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Hyers-Ulam Stability of Linear and Nonlinear Differential Equations of Second Order

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Abstract

In this paper we established the Hyers-Ulam stability of a nonlinear differential equation of second order with initial condition. We also proved the Hyers -Ulam stability of a linear differential equation of second order with initial condition.

Keywords: *Differential equation, Hyers -Ulam Stability, Linear, Nonlinear , Second-order.*

1 Introduction

In [1], Ulam posed the basic problem of the stability of functional equations: Give conditions in order for a linear mapping near an approximately linear mapping to exist . This problem was partially solved by Hyers in 1941, for approximately additive mappings on Banach spaces [2]. In 1978 Rassias in his work [3], has generalized that result obtained by Hyers.

After then, many mathematicians have extensively investigated the stability problems of functional equations (see [4, 5, 6]).

Alsina and Ger [7] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$.They proved that if a differentiable function $y : I \to R$ satisfies $|y' - y| \leq \varepsilon$ for all $t \in I$,then there exists a differentiable function $g : I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \leq 3\varepsilon$,for all $t \in I$. This result of Alsina and Ger has been generalized by Takahasi et al. [8] to the case of the complex Banach space valued differential equation $y' = \lambda y$.

Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura et al. [9], Wang et al. [10], and Jung [11]. In the paper [12] Jung proved the Hyers-Ulam stability for Legendre's differential equation $(1-x^2)y'' - 2xy + p(p+1)y = 0$ when the function $y(x)$ has a power series form. In his paper Li [13] has established the Hyers-Ulam stability of the equation $y'' = \lambda^2 y$, while Gavruta et al. [14] proved the Hyers-Ulam stability of the equation $y'' + \beta(x)y = 0$ with boundary and initial conditions. Li and Shen [15] proved the stability of the nonhomogeneous linear differential equation of second order $y'' + p(x)y' + q(x)y + r(x) = 0$ in the sense of the Hyers and Ulam . In the paper [16] Javadian et al. have proved the Hyers and Ulam stability of the nonhomogeneous linear differential equation of second order $y'' + p(x)y' + q(x)y = f(x)$ in a complex Banach space with the condition that there exists a solution of the corresponding homogeneous equation.

In this paper we investigate the Hyers-Ulam stability of the following nonlinear differential equation of second order

$$z'' + p(x)z' + q(x)z = h(x)\,|z|^{\beta}\, e^{\left(\frac{\beta-1}{2}\right)\int p(x)dx} sgn z \quad , \ \beta \in (0,1) \tag{1}$$

with the initial conditions

$$z(x_0) = 0 = z'(x_0) \tag{2}$$

where $q \in C^0(I)$, , h , $p \in C^1(I)$, $I = [x_0, x] \subseteq \mathbb{R}$, $x_0 > 0$, $p(x) > 0$, and $h(x)$ is a bounded for all sufficiently large x in R. Moreover we proved the Hyers-Ulam stability of the linear differential equation of second order

$$z'' + p(x)z' + (q(x) - \alpha(x))\ z = 0 \tag{3}$$

with the initial conditions

$$z(x_0) = 0 = z'(x_0) \tag{4}$$

where $\alpha(x)$ is a bounded function for all sufficiently large x in R.

It should be note here that we may assume that $z > 0$ in equation (1) because if $z < 0$ we set $z = -u$, $u > 0$. So we will consider in future the equation

$$z'' + p(x)z' + q(x)z = h(x)z^{\beta}e^{\left(\frac{\beta-1}{2}\right)\int p(x)dx} \quad , \beta \in (0,1) \tag{5}$$

2 Preliminaries and Auxiliary Results

Definition 2.1: We will say that the equation (3) has the Hyers -Ulam stability with the initial conditions (4) if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0,\ z \in C^2(I)$ where x is sufficiently large in $\mathbb{R}$, if

$$|z'' + p(x)z' + (q(x) - \alpha(x))\ z| \le \varepsilon \tag{6}$$

then there exists some solution $w \in C^2(I)$ of the equation (5), such that $|z(x) - w(x)| \le K\varepsilon$ and satisfies the initial conditions

$$w(x_0) = 0 = w'(x_0) \tag{7}$$

Definition 2.2: We say that equation (5) has the Hyers -Ulam stability with initial conditions (4) if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0,\ z \in C^2(I)$ where x is sufficiently large in $\mathbb{R}$, if

$$|z'' + p(x)z' + q(x)z - h(x)z^\beta e^{\left(\frac{\beta-1}{2}\right)\int p(x)dx}\ | \le \varepsilon \tag{8}$$

then there exists some solution $w \in C^2(I)$ of the equation (5) and

$$w(x_0) = w'(x_0) = 0 \tag{9}$$

such that $|z(x) - w(x)| \le K\varepsilon$.

Definition 2.3: We will say that the equations (3),(5) have the Hyers -Ulam asymptotic stability with the initial conditions (4) if the equation is stable in the sense of Hyers and Ulam and $\lim\limits_{x\to\infty}(z(x) - w(x)) = 0$.

The author in his work [17] has proved the following Lemma and Theorem.

Lemma 2.1: (see [17]) A substitution $z(x) = y(x)\exp(-\frac{1}{2}\int p(x)dx)$ reduces the equations (3) and (5) to the equations (10) and (11), respectively

$$y'' + y = \alpha(x)y \tag{10}$$

$$y'' + y = h(x)y^\beta \quad , \beta \in (-1,1)\backslash\{0\} \tag{11}$$

where

$$q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x) = 1. \tag{12}$$

Theorem 2.1(see [17]) Suppose that $h(x)$ is a continuously differentiable function, bounded for all sufficiently large $x \in R$, and that the integral $\int_{x_0}^{\infty} |h'(x)|\, dx$ is convergent then any solution of the equation (11) is bounded as $x \to \infty$.

Proof. Multiplying both sides of the equation (11) by y' and integrate the result we get

$$y'^2(x) + y'^2(x) = y'^2(x_0) + y^2(x_0) - \frac{2h(x_0)y^{\beta+1}(x_0)}{\beta+1}$$

$$+\frac{2h(x)y^{\beta+1}(x)}{\beta+1} - \frac{2}{\beta+1}\int_{x_0}^{x} h'(t).y^{\beta+1}(t)dt$$

Hence

$$y^2(x) \ \le y'^2(x) + y^2(x) \le A_{x_0} + \frac{2\left|h(x)\right|\left|y(x)\right|^{\beta+1}}{\beta+1} + \frac{2}{\beta+1}\int_{x_0}^{x} \left|h'(t)\right|.\left|y(t)\right|^{\beta+1} dt$$

where $A_{x_0} \ge 0$ is an expression dependent only on x_0.

Let $M = \max_{x_0 \le t \le x} |y(t)|$, and without loss of generality we may assume that $M \ge a_0 > 0$, otherwise the theorem is proved. Since $h(x)$ is bounded we get

$$M^{1-\beta} \le \frac{A_{x_0}}{M^{\beta+1}} + \frac{2B_0}{\beta+1} + \frac{2}{\beta+1}\int_{x_0}^{x} \left|h'(t)\right| dt \le \frac{A_{x_0}}{a_0} + \frac{2B_0}{\beta+1}$$

$$+\frac{2}{\beta+1}\int_{x_0}^{\infty} \left|h'(t)\right| dt$$

Since the integral $\int_{x_0}^{\infty} \left|h'(x)\right| dx$ converges , we obtain

$$|y(x)| \ \le \ M \le C^{\frac{1}{1-\beta}} \ , \ \beta \in (-1,1)\backslash\{0\}$$

Therefore $y(x)$ is bounded for $x \to \infty$.

In the following theorem the author has established sufficient conditions for boundedness of the solutions of the equation (10) which are similar to those obtained in [18]**.**

Theorem 2.2 Suppose that $|\alpha(x)| \le L$ for all $x \ge x_0$. If $L < 1$ then any solution of the equation (10) is bounded as $x \to \infty$.

Proof. Multiplying both sides of the equation (10) by y' and integrating the result, we obtain

$$\int_{x_0}^{x} y'(t).y''(t)dt + \int_{x_0}^{x} y(t).y'(t)dt = \quad 2\int_{x_0}^{x} \alpha(t).y(t)y'(t)dt$$

Since $\alpha(x)$ is bounded we get

$$\begin{aligned} y^2(x) &\le y'^2(x) + y^2(x) \le A_{x_0} + 2\int_{x_0}^{x} \alpha(t).y(t)y'(t)dt \\ &\le A_{x_0} + Ly^2(x) \end{aligned}$$

It follows that

$$y^2 \le \frac{A_{x_0}}{(1-L)}$$

Therefore $y(x)$ is bounded for $x \to \infty$.

3 Main Results on Hyers-Ulam stability

Theorem 3.1 Suppose $|\alpha(x)| \le L < 1$ for all $x \ge x_0$, and that $y \in C^2(I)$, such that satisfies the inequality

$$|y'' + y - \alpha(x)\ y| \le \varepsilon \tag{13}$$

with the initial condition

$$y(x_0) = 0 = y'(x_0) \tag{14}$$

Then the equation (10) has the Hyers-Ulam stability with initial condition (14).

Proof. suppose that $\varepsilon > 0$ and $y \in C^2(I)$ satisfies the inequation (13) with the initial conditions (14) and $M = \max\limits_{x \ge x_0} |y(x)|$.

We will show that there exists a function $w(x) \in c^2(I)$ satisfying the equation (10) and the initial condition (7) such that $|z(x) - w(x)| \le k\varepsilon$.

From the inequality (13) we have

$$-\varepsilon \le y'' + y - \alpha(x)\ y \le \varepsilon \tag{15}$$

Multiply the inequality (15) by y' and then integrate we obtain

$$-2\varepsilon y \le y'^2(x) + y^2(x) - 2\int_{x_0}^{x} \alpha(t)\ yy'dt \le 2\varepsilon y$$

From which we get that

$$\begin{aligned} y^2(x) &\le 2\varepsilon y + 2\int_{x_0}^{x} \alpha(t)\ yy'dt = 2\varepsilon y + \alpha(x^*)\ y^2 \le 2\varepsilon y + \alpha(x^*)\ y^2 \\ &\le 2\varepsilon M + L\ M^2 \end{aligned}$$

Therefore

$$M \leq \frac{2\varepsilon}{1-L}$$

Hence $|y(x)| \leq k\varepsilon, \ for \ all \ x \geq x_0$. Obviously , $w_0(x) = 0$ satisfies the equation (10) and the zero initial condition (14) such that

$$|y(x) - w_0(x)| \leq k\varepsilon$$

Hence the equation (10) has the Hyers-Ulam stability with initial condition (14).

Corollary 3.1: Suppose $|\alpha(x)| \leq L < 1$ for all $x \geq x_0$, $z \in C^2(I)$ and satisfies the inequality (6) with the initial condition (4) . If the integral $\int_{x_0}^{\infty} p(x)dx$ converges then the equation (3) has the Hyers-Ulam stability with initial condition (4).

Proof. Suppose that $z \in C^2(I)$ satisfies the inequality

$$|z'' + p(x)z' + (q(x) - \alpha(x)) \ z| \ \leq \varepsilon$$

From the Theorem 3.1 it follows that the equation (10) has the Hyers-Ulam stability with initial condition (14) and according to the substitution in Lemma 2.1 it follows that the equation (3) has the Hyers-Ulam stability with initial condition (4).

Corollary 3.2 Suppose $|\alpha(x)| \leq L < 1$ for all $x \geq x_0$, $z \in C^2(I)$ and satisfies the inequality (6) with the initial condition (4) and $\int_{x_0}^{\infty} p(x)dx = \infty$, then the equation (3) has the Hyers-Ulam asymptotic stability with initial condition (4).

Proof. From the Corollary 3.1 it follows that the equation (3) has the Hyers-Ulam stability with initial condition (4). Since $\int_{x_0}^{\infty} p(x)dx = \infty$ then according to the substitution in Lemma 2.1 it follows that the equation (3) has the Hyers-Ulam asymptotic stability with initial condition (4).

Theorem 3.2 Suppose $|h(x)| \leq A$ for all $x \geq x_0$, and that $y \in C^2(I)$, such that satisfies the inequality

$$|y'' + y - \ h(x) \ y^{\beta}| \ \leq \varepsilon \qquad , \beta \in (0,1) \tag{16}$$

with the initial condition

$$y(x_0) = 0 = y'(x_0) \tag{17}$$

If $A < \frac{(\beta+1)}{2} \left(\max_{x \geq x_0} |y(x)| \right)^{-\beta}$,for $x \geq x_0$,then the equation

$$y'' + y = \ h(x) \ y^{\beta} \qquad , \beta \in (0,1) \tag{18}$$

has the Hyers-Ulam stability with initial condition (17).

Proof. suppose that $\varepsilon > 0$, $y \in C^2(I)$ satisfies the inequation (16) with the initial conditions (17) and that $M = \max_{x \geq x_0} |y(x)|$.

We will show that there exists a function $w(x) \in c^2(I)$ satisfying the equation (18) and the initial condition (17) such that $|z(x) - w(x)| \leq k\varepsilon$.

From the inequality (16) we have

$$-\varepsilon \leq y'' + y - h(x)\ y^{\beta} \leq \varepsilon \tag{19}$$

Multiply the inequality (19) by y' and then integrate we obtain

$$-2\varepsilon y \leq y'^2(x) + y^2(x) - 2\int_{x_0}^{x} h(x)\ y^{\beta} y' dt \leq 2\varepsilon y$$

From which we get that

$$y^2(x) \leq 2\varepsilon y + 2\int_{x_0}^{x} h(t)\ y^{\beta} y' dt = 2\varepsilon y + \frac{2h(x^*)\ y^{\beta+1}}{\beta+1} \leq 2\varepsilon M + \frac{2A\ M^{\beta+1}}{\beta+1}$$

Therefore

$$M \leq \frac{2\varepsilon}{1 - \frac{2AM^{\beta}}{\beta+1}}$$

Hence $|y(x)| \leq k\varepsilon$, for all $x \geq x_0$. Obviously , $w_0(x) = 0$ satisfies the equation (18) and the zero initial condition (17) such that

$$|y(x) - w_0(x)| \leq k\varepsilon$$

Thus the equation (18) has the Hyers-Ulam stability with initial condition (17).

Corollary 3.3 Assume that $h(x)$ and $z(x)$ satisfy the conditions of Theorem 3.2, and the inequality (8) with the initial condition (2).

If $A < \frac{(\beta+1)}{2}\left(\max_{x \geq x_0} |y(x)|\right)^{-\beta}$,for $x \geq x_0$ and the integral $\int_{x_0}^{\infty} p(x)dx$ converges then the equation (5) has the Hyers-Ulam stability with initial condition (2). Moreover, if the integral $\int_{x_0}^{\infty} p(x)dx = \infty$ then the equation (5) has the Hyers-Ulam asymptotic stability with initial condition (2).

Proof. Suppose that $z \in C^2(I)$ satisfies the inequality (8) with the initial condition (2).

Then from the Theorem 3.2 it follows that the equation (18) has the Hyers-Ulam stability with initial condition (17), and according to the substitution used in Lemma 2.1 it follows that the equation (5) has the Hyers-Ulam stability with initial condition (2). Now if $\int_{x_0}^{\infty} p(x)dx = \infty$, then the equation (5) has the Hyers-Ulam asymptotic stability with initial condition (2).

Now we illustrate the Theorem by the following example.

Example 3.1 Consider the equation

$$z'' + \frac{2}{x}z' + z = \frac{e^{-x/2}z^{1/2}}{\sqrt{x}} \tag{20}$$

with the initial condition

$$z(x_0) = 0 = z'(x_0) \tag{21}$$

If we set $z(x) = \frac{y(x)}{x}$ in the the equation (20) we obtain

$$y''(x) + y(x) = e^{-x/2}y^{1l2} \tag{22}$$

We let $y(x) = (x - x_0)^2 e^{-x}$ and estimate the difference

$$\left|y''(x) + y(x) - e^{-x/2}y^{1l2}\right| = \left|\frac{2 - 5(x - x_0) + 2(x - x_0)^2}{e^x}\right| \leq \varepsilon \tag{23}$$

Now we may choose the number x_0 sufficiently large such that the inequality (23) will satisfy for any $x \geq x_0$ and for any $\varepsilon > 0$.

Hence $y(x) = (x - x_0)^2 e^{-x}$ is an approximate solution of the equation (20) satisfying the zero initial condition

$$y(x_0) = 0 = y'(x_0) \tag{24}$$

Now we have

$$h(x) = e^{-x/2} \leq 1 < \frac{3e}{8} < \frac{3}{4}\left(\max_{x \geq x_0}|y(x)|\right)^{-\frac{1}{2}} = \frac{3e^{1+\frac{x_0}{2}}}{8}.$$

Therefore

$$M \leq k\varepsilon \ \ , \text{ where } \frac{6e^{(1+\frac{x_0}{2})}}{3e^{(1+\frac{x_0}{2})} - 8} > 0$$

It is clear that $z_0 \equiv 0$ satisfies the zero initial condition and the inequality $|y(x) - z_0(x)| \leq k\varepsilon$. Thus the equation (20) has the Hyers-Ulam stability. Moreover, since $\lim\limits_{x\to\infty} |y(x) - z_0(x)| = 0$,then it also is asymptotically stable in the sense of Hyers and Ulam as x$\to \infty$. Now since the integral $\int_1^\infty p(x)dx = \int_1^\infty \frac{2}{x}dx = \infty$, then by Lemma it follows that the equation (20) has the Hyers-Ulam stability with zero initial condition (21). Moreover the equation (20) is asymptotically stable in the sense of Hyers and Ulam as $x \to \infty$.

4 Special Case of the equation (5)

Now consider a special case of the equation (5)

$$x^2z'' + 2\lambda xz' + [x^2 + \lambda(\lambda - 1)]z = h(x)x^{2+\lambda(\beta-1)}z^{\beta} \tag{25}$$

where $\lambda > 0$, $\beta \in (0,1)$, and it satisfies the initial condition

$$z(x_0) = 0 = z'(x_0) \tag{26}$$

It should be note that the equation (25) is a special case of the equation (5) with $p(x) = \frac{2\lambda}{x}$ and $q(x) = \frac{x^2+\lambda(\lambda-1)}{x^2}$. So if we let $z(x) = \frac{y(x)}{x^{\lambda}}$, $\lambda \succ 0$, then the equation (25) is reduced to the equation (18) with $y(x_0) = 0 = y'(x_0)$.

Theorem 4.1 Suppose that the conditions of the Theorem 3.2 hold, the integral $\int_{x_0}^{\infty} p(x)dx$ converges and that $z \in C^2(I)$ and satisfies the inequality

$$\left|x^2z'' + 2\lambda xz' + [x^2 + \lambda(\lambda - 1)]z - h(x)x^{2+\lambda(\beta-1)}z^{\beta}\right| \leq \varepsilon$$

then the equation (25) has the Hyers-Ulam stability with initial condition (26). Moreover, if the integral $\int_{x_0}^{\infty} p(x)dx = \infty$ then the equation (25) has the Hyers-Ulam asymptotic stability with the initial condition (26).

Proof. It follows from the Theorem 3.2 and Corollary 3.3

Example 4.1 Consider the equation

$$x^2z'' + xz' + \left(x^2 - \frac{1}{4}\right)z = x^{7/4}e^{-x/2}z^{1/2} \tag{27}$$

with the initial condition

$$z(x_0) = 0 = z'(x_0) \tag{28}$$

Setting $z(x) = \frac{y(x)}{\sqrt{x}}$ in the equation (27) we get

$$y''(x) + y(x) = e^{-x/2}y^{1l2} \tag{29}$$

If we apply the same argument used in Example 3.1 for the function $y(x) = (x - x_0)^2 e^{-x}$ we can show that it satisfies the inequality

$$\left|y''(x) + y(x) - e^{-x/2}y^{1l2}\right| < \varepsilon$$

with initial condition $y(x_0) = 0 = y'(x_0)$, and the inequality

$$M \leq k\varepsilon \ , \text{ where } k = \frac{6e^{(1+\frac{x_0}{2})}}{3e^{(1+\frac{x_0}{2})} - 8} > 0$$

Therefore, we get the Hyers-Ulam stability and asymptotic stability for the equation (27).

5 Conclusion

In this paper we obtained sufficient criteria for Hyers-Ulam stability of linear and nonlinear differential equations of second Order with zero initial conditions.

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