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Weak and strong convergence of implicit iterative process for a finite family of asymptotically TJ mappings

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Abstract

In this paper we study the weak and strong convergence of implicit iteration process to a common fixed point for a finite family of asymptotically TJ mappings in Hilbert spaces. This paper is motivated by [Lin, Lai-Jiu; Chuang, Chih-Sheng; Yu, Zenn-Tsun *Fixed point theorems for some new nonlinear mappings in Hilbert spaces*, Fixed Point Theory Appl. (2011), 2011:51, 16 pp.].

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1 Introduction

Throughout this paper, let H be a real Hilbert space and C be a arbitrary non-empty closed convex subset of H. Let T is a mapping on H. We denotes the set of fixed points of T by $F(T)$. The mappings that we work by those are asymptotic TJ mappings that defined as following (see [2])

Definition 1.1. *We say $T : C \to C$ is an asymptotic TJ mapping if there exist two functions $\alpha : C \to [0, 2]$ and $\beta : C \to [0, k]$, $k < 2$, such that*

(i) $2\|Tx - Ty\|^2 \leq \alpha(x)\|x - y\|^2 + \beta(x)\ \|Tx - y\|^2 \quad for\ all \quad x, y \in C;$
(ii) $\alpha(x) + \beta(x) \leq 2 \quad for\ all \quad x \in C.$

Let $T: C \to C$ be a mapping, $x_0 \in C$ be arbitrary and $\{\alpha_n\}$ be a sequence of real numbers in the interval $(0,1)$, we define

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \qquad n = 0, 1, 2, \ldots. \tag{1}$$

The iteration process (1) is known as the Mann's type iteration process, see [3]. We extend the iteration process (1) by a finite family of asymptotically TJ mappings. Let $T_1, T_2, \ldots, T_N$ be N asymptotically TJ mappings of C into itself such that

$$F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset.$$

We can define a sequence $\{x_n\}$ as follows

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1} \\ &\vdots \end{aligned}$$

We write the above iteration process in the following compact form

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \qquad n = 1, 2, \ldots, \tag{2}$$

where $T_n = T_{n(\text{mod} N)}$ that the mod N function takes values in $\{1, 2, \ldots, N\}$. We'll show that under suitable conditions the sequence $\{x_n\}$ is weakly convergence to an element of F.

2 Preliminaries

In this section we collect some well-known results.

Throughout this paper, we denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [5], for each $x, y \in H$ and $\alpha \in (0,1)$, we have

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{3}$$

It's well-known that each Hilbert space H satisfies the *Opial's condition*, see [4]; that is, for any sequence $\{x_n\} \subset H$ that $x_n \rightharpoonup x$ we have

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\| \quad \textit{for all} \quad y \in H \setminus \{x\}.$$

Definition 2.1. *Let C be a closed subset of H. The mapping $T : C \to C$ is semi-compact whenever for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to x \in C$ as $j \to \infty$.*

Now we recall the Demi-closedness principle, see [1], in the following lemma.

Lemma 2.2. *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $T : C \to C$ be an asymptotically TJ mapping. Then $I - T$ is demi-closed at zero, i.e. for each sequence $\{x_n\}$ in H, if $\{x_n\}$ converges weakly to $p \in C$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)p = 0$.*

3 Main Results

In this section, we state our main results. We begin by the following theorem.

Theorem 3.1. *Let H be a Hilbert space that satisfying Opial's condition and C be a nonempty closed convex subset of H. Let $T_1, T_2, \dots, T_N : C \to C$ be N asymptotic TJ mappings with $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\{\alpha_n\}$ be a sequence in $(0, 1)$. If the sequence $\{x_n\}$ defined as (2) and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, then $x_n \rightharpoonup p$ as $n \to \infty$, for some $p \in F$.*

Proof. Assume $p \in F$. Since $T_{n(\text{mod}N)}$ is asymptotic TJ mapping, then for each $n \in \mathbb{N}$, we have

$$\|T_{n(\text{mod}N)}x_n - p\| \leq \|x_n - p\|. \tag{4}$$

From (3) and (4) we infer that

$$\begin{aligned} \|x_n - p\|^2 &= \|\alpha_n x_{n-1} + (1 - \alpha_n)T_{n(\text{mod}N)}x_n - p\|^2 \\ &= \alpha_n\|x_{n-1} - p\|^2 + (1 - \alpha_n)\|T_{n(\text{mod}N)}x_n - p\|^2 \\ &- \alpha_n(1 - \alpha_n)\|x_{n-1} - T_{n(\text{mod}N)}x_n\|^2 \\ &\leq \alpha_n\|x_{n-1} - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\ &- \alpha_n(1 - \alpha_n)\|x_{n-1} - T_{n(\text{mod}N)}x_n\|^2. \end{aligned} \tag{5}$$

Hence, we have

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2.$$

Thus $\{\|x_n - p\|\}$ is a decreasing sequence, so $\lim_{n\to\infty} \|x_n - p\|$ exists. Also, by ineq uality (5) we have

$$\alpha_n(1 - \alpha_n)\|x_{n-1} - T_{n(\text{mod}N)}x_n\|^2 \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2,$$

since $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, we deduce

$$\lim_{n\to\infty} \|x_{n-1} - T_{n(\text{mod}N)}x_n\| = 0. \tag{6}$$

On the other hand

$$\begin{aligned}\|x_n - x_{n-1}\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_{n(\mathrm{mod}N)}x_n - x_{n-1}\| \\ &\leq (1-\alpha_n)\|T_{n(\mathrm{mod}N)}x_n - x_{n-1}\|.\end{aligned}$$

From the last inequality and (6) we obtain that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{7}$$

Triangle inequality, (6) and (7) imply that

$$\lim_{n\to\infty} \|x_n - T_{n(\mathrm{mod}N)}x_n\| = 0, \tag{8}$$

and

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \qquad \forall j \in \{1, 2, \ldots, N\}. \tag{9}$$

Assume $j \in \{1, 2, \ldots, N\}$. Since $T_{n(\mathrm{mod}N)+j}$ is an asymptotic TJ mapping, there are two functions $\alpha := \alpha_{n(\mathrm{mod}N)+j}$ and $\beta := \beta_{n(\mathrm{mod}N)+j}$ that satisfy conditions of Definition 1.1. We have

$$\begin{aligned}&\|x_n - T_{n(\mathrm{mod}N)+j}x_n\| \\ &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n(\mathrm{mod}N)+j}x_{n+j}\| + \|T_{n(\mathrm{mod}N)+j}x_{n+j} - T_{n(\mathrm{mod}N)+j}x_n\| \\ &\leq (1 + \sqrt{\frac{\alpha(x)}{2}})\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n(\mathrm{mod}N)+j}x_{n+j}\| \\ &+ \sqrt{\frac{\beta(x)}{2}}\|T_{n(\mathrm{mod}N)+j}x_{n+j} - x_n\| \\ &\leq (1 + \sqrt{\frac{\alpha(x)}{2}} + \sqrt{\frac{\beta(x)}{2}})\|x_n - x_{n+j}\| \\ &+ (1 + \sqrt{\frac{\beta(x)}{2}})\|x_{n+j} - T_{n(\mathrm{mod}N)+j}x_{n+j}\|.\end{aligned}$$

So by (8) and (9) we infer that

$$\lim_{n\to\infty} \|x_n - T_{n(modN)+j}x_n\| = 0 \qquad (1 \leq j \leq N). \tag{10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of it such that $\{x_{n_k}\}$ converges weakly to $p \in C$. For any $l \in \{1, 2, \ldots, N\}$, (10) implies that

$$\lim_{k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0. \tag{11}$$

Now we can apply Lemma 2.2 to infer that $(I - T_l)p = 0$, for any $l \in \{1, 2, \ldots, N\}$. Therefore $p \in F$. Now we prove $x_n \rightharpoonup p$ as $n \to \infty$. Let

$\{x_{n_i}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$ then we show that $q = p$. Assume $p \neq q$, then by Opial's condition we deduce

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| = \lim_{i\to\infty} \|x_{n_i} - q\| \\ &< \lim_{i\to\infty} \|x_{n_i} - p\| = \lim_{n\to\infty} \|x_n - p\|,\end{aligned}$$

which is a contradiction. Therefore we conclude that $x_n \rightharpoonup p$. □

In the following theorem we infer the strong convergence.

Theorem 3.2. *Let H be a Hilbert space satisfying Opial's condition and C be a nonempty closed convex subset of H. Let $T_1, T_2, \ldots, T_N : C \to C$ be N semi-compact asymptotic TJ mappings with $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\{\alpha_n\}$ be a sequence in $(0, 1)$. If $\{x_n\}$ defined as (2) and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, then $x_n \to p$ as $n \to \infty$ for some $p \in F$.*

Proof. From the proof of Theorem 3.1, we know that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ as $k \to \infty$ for some $p \in C$ and satisfies (11). Since T_l, $l = 1, 2, \ldots, N$, is semi-compact we have $\lim_{k\to\infty} \|x_{n_k} - p\| = 0$. Therefore $x_n \to p$ as $n \to \infty$, because $\{\|x_n - p\|\}$ is convergence. □

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