

On the Numerical Solution of Hammerstein Integral Equations using Legendre Approximation

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Abstract

In this study, Legendre collocation method is presented to solve numerically the Fredholm-Hammerstein integral equations. This method is based on replacement of the unknown function by truncated series of well known Legendre expansion of functions. The proposed method converts the equation to matrix equation, by means of collocation points on the interval $[-1, 1]$ which corresponding to system of algebraic equations with Legendre coefficients. Thus, by solving the matrix equation, Legendre coefficients are obtained. Some numerical examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords: *Fredholm-Hammerstein integral equations, Integral equation, Legendre collocation matrix method, Legendre polynomials, Volterra integral equation.*

1 Introduction

In recent years, there has been a growing interest in the Fredholm and Volterra integral equations. This is an important branch of modern mathematics and arise frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology [1]-[5]. There are several methods for

approximating the solution of linear and non-linear integral equations [10]-[14]. We consider the Hammerstein integral equations in the forms

$$x(t) = f(t) + \lambda_1 \int_0^1 K_1(t, s)F(x(s))ds + \lambda_2 \int_0^t K_2(t, s)G(x(s))ds, \quad (1)$$

where $f(t)$, $K_1(t, s)$ and $K_2(t, s)$ are given functions, $0 \leq t, s \leq 1$, and λ_1, λ_2 are arbitrary constants.

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common set of the Legendre polynomials $P_0(t), P_1(t), \dots, P_N(t)$ which are orthogonal on $[-1, 1]$ with respect to the weight function $w(t) = 1$. The Legendre polynomials $P_n(t)$ satisfy the Legendre differential equation

$$(1 - t^2)u''(t) - 2tu'(t) + n(n + 1)u(t) = 0, \quad -1 < t < 1, \quad n \geq 0,$$

and are given by the following relation

$$P_n(t) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n - 2k)!}{(n - k)!(n - 2k)k!} t^{n-2k}, \quad n = 0, 1, 2, \dots \quad (2)$$

Also, the recurrence formula associated with Legendre polynomials is given with the relations,

$$P_0(t) = 1, P_1(t) = \frac{1}{2}(3t^2 - 1), (n+1)P_{n+1}(t) = (2n + 1)tP_n(t) - nP_{n-1}(t), \quad n \geq 1. \quad (3)$$

Legendre polynomials occur in the solution of Laplace equation of the potential, $\nabla^2\Phi(x) = 0$, in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry, the solution for the potential will be

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta),$$

A_l and B_l are to be determined according to the boundary condition of each problem. They also appear when solving Schrödinger equation in three dimensions for a central force.

The Legendre and Chebyshev polynomials are mostly used to solve several problems of differential equations or integral equations, for example, the Legendre pseudo-spectral method is used to solve the delay and the diffusion differential equations ([6], [7]). Chebyshev polynomials are used to introduce an efficient modification of homotopy perturbation method [8]. Also, the polynomial approximation is used to solve high-order linear Fredholm integro-differential equations with constant coefficient [11] and others ([9], [15]-[17]).

2 Procedure Solution using the Proposed Numerical Method

We consider the Fredholm-Volterra integral equation (1). The function $x(t)$ may be expanded by infinite series of Legendre polynomials as follows

$$x(t) = \sum_{n=0}^{\infty} c_n P_n(t), \quad (4)$$

where $c_n = (x(t), P_n(t))$. If we consider truncated series in Eq.(4), we obtain

$$x(t) \simeq \sum_{n=0}^N c_n P_n(t) = C^T P(t), \quad (5)$$

such that C and P are matrices given by

$$C = [c_0 \ c_1 \ \dots \ c_N], \quad P(t) = [P_0(t) \ P_1(t) \ \dots \ P_N(t)]^T. \quad (6)$$

Then we substitute the approximation Eq.(5) into Eq.(1) we get

$$C^T P(t) = f(t) + \lambda_1 \int_0^1 K_1(t, s) F(C^T P(s)) ds + \lambda_2 \int_0^t K_2(t, s) G(C^T P(s)) ds. \quad (7)$$

Now, to use the Legendre collocation method which is a matrix method based on the Legendre collocation points depended by

$$t_i = -1 + \frac{2}{N}i, \quad i = 0, 1, \dots, N, \quad (8)$$

we collocate Eq.(7) with the points (8) to obtain

$$C^T P(t_j) = f(t_j) + \lambda_1 \int_0^1 K_1(t_j, s) F(C^T P(s)) ds + \lambda_2 \int_0^{t_j} K_2(t_j, s) G(C^T P(s)) ds. \quad (9)$$

The integral terms in Eq.(9) can be found using composite trapezoidal integration technique as:

$$\int_0^1 K_1(t_j, s) F(C^T P(s)) ds \cong \frac{h}{2} (\Omega_1(s_0) + \Omega_1(s_m) + 2 \sum_{k=1}^{m-1} \Omega_1(s_k)), \quad (10)$$

where $\Omega_1(s) = K_1(t_j, s) F(C^T P(s))$, $h = \frac{1}{m}$, for an arbitrary integer m , $s_i = ih$, $i = 0, 1, \dots, m$ and

$$\int_0^{t_j} K_2(t_j, s) G(C^T P(s)) ds \cong \frac{h_j}{2} (\Omega_2(\bar{s}_0) + \Omega_2(\bar{s}_m) + 2 \sum_{k=1}^{m-1} \Omega_2(\bar{s}_k)), \quad (11)$$

where $\Omega_2(s) = K_2(t_j, s)G(C^T P(s))$, $h_j = \frac{t_j}{m}$, for an arbitrary integer m , $\bar{s}_i = ih$.

Eq.(9) gives $(N + 1)$ system of linear or non-linear algebraic equations, which can be solved for c_k , $k = 0, 1, \dots, N$. So the unknown function $x(t)$ can be found.

3 Numerical Implementation

In this section, to achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results of numerical examples.

Example 1.

Consider Eq.(1) with the following functions and coefficients

$$f(t) = t^3 - (6 - 2e)e^t, \quad \lambda_1 = 1, \quad \lambda_2 = 1,$$

$$K_1(t, s) = e^{(t+s)}, \quad K_2(t, s) = 0, \quad F(x(s)) = x(s), \quad G(x(s)) = 0.$$

Eq.(1) takes the form

$$x(t) = t^3 - (6 - 2e)e^t + \int_0^1 e^{(s+t)} x(s) ds. \quad (12)$$

We apply the suggested method with $N = 4$, and approximate the solution $x(t)$ as follows

$$x_N(t) = \sum_{i=0}^4 c_i P_i(t) = C^T P(t). \quad (13)$$

By the same procedure in the previous section and using Eq.(9) we have

$$\sum_{i=0}^4 c_i P_i(t_j) - (t_j^3 - (6 - 2e)e^{t_j}) - \frac{h}{2}(\Omega(s_0) + \Omega(s_m) + 2 \sum_{k=1}^{m-1} \Omega(s_k)) = 0, \quad j = 0, 1, 2, 3, 4, \quad (14)$$

where $\Omega(s) = e^{(s+t_j)} \sum_{i=0}^4 c_i P_i(s)$ and the nodes $s_{l+1} = s_l + h$, $l = 0, 1, \dots, m$, $s_0 = 0$ and $h = \frac{1}{m}$.

Eq.(14) represents linear system of $N + 1$ algebraic equations in the coefficients c_i , by solving it using the conjugate gradient method, we obtain

$$c_0 = -0.0048, \quad c_1 = 0.5955, \quad c_2 = -0.0015, \quad c_3 = 0.3998, \quad c_4 = -0.0001.$$

Therefore, the approximate solution of this example using Eq.(13) is given by

$$x(t) = -0.0048P_0(t) + 0.5955P_1(t) - 0.0015P_2(t) + 0.3998P_3(t) - 0.0001P_4(t).$$

The exact solution of this example is $x(t) = t^3$.

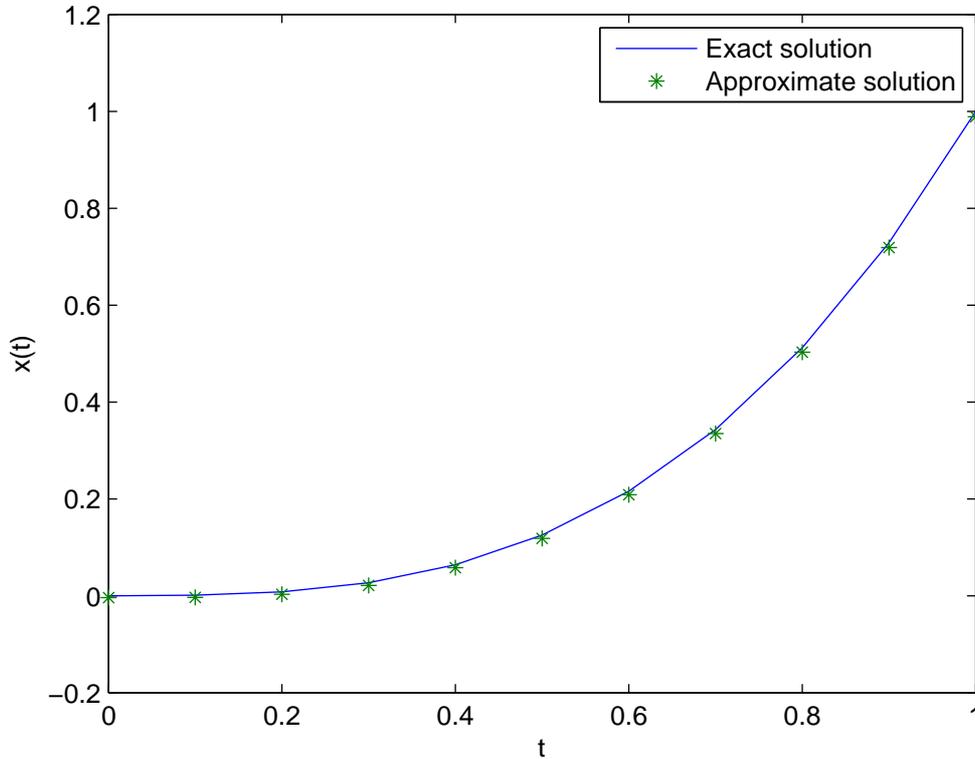


Fig.1. The behavior of the exact solution and the approximate solution at $N = 4$.

The behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution are presented in Fig.1. From this Fig., it is clear that the proposed method can be considered as an efficient method to solve the linear integral equations.

Example 2.

Consider Eq.(1) with the following functions and coefficients

$$f(t) = 2te^t - e^t + 1, \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad K_1(t, s) = 0,$$

$$K_2(t, s) = (s + t), \quad F(x(s)) = 0, \quad G(x(s)) = e^{x(s)}.$$

Eq.(1) takes the following form

$$x(t) = 2te^t - e^t + 1 - \int_0^t (s + t)e^{x(s)} ds. \tag{15}$$

We apply the suggested method with $N = 4$, and approximate the solution $x(t)$ as follows

$$x_N(t) = \sum_{i=0}^4 c_i P_i(t) = C^T P(t). \quad (16)$$

By the same procedure in the previous section and using Eq.(9) we have

$$\sum_{i=0}^4 c_i P_i(t_j) - f(t_j) + \frac{h_j}{2} (\Omega(s_0) + \Omega(s_m)) + 2 \sum_{k=1}^{m-1} \Omega(s_k) = 0, \quad j = 0, 1, 2, 3, 4, \quad (17)$$

where the nodes $s_{l+1} = s_l + h$, $l = 0, 1, \dots, m$, $s_0 = 0$ and $h_j = \frac{t_j}{m}$, $\Omega(s) = (s + t_j)e^{C^T P(s)}$.

Eq.(17) presents non-linear system of $N + 1$ algebraic equations in the coefficients c_i . By solving it by using the Newton iteration method with suitable initial solution we obtain

$$c_0 = 0.0002, \quad c_1 = 0.9895, \quad c_2 = 0.0022, \quad c_3 = -0.0088, \quad c_4 = 0.0023.$$

Therefore, the approximate solution of this example can be found using (16) as follows

$$x(t) = 0.0002P_0(t) + 0.9895P_1(t) + 0.0022P_2(t) - 0.0088P_3(t) + 0.0023P_4(t).$$

The exact solution of this problem is $x(t) = t$.

The behavior of the approximate solution using the proposed method with $N = 4$ and the exact solution are presented in Fig.2. From this Fig., it is clear that the proposed method can be considered as an efficient method to solve the nonlinear integral equations.

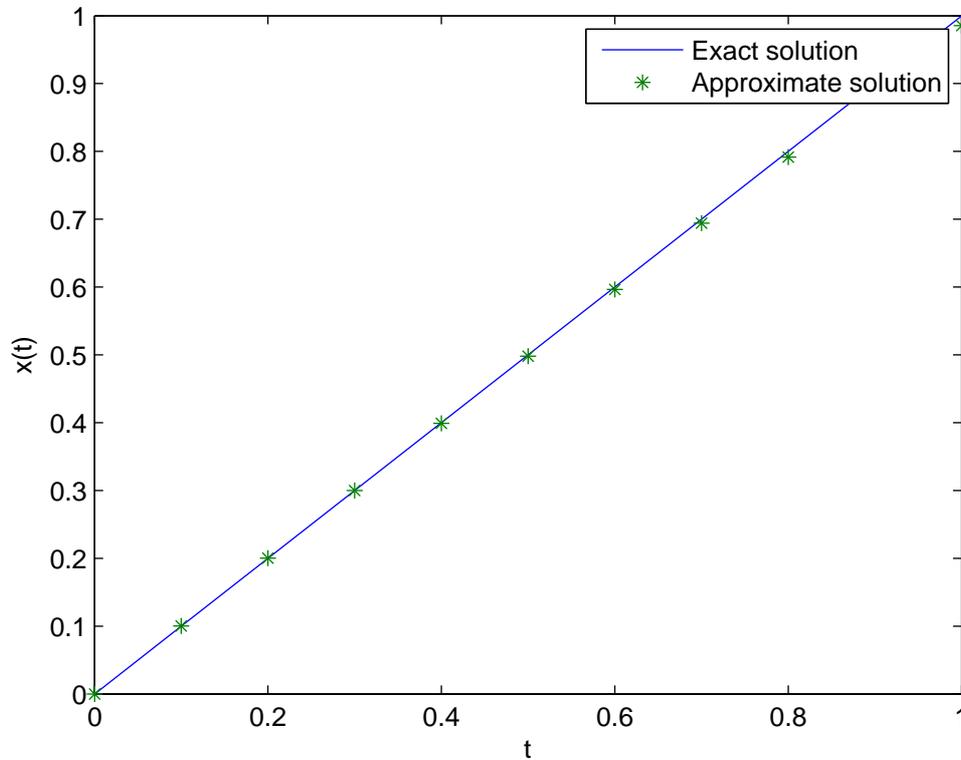


Fig.2. The behavior of the exact solution and the approximate solution at $N = 4$.

Example 3.

Consider Eq.(1) with the following functions and coefficients

$$f(t) = te + 1, \quad \lambda_1 = -1, \quad \lambda_2 = 1, \quad K_1(t, s) = s + t, \\ K_2(t, s) = 0, \quad F(x(s)) = e^{x(s)}, \quad G(x(s)) = 0.$$

Eq.(1) takes the following form

$$x(t) = te + 1 - \int_0^1 (s + t)e^{x(s)} ds. \tag{18}$$

The exact solution of this problem is $x(t) = t$.

We apply the suggested method with $N = 3$, and approximate the solution $x(t)$ as in (16), and by the same procedure in the previous section with using Eq.(9) we have

$$\sum_{i=0}^3 c_i P_i(t_j) - f(t_j) + \frac{h}{2}(\Omega(s_0) + \Omega(s_m)) + 2 \sum_{k=1}^{m-1} \Omega(s_k) = 0, \quad j = 0, 1, 2, 3, \tag{19}$$

where $s_{l+1} = s_l + h$, $l = 0, 1, \dots, m$, $s_0 = 0$ and $h = \frac{1}{m}$ and $\Omega(s) = (s + t_j)$.
 $e^{(\sum_{i=0}^3 c_i P_i(s))}$.

Eq.(19) presents non-linear system of algebraic equations. By solving it using the well known Newton iteration method with suitable initial solution we obtain

$$c_0 = -0.0023, \quad c_1 = 1.0013, \quad c_2 = 0.0 \quad c_3 = 0.0.$$

Therefore, the approximate solution of this example can be found using (16)

$$x(t) = -0.0023P_0(t) + 1.0013P_1(t) + 0.0P_2(t) + 0.0P_3(t).$$

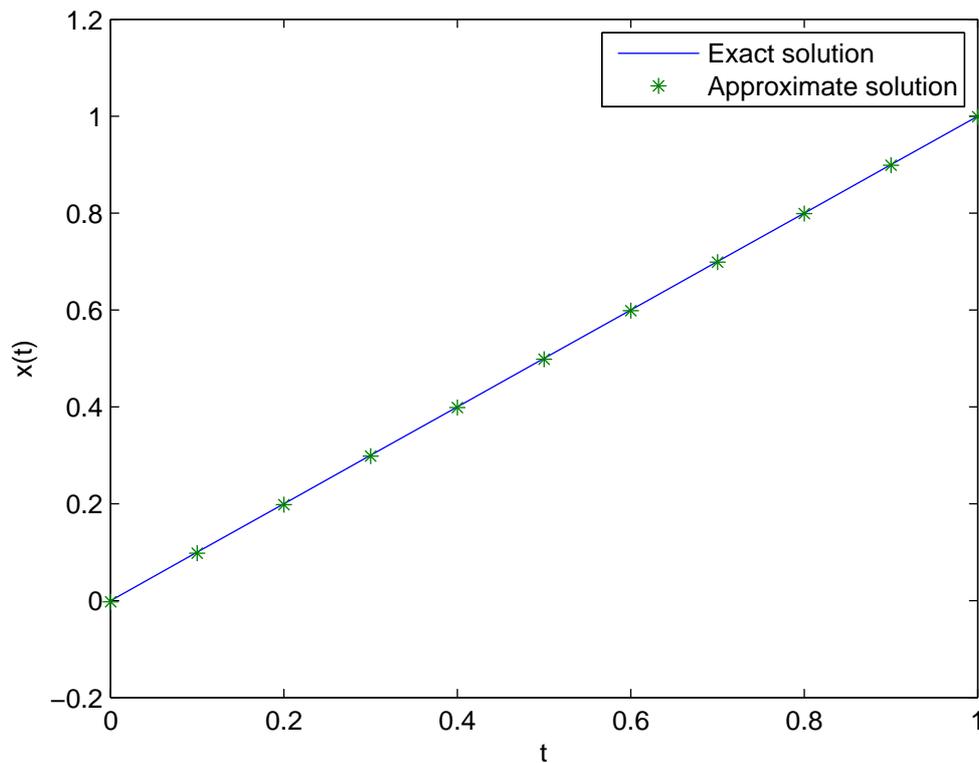


Fig.3. The behavior of the exact solution and the approximate solution at $N = 3$.

The behavior of the approximate solution using the proposed method with $N = 3$ and the exact solution are presented in Fig.3.

Example 4.

Consider Eq.(1) with the following functions and coefficients

$$f(t) = \frac{t}{2} - \frac{t^4}{12} - \frac{1}{3}, \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad K_1(t, s) = s + t,$$

$$K_2(t, s) = s - t, \quad F(x(s)) = x(s), \quad G(x(s)) = x^2(s).$$

Eq.(1) takes the following form

$$x(t) = \frac{t}{2} - \frac{t^4}{12} - \frac{1}{3} + \int_0^1 (s + t)x(s)ds + \int_0^t (s - t)x^2(s)ds. \quad (20)$$

We apply the suggested method with $N = 4$, and approximate the solution $x(t)$ as follows:

$$x_N(t) = \sum_{i=0}^4 c_i P_i(t) = C^T P(t). \quad (21)$$

By the same procedure in the previous section and using Eq.(9) we have

$$\begin{aligned} \sum_{i=0}^4 c_i P_i(t_j) - f(t_j) - \frac{h}{2}(\Omega_1(\bar{s}_0) + \Omega_1(\bar{s}_m) + 2 \sum_{k=1}^{m-1} \Omega_1(\bar{s}_k)) - \frac{h_j}{2}(\Omega_2(s_0) + \Omega_2(s_m) \\ + 2 \sum_{k=1}^{m-1} \Omega_2(s_k)) = 0, \end{aligned} \quad (22)$$

where $\bar{s}_{l+1} = \bar{s}_l + h$, $s_{l+1} = s_l + h_j$, $l = 0, 1, \dots, m$, $s_0 = \bar{s}_0 = 0$, $h = \frac{1}{m}$, $h_j = \frac{t_j}{m}$, and $\Omega_1(s) = (s + t_j)(\sum_{i=0}^4 c_i P_i(s))$, $\Omega_2(s) = (t_j - s)(C^T P(s))^2$.

Eq.(22) presents non-linear system of $N + 1$ algebraic equations. By solving it using Newton iteration method we obtain

$$c_0 = -0.0012, \quad c_1 = 0.9987, \quad c_2 = -0.0039, \quad c_3 = 0.0007, \quad c_4 = -0.0017.$$

Therefore, the approximate solution of this example can be found using (21):

$$x(t) = -0.0012P_0(t) + 0.9987P_1(t) - 0.0039P_2(t) + 0.0007P_3(t) - 0.0017P_4(t).$$

The exact solution of this problem is $x(t) = t$.

The behavior of the approximate solution using the proposed method using $N = 4$ and the exact solution are presented in Fig.4. From Fig.4, it is clear that the proposed method can be considered as an efficient method to solve the nonlinear Hammerstein integral equations.

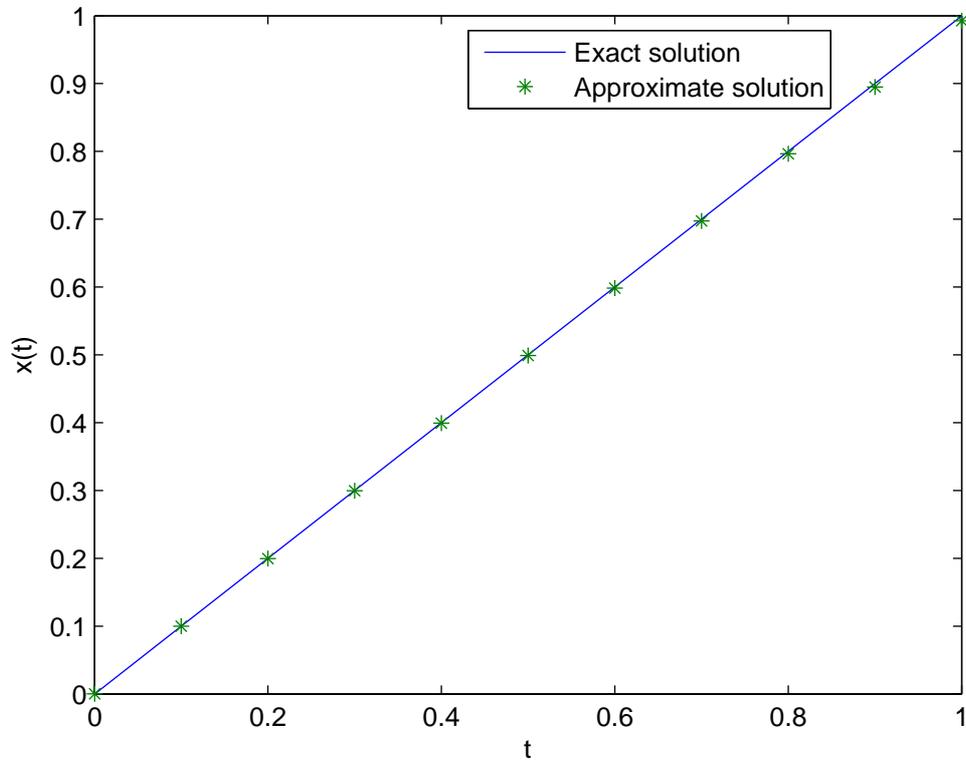


Fig.4. The behavior of the exact solution and the approximate solution at $N = 4$.

4 Conclusion

An approximate method for the solution of linear and non-linear Fredholm-Volterra integral equations in the most general form has been proposed and investigated. For this purpose, the presented method which is based on the Legendre polynomials is proposed to find the approximate solution. A comparison of the exact solution reveals that the presented method is very effective and convenient. The numerical results show that the accuracy improves with increasing N , hence for better results, using number N is recommended. Also, from the obtained approximate solution, we can conclude that the proposed method gives the solution in an excellent agreement with the exact solution. All computations are done using Matlab programming.

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