On the Mazur-Ulam problem
in fuzzy anti-normed spaces

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Abstract
The aim of this article is to proved a Mazur-Ulam type theorem in the strictly convex fuzzy anti-normed spaces.

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1. Introduction and preliminaries

The theory of fuzzy sets was introduced by L. Zadeh [11] in 1965 and thereafter several authors applied it different branches of pure and applied mathematics. Many mathematicians considered the fuzzy normed spaces in several angels (see [1], [4], [10]). In [6] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [2] and investigated their important properties. In 1932, the theory of isometric mappings was originated in the classical paper [8] by Mazur and Ulam. They proved that every isometry f of a normed real vector space X onto another normed real vector space X is a linear mapping up to translation, that is, \( x \mapsto f(x) - f(0) \) is linear, which amounts to the definition that f is affine. We call this the Mazur-Ulam theorem. The property is not true for normed complex vector spaces. The hypothesis of surjectivity is essential. Without this assumption, Baker [3] proved that every isometry from a normed real space into a strictly convex normed real space is linear up to translation. A number of mathematicians have had deal with the Mazur-Ulam theorem; see [7, 9] and references therein. In this paper, we prove that the Mazur-Ulam theorem holds under some conditions in the fuzzy anti-normed spaces. We establish a Mazur-Ulam type theorem in the framework of strictly convex normed spaces by using some ideas of [5]. Now we recall some notations and definitions used in this paper.

Definition 1.1 [6] Let X be a linear space over a real field F. A fuzzy subset N of \( X \times \mathbb{R} \) is called a fuzzy anti-norm on X if the following conditions are satisfied for all \( x, y \in X \):

(a) \( N(\alpha x, t) = N(x, t/|\alpha|) \) for all \( \alpha \neq 0, \alpha \in F \).

(b) \( N(x, t) \) is a non-increasing function of \( t \in \mathbb{R} \) and \( \lim_{t \to \infty} N(x, t) = 0 \).

Then the pair \( (X, N) \) is called a fuzzy anti-normed linear space.
Example 1.2 Let \((X, \| \cdot \|)\) be a normed space. If for all \(k, m, n \in \mathbb{R}^+\) we define

\[
N(x, t) = \begin{cases} 
\frac{m\|x\|}{kt^n + m\|x\|} & \text{if } t > 0 \\
1 & \text{if } t \leq 0.
\end{cases}
\]

In particular if \(k = m = n = 1\) we have

\[
N(x, t) = \begin{cases} 
\|x\| & \text{if } t > 0 \\
1 & \text{if } t \leq 0
\end{cases}
\]

which is called the standard fuzzy anti-norm induced by the norm \(\| \cdot \|\).

Definition 1.3 A fuzzy anti-normed space \(X\) is called strictly convex if \(N(x + y, s + t) = \max\{N(x, s), N(y, t)\}\) and \(N(x, s) = N(y, t)\) implies that \(x = y\) and \(s = t\).

Definition 1.4 Let \((X, N)\) and \((Y, N)\) be two fuzzy anti-normed spaces. We call that \(f : (X, N) \rightarrow (Y, N)\) is a fuzzy isometry if \(N(x + y, s + t) = \max\{N(x, s), N(y, t)\}\) and \(N(x, s) = N(y, t)\) implies that \(x = y\) and \(s = t\).

Definition 1.5 Let \(X\) be a real linear space and \(x, y, z\) mutually disjoint elements of \(X\). Then \(x, y\) and \(z\) are said to be collinear if \(y - z = k(x - z)\) for some real number \(k\).

2. Main results

In this section we will prove that the MazurUlam theorem under some conditions in the fuzzy real anti-normed strictly convex spaces. First, we prove the following lemma that is require for the main theorem of our paper.

Lemma 2.1 Let \(X\) be a fuzzy anti-normed space which is strictly convex and let \(x, y, z \in X\) and \(s > 0\). Then \(x = y + z\) is unique element of \(X\) such that

\[
N(y - x, s) = N(y - z, 2s)
\]

and

\[
N(z - x, s) = N(y - z, 2s).
\]

Proof. There is nothing to prove if \(y = z\). Let \(y \neq z\). Then by \((a - N_3)\), we have

\[
N(y - x, s) = N(y - \frac{y + z}{2}, s) = N(y - z, 2s)
\]

and

\[
N(z - x, s) = N(z - \frac{y + z}{2}, s) = N(y - z, 2s),
\]

that is the existence holds. For the uniqueness, we may assume that \(u\) and \(v\) are two elements of \(X\) such that

\[
N(y - u, s) = N(y - v, s) = N(z - u, s) = N(z - v, s) = N(y - z, 2s).
\]

Then

\[
N(y - \frac{u + v}{2}, s) \leq \max\{N(y - u, s), N(y - v, s)\} = N(y - z, 2s)
\]

(1.2)

and

\[
N(z - \frac{u + v}{2}, s) \leq \max\{N(z - u, s), N(z - v, s)\} = N(y - z, 2s).
\]

(2.2)
If both of inequalities (2.1) and (2.2) were strict we would have
\[
N(y - z, 2s) = N\left(y - \frac{u + v}{2} + \frac{u + v}{2} - z, 2s\right)
\leq \max\{N\left(y - \frac{u + v}{2}, N\left(z - \frac{u + v}{2}, s\right)\right)\}
< N(y - z, 2s),
\]
which is a contradiction. So at least one of the equalities holds in (2.1) and (2.2). Without lose of generality assume that equality holds in (2.1). Then
\[
N(y - \frac{u + v}{2}, s) = \max\{N(y - u, s), N(y - v, s)\}.
\]
The strict convexity of \(X\) implies that, \(N(y - u, s) = N(y - v, s)\), and so, \(u = v\). Therefore the proof is completed.

**Theorem 2.2** Let \(X\) and \(Y\) be real fuzzy anti-normed spaces and let \(Y\) be strictly convex. Suppose \(f : X \to Y\) be a fuzzy isometry satisfies \(f(x), f(y)\) and \(f(z)\) are collinear when \(x, y\) and \(z\) are collinear. Then \(f\) is affine.

**Proof.** Let \(g(x) := f(x) - f(0)\). Then \(g\) is fuzzy isometry and \(g(0) = 0\). It is easy to check that if \(x, y\) and \(z\) are collinear, then \(g(x), g(y)\) and \(g(z)\) are also collinear. So it suffices to show that \(g\) is linear. We have
\[
N(g\left(\frac{y + z}{2}\right) - g(y), s) = N\left(\frac{y + z}{2} - y, s\right) = N(y - z, 2s)
\]
and similarly
\[
N(g\left(\frac{y + z}{2}\right) - g(z), s) = N\left(\frac{y + z}{2} - z, s\right) = N(y - z, 2s)
\]
for all \(y, z \in X\) and \(s > 0\). By lemma (2.1) we have
\[
g\left(\frac{y + z}{2}\right) = \frac{1}{2}g(y) + \frac{1}{2}g(z).
\]
Since \(g(0) = 0\), we can easily show that \(g\) is additive. It follows that \(g\) is \(Q\)-linear. We have to show that \(g\) is \(R\)-linear.
Let \(r \in \mathbb{R}^+\) with \(r \neq 1\) and \(y \in X\). Since \(0, y\) and \(ry\) are collinear \(g(0), g(y)\) and \(g(r)y\) are also collinear. Since \(g(0) = 0\), there exists \(r' \in \mathbb{R}\) such that \(g(ry) = r'g(y)\). Now, we will proved that \(r = r'\). Since \(y\) and \(z\) are collinear, then \(y \neq z\). Hence,
\[
N(r(y - z), s) = N(g(ry) - g(rz), s)
= N(g(r'y) - g(r'z), s)
= N(r'(g(y) - g(z)), s)
= N(r'(y - z), s).
\]
By the strict convexity we obtain \(r(y - z) = r'(y - z)\). Thus \(g(ry) = rg(y)\) for all \(y \in X\) and all \(r \in \mathbb{R}\). Therefore \(g\) is affine and the proof is complete.

**References**


