International Journal of Applied Mathematical Research, 3 (2) (2014) 93-96
©Science Publishing Corporation
www.sciencepubco.com/index.php/IJAMR
doi: 10.14419/ijamr.v3i2.1936
Research Paper

On the Mazur-Ulam problem in fuzzy anti-normed spaces

Majid Abrishami-Moghaddam

Department of Mathematics, Birjand Branch, Islamic Azad University, Birjand, Iran
E-mail: m.abrishami.m@gmail.com

Abstract

The aim of this article is to proved a Mazur-Ulam type theorem in the strictly convex fuzzy anti-normed spaces.

Keywords: *Fuzzy anti-normed space, Mazur-Ulam theorem, strictly convex.*

1. Introduction and preliminaries

The theory of fuzzy sets was introduced by L. Zadeh [11] in 1965 and thereafter several authors applied it different branches of pure and applied mathematics. Many mathematicians considered the fuzzy normed spaces in several angels (see [1], [4], [10]). In [6] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [2] and investigated their important properties. In 1932, the theory of isometric mappings was originated in the classical paper [8] by Mazur and Ulam. They proved that every isometry f of a normed real vector space X onto another normed real vector space X is a linear mapping up to translation, that is, $x \mapsto f(x) - f(0)$ is linear, which amounts to the definition that f is affine. We call this the Mazur-Ulam theorem. The property is not true for normed complex vector spaces. The hypothesis of surjectivity is essential. Without this assumption, Baker [3] proved that every isometry from a normed real space into a strictly convex normed real space is linear up to translation. A number of mathematicians have had deal with the Mazur-Ulam theorem; see [7, 9] and references therein. In this paper, we prove that the Mazur-Ulam theorem holds under some conditions in the fuzzy anti-normed spaces. We establish a Mazur-Ulam type theorem in the framework of strictly convex normed spaces by using some ideas of [5]. Now we recall some notations and definitions used in this paper.

Definition 1.1 *[6] Let X be a linear space over a real field F. A fuzzy subset N of $X \times \mathcal{R}$ is called a fuzzy anti-norm on X if the following conditions are satisfied for all $x, y \in X$*
$(a-N_1)$ *For all $t \in \mathcal{R}$ with $t \leq 0, N(x,t) = 1$,*
$(a-N_2)$ *For all $t \in \mathcal{R}$ with $t > 0, N(x,t) = 0$ if and only if $x = \bar{0}$,*
$(a-N_3)$ *For all $t \in \mathcal{R}$ with $t > 0, N(\alpha x, t) = N(x, t/|\alpha|)$, for all $\alpha \neq 0, \alpha \in F$,*
$(a-N_4)$ *For all $s, t \in \mathcal{R}, N(x+y, t+s) \leq \max\{N(x,s), N(y,t)\}$,*
$(a-N_5)$ *$N(x,t)$ is a non-increasing function of $t \in \mathcal{R}$ and $\lim_{t\to\infty} N(x,t) = 0$.*
Then the pair (X, N) is called a fuzzy anti-normed linear space.

Example 1.2 *Let* $(X, \|.\|)$ *be a normed space. If for all* $k, m, n \in \mathcal{R}^+$ *we define*

$$\mathcal{N}(x,t) = \begin{cases} \frac{m\|x\|}{kt^n + m\|x\|} & if \quad t > 0 \\ 1 & if \quad t \leq 0. \end{cases}$$

In particular if $k = m = n = 1$ *we have*

$$\mathcal{N}(x,t) = \begin{cases} \frac{\|x\|}{t + \|x\|} & if \quad t > 0 \\ 1 & if \quad t \leq 0. \end{cases}$$

which is called the standard fuzzy anti-norm induced by the norm $\|.\|$.

Definition 1.3 *A fuzzy anti-normed space* X *is called strictly convex if* $N(x + y, s + t) = \max\{N(x, s), N(y, t)\}$ *and* $N(x, s) = N(y, t)$ *implies that* $x = y$ *and* $s = t$.

Definition 1.4 *Let* (X, N) *and* (Y, N) *be two fuzzy anti-normed spaces. We call that* $f : (X, N) \to (Y, N)$ *is a fuzzy isometry if* $N(x - y, s) = N(f(x) - f(y), s)$ *for all* $x, y \in X$ *and* $s > 0$.

Definition 1.5 *Let* X *be a real linear space and* x, y, z *mutually disjoint elements of* X. *Then* x, y *and* z *are said to be collinear if* $y - z = k(x - z)$ *for some real number* k.

2. Main results

In this section we will prove that the MazurUlam theorem under some conditions in the fuzzy real anti-normed strictly convex spaces. First, we prove the following lemma that is require for the main theorem of our paper.

Lemma 2.1 *Let* X *be a fuzzy anti-normed space which is strictly convex and let* $y, z \in X$ *and* $s > 0$. *Then* $x = \frac{y+z}{2}$ *is unique element of* X *such that*

$$N(y - x, s) = N(y - z, 2s)$$

and

$$N(z - x, s) = N(y - z, 2s).$$

Proof. There is nothing to prove if $y = z$. Let $y \neq z$. Then by $(a - N_3)$, we have

$$N(y - x, s) = N(y - \frac{y+z}{2}, s) = N(y - z, 2s)$$

and

$$N(z - x, s) = N(z - \frac{y+z}{2}, s) = N(y - z, 2s),$$

that is the existence holds. For the uniqueness, we may assume that u and v are two elements of X such that

$$N(y - u, s) = N(y - v, s) = N(z - u, s) = N(z - v, s) = N(y - z, 2s).$$

Then

$$\begin{aligned} N(y - \frac{u+v}{2}, s) &\leq \max\{N(y - u, s), N(y - v, s)\} \\ &= N(y - z, 2s) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} N(z - \frac{u+v}{2}, s) &\leq \max\{N(z - u, s), N(z - v, s)\} \\ &= N(y - z, 2s). \end{aligned} \tag{2.2}$$

If both of inequalities (2.1) and (2.2) were strict we would have

$$\begin{aligned} N(y-z,2s) &= N(y-\frac{u+v}{2}+\frac{u+v}{2}-z,2s) \\ &\leq \max\{N(y-\frac{u+v}{2}),N(z-\frac{u+v}{2},s)\} \\ &< N(y-z,2s), \end{aligned}$$

which is a contradiction. So at least one of the equalities holds in (2.1) and (2.2). Without lose of generality assume that equality holds in (2.1). Then

$$N(y-\frac{u+v}{2},s)=\max\{N(y-u,s),N(y-v,s)\}.$$

The strict convexity of X implies that, $N(y-u,s)=N(y-v,s)$, and so, $u=v$. Therefore the proof is completed.

Theorem 2.2 *Let X and Y be real fuzzy anti-normed spaces and let Y be strictly convex. Suppose $f:X\to Y$ be a fuzzy isometry satisfies $f(x),f(y)$ and $f(z)$ are collinear when x,y and z are collinear. Then f is affine.*

Proof. Let $g(x):=f(x)-f(0)$. Then g is fuzzy isometry and $g(0)=0$. It is easy to check that if x,y and z are collinear, then $g(x),g(y)$ and $g(z)$ are also collinear. So it suffices to show that g is linear. We have

$$N(g(\frac{y+z}{2})-g(y),s)=N((\frac{y+z}{2})-y,s)=N(y-z,2s)$$

and similarly

$$N(g(\frac{y+z}{2})-g(z),s)=N((\frac{y+z}{2})-z,s)=N(y-z,2s)$$

for all $y,z\in X$ and $s>0$. By lemma (2.1) we have

$$g(\frac{y+z}{2})=\frac{1}{2}g(y)+\frac{1}{2}g(z).$$

Since $g(0)=0$, we can easily show that g is additive. It follows that g is $\mathcal{Q}$-linear. We have to show that g is $\mathcal{R}-$linear.
Let $r\in\mathcal{R}^+$ with $r\neq 1$ and $y\in X$. Since $0,y$ and ry are collinear $g(0),g(y)$ and $g(r)y$ are also collinear. Since $g(0)=0$, there exists $r'\in\mathcal{R}$ such that $g(ry)=r'g(y)$. Now, we will proved that $r=r'$. Since y and z are collinear, then $y\neq z$. Hence,

$$\begin{aligned} N(r(y-z),s) &= N(g(ry)-g(rz),s) \\ &= N(g(r'y)-g(r'z),s) \\ &= N(r'(g(y)-g(z)),s) \\ &= N(r'(y-z),s). \end{aligned}$$

By the strict convexity we obtain $r(y-z)=r'(y-z)$. Thus $g(ry)=rg(y)$ for all $y\in X$ and all $r\in\mathcal{R}$. Therefore g is affine and the proof is complete.

References

[1] C. Felbin, "Finite dimensional fuzzy normed linear spaces", *Fuzzy Sets and Systems*, Vol.48, No.2, (1992), pp.239-248.

[2] T. Bag, S.K. Samanta, "A comparative study of fuzzy norms on a linear space", *Fuzzy Sets and Systems*, Vol.159, No.6, (2008), 670-684.

[3] J.A. Baker, "Isometries in normed spaces", *Amer. Math. Monthly*, Vol.78, No.6, (1971), 655-658.

[4] S.C. Cheng, J.N. Mordeson, "Fuzzy linear operator and fuzzy normed linear spaces", *Bull. Calcutta Math. Soc.* Vol.86, (1994), 429-436.

[5] M. Eshaghi Gordji, N. Ghobadipour, "On the Mazur-Ulam theorem in fuzzy normed spaces", arXiv:0905.2166 (2009).

[6] Iqbal H. Jebril, T.K. Samanta, "Fuzzy anti-normed linear space", *J. math. & Tech.*, February (2010), 66-77.

[7] D. Kangb,., H. Kohb, I. G. Choa, "On the Mazur-Ulam theorem in non-Archimedean fuzzy normed spaces", *Appl. Math. Let.* Vol.25, No.3, (2012), 301-304.

[8] S. Mazur, S. Ulam, "Sur les transformation isomtriques despaces vectoriels norms", *C. R. Acad. Sci. Paris*, Vol.194 (1932), 946-948.

[9] M. S. Moslehian and Gh. Sadeghi, "A Mazur-Ulam theorem in non-Archimedean normed spaces", *Nonlinear Anal.* Vol.69 (2008), 3405-3408.

[10] R. Saadati, S.M. Vaezpour, "Some results on fuzzy Banach spaces", *J. Appl. Math. Computing*, Vol.17, No. 1-2, (2005), 475-484.

[11] L.A. Zadeh, "Fuzzy sets", *Inf. & Cont.*, Vol.8 (1965), 338-353.