \quad U_{2k+1}j_{2k} = 4J_{4k} \quad (11)$$

$$(iv) \quad U_{2k+1}j_{2k+2} = 4(J_{4k+2} - 4^k) \quad (12)$$

Corollary 3.4:

$$(i) \quad V_{2k+1}j_{2k+2} = V_{4k+2} \quad (13)$$

$$(ii) \quad V_{2k+2}j_{2k} = V_{4k+1} + 2^{2k+1} \quad (14)$$

$$(iii) \quad U_{2k+1}j_{2k} = U_{4k+1} \quad (15)$$

$$(iv) \quad U_{2k+2}j_{2k+2} = U_{4k+3} - 4^{k+1} \quad (16)$$

Theorem 3.5:

$$(i) \quad V_{2k+2} j_{2k} = V_{4k+2} + 3(2^{2k+1}) \quad (17)$$

$$(ii) \quad V_{2k} j_{2k} = V_{4k} + 2^{2k+1} \quad (18)$$

$$(iii) \quad U_{2k+2} j_{2k} = U_{4k+2} + 4^{k+1} \quad (19)$$

$$(iv) \quad U_{2k} j_{2k} = U_{4k} + 2^{2k+1} \quad (20)$$

Corollary 3.6:

$$(i) \quad V_{2k+2} j_{2k} = 2(J_{4k+3} + 3(4^k)) \quad (21)$$

$$(ii) \quad V_{2k} j_{2k} = 2(J_{4k+1} - 4^k) \quad (22)$$

$$(iii) \quad U_{2k+2} j_{2k} = 4(J_{4k+1} + 4^k) \quad (23)$$

$$(iv) \quad U_{2k} j_{2k} = 2(2J_{4k-1} + 4^k) \quad (24)$$

Theorem 3.7:

$$(i) \quad V_{2k-1} j_{2k+1} = V_{4k} - 2^{2k+1} \quad (25)$$

$$(ii) \quad V_{2k-1} j_{2k-1} = V_{4k-2} - 4^k \quad (26)$$

$$(iii) \quad U_{2k-1} j_{2k+1} = U_{4k} - 3(4^k) \quad (27)$$

$$(iv) \quad U_{2k-1} j_{2k-1} = U_{4k-2} - 4^k \quad (28)$$

Corollary 3.6:

$$(i) \quad V_{2k-1} j_{2k+1} = 2(J_{4k+1} - 4^k) \quad (29)$$

$$(ii) \quad V_{2k-1} j_{2k-1} = 2J_{4k-1} - 4^k \quad (30)$$

$$(iii) \quad U_{2k-1} j_{2k+1} = 4J_{4k-1} - 3(4^k) \quad (31)$$

$$(iv) \quad U_{2k-1} j_{2k-1} = 4J_{4k-3} - 4^k \quad (32)$$

Theorem 3.8:

$$(i) \quad V_{2k} j_{2k+1} = V_{4k+1} \quad (33)$$

$$(ii) \quad V_{2k-1} j_{2k} = V_{4k-1} \quad (34)$$

$$(iii) \quad U_{2k} j_{2k+1} = U_{4k+1} + 2^{2k+1} \quad (35)$$

$$(iv) \quad U_{2k-1} j_{2k} = U_{4k-1} - 4^k \quad (36)$$

Corollary 3.6:

$$(i) \quad V_{2k} j_{2k+1} = 2J_{4k+2} \quad (37)$$

$$(ii) \quad V_{2k-1} j_{2k} = 2J_{4k} \quad (38)$$

$$(iii) \quad U_{2k} j_{2k+1} = 2(2J_{4k} + 4^k) \quad (39)$$

$$(iv) \quad U_{2k-1} j_{2k} = 4J_{4k-2} - 4^k \quad (40)$$

4 Conclusion

In this paper we have stated and derived many identities of common factors of generalized Fibonacci and Jacobsthal and Jacobsthal-Lucas numbers with the help of their Binet's formula. The concept can be executed for generalized Fibonacci sequences as well as polynomials.

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