

A class of exact solutions of equations for plane steady motion of incompressible fluids of variable viscosity in presence of body force

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Abstract

This paper determines a class of exact solutions for plane steady motion of incompressible fluids of variable viscosity with body force term in the Navier-Stokes equations. The class consists of stream function ψ characterized by equation $\theta = f(r) + g(r)v(\psi)$, in polar coordinates r, θ where $f(r)$, $g(r)$ and $v(\psi)$ are continuously differentiable functions, derivative of $g(r)$ is non-zero but double derivative of $v(\psi)$ is zero. We find exact solutions, for a suitable component of body force, considering two cases based on velocity profile. The first case fixes both the functions $g(r)$, $f(r)$ and provides viscosity as function of temperature. Where as the second case fixes the function $g(r)$, leaves $f(r)$ arbitrary and provides viscosity and temperature for the arbitrary function $f(r)$. In both the cases, we can create infinite set of expressions for streamlines, viscosity function, generalized energy function and temperature distribution in the presence of body force.

Keywords: Exact Solutions in the Presence of Body Force; Exact Solutions for Incompressible Fluids; Variable Viscosity Fluids; Na-Vier-Stokes Equations with Body Force; Martin's Coordinates.

1. Introduction

For the study of motion of a fluid element, we consider its momentum, the energy and conservation of mass equations. The momentum equations are the Navier-Stokes equations (NSE) having capacity to incorporate surface forces as well as body forces. The mathematical structure of these equations in the presence of surface forces is complex and it is extremely difficult to reach exact solutions in general. The complexity for exact solutions increases when we add more terms to the right-hand side of Navier-Stokes equations like body force, as it appears in geophysical fluid dynamics (GFD) due to the earth's rotation and in magneto-hydrodynamic (MHD) where the magnetic field induces currents in a moving fluid for creating force on the fluid [1-3]. For the exact solutions of these basic equations with and without body force, we find some appropriate dimension analysis method and coordinates transformation techniques in the available literature. The readers interested in these methods/techniques may refer to [4]-[17] and the references therein. Another technique may be the computer technique through which we can integrate NSE yet determination of accuracy, verification of numerical and empirical methods demands exact solutions. We are using coordinate transformations technique on non-dimensional equations for determining the exact solutions in this communication.

The fundamental non-dimensional equations for plane steady motion of incompressible variable viscosity fluid in the presence of external force in Cartesian space (x, y) , given in [5-7] are following

$$u_x + v_y = 0 \tag{1}$$

$$\mu u_x + \nu u_y = F_1 - p_x + [(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{2}$$

$$\mu v_x + \nu v_y = F_2 - p_y + \frac{1}{Re} [(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{3}$$

$$\mu T_x + \nu T_y = \frac{1}{Re Pr} (T_{xx} + T_{yy}) + \frac{Ec}{Re} [2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2] \tag{4}$$

Where the coefficient of viscosity is $\mu > 0$, the vector field $\mathbf{q} = (u, v)$ and $\rho = \rho(x, y)$ are fluid velocity and density respectively. The function $p = p(x, y)$ is pressure and the body force per unit mass is $\mathbf{F} = (F_1(x, y), F_2(x, y))$ acting on the fluid. The dimensionless quantities Re , Pr and Ec are respectively the Reynolds number, the Prandtl number and the Eckert number. A stream function $\psi = \psi(x, y)$, such that $\psi_{y,x} = \psi_{x,y}$, satisfies equation (1) identically when

$$\psi_y = u, \quad \psi_x = -v \tag{5}$$

whereas for the solution of equations (2-4), we transform them into Martin's coordinate system. Martin [17] on a curvilinear net (ϕ, ψ)

has defined the curves $\psi = const$ as streamlines and left the curves $\varphi = const$ arbitrary we will call it Martin's coordinate system (φ, ψ) . Since the curves $\varphi = const$ are arbitrary in Martin's system, therefore we set

$$\varphi = r(x, y) \tag{6}$$

And characterize the streamlines $\psi = const$ by

$$\frac{\theta - f(r)}{g(r)} = const \tag{7}$$

in polar coordinates r, θ . Where

$$x = r \cos \theta, \quad y = r \sin \theta \tag{8}$$

The equation (7) implies

$$\theta = f(r) + g(r)v(\psi) \tag{9}$$

Where $f(r)$ and $g(r)$ are continuously differentiable functions. In this communication we consider $g'(r) \neq 0$ and $v''(\psi) = 0$. We organize this paper as follow: In section (2) we convert the basic non-dimensional flow equations from Cartesian space (x, y) into (r, ψ) coordinates with the help of Martin's coordinates (φ, ψ) . In section (3), we find exact solutions of fundamental equations. In last section, we present conclusion.

2. Basic equations in (r, ψ) coordinates

For the exact solutions of fundamental equations, we follow [4-7]. First, we write the basic equations in some manageable form by introducing the vorticity function w and the total energy function L defined by

$$w = v_x - u_y \tag{10}$$

$$L = p + \frac{1}{2}(u^2 + v^2) - \frac{2\mu u_x}{Re} \tag{11}$$

And functions A and B as follow

$$A = \mu(u_y + v_x), \quad B = 4\mu u_x \tag{12}$$

Then, we transform the resulting equations into Martin's coordinates system (φ, ψ) through the transformations

$$x = x(\varphi, \psi), \quad y = y(\varphi, \psi) \tag{13}$$

Such that the Jacobian $J = \frac{\partial(x, y)}{\partial(\varphi, \psi)}$ of the transformation is non-zero and finite. We have given the resulting fundamental equations in Martin's coordinates system (φ, ψ) in [5-7] considering a common point $P(x, y)$ on the curves $\psi = const$ and $\varphi = const$, letting α the angle between the tangents to these curves. We transform the fundamental equations from Martin's coordinates system into (r, ψ) coordinates utilizing equations (6), (8) and (9) and find

$$q = \frac{\sqrt{1 + (M + Nv)^2}}{J} \tag{14}$$

$$-R_e w = -R_e J F_2 + R_e L_{\psi} - J A_r + \sqrt{E-1} A_{\psi} + B_{\psi} \tag{15}$$

$$0 = R_e (F_1 + F_2 \sqrt{E-1}) - R_e L_r + \frac{A_{\psi}(2-E)}{J} + A_r \sqrt{E-1} - \frac{\sqrt{E-1} B_{\psi}}{J} \tag{16}$$

$$J T_{rr} - 2a \sqrt{E-1} T_{vr} + \frac{a^2 E}{J} T_{vv} + \left(J_r - \frac{E_{\psi}}{2\sqrt{E-1}} - R_e P_r \right) T_r + a \left(\frac{E_{\psi}}{J} - \frac{E_r}{2\sqrt{E-1}} - \frac{E J_{\psi}}{J^2} \right) T_v = -\frac{J E_c P_r}{4\mu} (B^2 + 4A^2) \tag{17}$$

$$w = \left[\frac{M'}{rg} - \frac{2MN}{(rg)^2} \right] \left(\frac{1}{a} \right) + \left[\frac{N'}{rg} - \frac{2N^2}{(rg)^2} \right] \left(\frac{v}{a} \right) \tag{18}$$

$$A = \frac{\mu}{a(rg)^2} [rg(M' + N'v) - 2(M + Nv)(N + g)] \tag{19}$$

$$B = \frac{-4\mu(N + g)}{a r^2 g^2} \tag{20}$$

$$E = 1 + (M + Nv)^2 \tag{21}$$

$$J = r g v' = a(rg) \tag{22}$$

$$N(r) = r g'(r) \quad M(r) = r f'(r), \tag{23}$$

Where E is the coefficient of first fundamental form, the magnitude of velocity vector $\mathbf{q} = (u, v)$ is q and $v = a\psi + b$ with $a \neq 0$, b as constants. In order to determine the solution of the flow equations (15-16), we follow [4-7] and form a compatibility equation applying the natural integrability condition $L_{r\psi} = L_{\psi r}$ on the function L . The natural condition $L_{r\psi} = L_{\psi r}$ on equations (15-16) yields the following compatibility equation

$$a r g A_{rr} - 2r(f' + g'v) A_{r\psi} - \frac{[1 - r^2(f' + g'v)^2]}{a r g} A_{\psi\psi} + a g A_r - A_{\psi}((f' + g'v) + r(f'' + g''v)) - A_{\psi}((f' + g'v) + r(f'' + g''v)) - \left\{ B_r - \frac{(f' + g'v) B_{\psi}}{a g} \right\}_{\psi} = R_e w_r + R_e (F_1 + F_2 r(f' + g'v))_{\psi} - R_e (a r g F_2)_r \tag{24}$$

The equation (24) involves the functions $f(r), g(r), v(\psi)$, the body force components $F_1(r, \psi), F_2(r, \psi)$, their derivatives and viscosity μ . Once a solution of this equation (24) is determined, the energy function L and temperature distribution T are obtained from equations (15-16) and (17) respectively, the viscosity μ is obtained either from equation (19) or (20), the pressure p from equation (10) and velocity components u, v from (5).

In [5-7], we used a relation between the two functions A and B for the solution of compatible equation; however we find that such a momentous relation from equations (19) and (20) for this communication is not possible. Therefore, we consider the following two cases

- Case I: $A = 0$
- Case II: $B = 0$

3. Exact solutions

Let us consider the case

$$A = 0 \tag{25}$$

On substituting equations (19) and (23) in equation (25) and comparing the coefficient of v , we find

$$\frac{2(rg)'g'}{g} - (rg)' = 0 \tag{26}$$

And

$$\frac{-2(rg)'f'}{g} + (rf)' = 0 \tag{27}$$

The solution of equation (26) is

$$g(r) = \frac{-1}{(C_0 r^2 + C_1)} \tag{28}$$

Utilizing equation (28) in equation (27) and solving for $f(r)$ we get

$$f(r) = \frac{-n_1}{(C_0 r^2 + C_1)} + n_2 \tag{29}$$

Where $C_0 \neq 0$, C_1 , n_1 and n_2 are constants. Utilizing equation (25) in equation (24), we get

$$\left\{ -B_r + \frac{(f' + g'v)B_\psi}{ag} \right\}_\psi = R_e w_r + R_e (F_1 + F_2 r (f' + g'v))_\psi - R_e (a r g F_2)_r \tag{30}$$

Here equation (30) involves the function B , vorticity function w and the components of unknown body force $F_1(r, \psi)$, $F_2(r, \psi)$ and their derivatives therefore its solution will depend upon the form of F_1 and F_2 . We can select many possible forms of F_1 and F_2 leading to the solution of equation (30), however we find that arbitrarily selected forms does not lead to the solution of the momentum equations (15-16) for the function L and the energy equation (17) for T . Our search for the appropriate form of F_1 and F_2 has discovered that the solution of the momentum equations (15-16) is obtainable if the component of body force F_2 has following relation with vorticity function w

$$R_e (r g a F_2) = R_e w + G_3(\psi) \tag{31}$$

Where $G_3(\psi)$ is the function of integration. On substituting equation (31) in equation (30), we find

$$R_e F_1 = -(f' + g'v) \left\{ \frac{R_e w}{ag} + \frac{G_3(\psi)}{ag} \right\} - B_r + \frac{(f' + g'v)B_\psi}{ag} + H_3(r) \tag{32}$$

Where $H_3(r)$ is function of integration. Utilizing equations (31-32), in equations (15-16) and solving for the function L , we obtain

$$R_e L = -B + \int H_3(r) dr + \int G_3(\psi) d\psi + m_1 \tag{33}$$

Where m_1 is constant. We find viscosity from equation (20)

$$\mu = \frac{-ar^2 g^2}{4(N + g)} B \tag{34}$$

The energy equation (17), on utilizing equation (19-21) and equation (25), becomes

$$\begin{aligned} & (ar g) T_{rr} - 2ar(\eta_1 g' + g') T_{vr} \\ & + \frac{a \{1 + r^2(\eta_1 g' + g'v)^2\}}{rg} T_{vv} + (ag - R_e P_r) T_r \\ & \left(\frac{2r(\eta_1 g' + g'v)g'}{g} - (\eta_1 g' + g'v) + (\eta_1 g'' + g''v) \right) a T_v \\ & = \frac{E_c P_r (N + g)}{rg} B \end{aligned} \tag{35}$$

Substituting B from equation (34) in equation (35), we find following relation between viscosity and temperature

$$\begin{aligned} \mu & = \frac{-ar^3 g^3}{4E_c P_r (N + g)^2} [(ar g) T_{rr} - 2ar(\eta_1 g' + g') T_{vr} \\ & + \frac{a \{1 + r^2(\eta_1 g' + g'v)^2\}}{rg} T_{vv} + (ag - R_e P_r) T_r \\ & + \left(\frac{2r(\eta_1 g' + g'v)g'}{g} - (\eta_1 g' + g'v) + (\eta_1 g'' + g''v) \right) a T_v] \end{aligned} \tag{36}$$

Thus, we find viscosity from equation (36) as function of temperature T , pressure p using equation (33) in equation (11) and velocity components from (5) using (28-29) in (9).

Now consider the case II

$$B = 0 \tag{37}$$

Substituting (23) in equation (37), we get

$$g = \frac{c}{r} \tag{38}$$

Where c is a non-zero constant. Substituting equation (37-38) in equation (24), we get

$$\begin{aligned} & ac A_{rr} - 2a \left(M - \frac{cv}{r} \right) A_{rv} - \frac{a \left[1 - \left(M - \frac{cv}{r} \right)^2 \right]}{c} A_{vv} \\ & + \frac{ac}{r} A_r - a A_v \left(M' + \frac{cv}{r^2} \right) + \frac{2 \left(M - \frac{cv}{r} \right)}{r} \\ & = R_e w_r + a R_e \left(F_1 + \left(M - \frac{cv}{r} \right) F_2 \right)_v - R_e (ac F_2)_r \end{aligned} \tag{39}$$

The coefficients in equation (39) suggest searching for the solution of the type

$$A = R(r) + d\nu \quad (40)$$

Where d is constant. Inserting equation (40) in equation (39), we have

$$\begin{aligned} & \left\{ acR'' + \frac{ac}{r}R' \right\} - ad \left(M' + \frac{2M}{r} \right) \\ &= R_e \left(\frac{1}{ac} \right) \left[M' + \frac{2M}{r} \right]' + \frac{2R_e \nu}{a r^3} \\ &+ a R_e \left(F_1 + \left(M - \frac{c\nu}{r} \right) F_2 \right)' - R_e (acF_2)'_r \end{aligned} \quad (41)$$

Here equation (41) involves the function R , the components of unknown body force F_1 , F_2 and their derivatives therefore its solution will depend upon the form of F_1 and F_2 . We can select many possible forms of F_1 and F_2 leading to the solution of equation (41), however we have determined that arbitrarily selected forms does not lead to the solution of the momentum equations (15-16) for the function L and the energy equation (17) for T . Our search for the appropriate form of F_1 and F_2 discover that the solution of the momentum equations (15-16) is obtainable if the function F_2 is

$$\begin{aligned} R_e acF_2 &= ad \left(M + 2 \int \left(\frac{M}{r} \right) dr \right) + \left(\frac{R_e}{ac} \right) \left[M' + \frac{2M}{r} \right] \\ &- acR' - ac \int \left(\frac{R'}{r} \right) dr + R_1(\nu) \end{aligned} \quad (42)$$

On substituting equation (42) in equation (39), we find

$$\begin{aligned} a R_e F_1 &= -\frac{1}{c} \left(M - \frac{c\nu}{r} \right) \left[ad \left(M + 2 \int \left(\frac{M}{r} \right) dr \right) \right. \\ &+ \left. \left(\frac{R_e}{ac} \right) \left[M' + \frac{2M}{r} \right] - acR' - ac \int \left(\frac{R'}{r} \right) dr + R_1(\nu) \right] \\ &- \left(\frac{R_e}{a r^3} \right) \nu^2 + e^r a \int MR' e^{-r} dr + E_1 e^r \end{aligned} \quad (43)$$

Utilizing (42-43), in equation (15-16), we get

$$\begin{aligned} R_e aL &= \nu \left(\frac{R_e}{ac} \right) \left[M' + \frac{2M}{r} \right] - \nu ac \int \left(\frac{R'}{r} \right) dr \\ &- \nu \left[M' + \frac{2M}{r} \right] \left(\frac{R_e}{ac} \right) + \frac{R_e \nu^2}{2ar^2} + e^r a \int MR' e^{-r} dr \\ &+ E_1 e^r + \int R_1(\nu) d\nu \end{aligned} \quad (44)$$

And

$$d = 0 \quad (45)$$

Equation (19) gives the viscosity

$$\mu = \frac{acR(r)}{(M' - g'\nu)} \quad (46)$$

On utilizing equations (37-38), equation (40), equations (45-46) in the energy equation (17) we have

$$\begin{aligned} & acT_{rr} - 2a \left(M - \frac{c\nu}{r} \right) T_{\nu r} + \frac{a \left[1 + \left(M - \frac{c\nu}{r} \right)^2 \right]}{c} T_{\nu\nu} \\ &+ \left(\frac{ac}{r} - R_e P_r \right) T_r - \left(\frac{2(M+N\nu)}{r} + \left(M' + \frac{c\nu}{r^2} \right) \right) a T_\nu \\ &= -E_c P_r \left(RM' + \frac{cR}{r^2} \nu \right) \end{aligned} \quad (47)$$

The right-hand side of equation (47) suggests searching for the solution of the form

$$T = R_1(r) + R_2(r)\nu + R_3(r)\nu^2 \quad (48)$$

On substituting equation (48) in equation (47), we get

$$\begin{aligned} & \nu^2 \left[acR_3'' + 4agR_3' + \frac{2aR_3g^2}{c} \right. \\ & \left. + 2aR_3 \left(g' + \frac{2g}{r} \right) + \left(\frac{ac}{r} - R_e P_r \right) R_3' \right] \\ &+ \nu \left[acR_2'' - 4aMR_3' + 2agR_2' - \frac{4aR_3Mg}{c} \right. \\ & \left. + \left(\frac{ac}{r} - R_e P_r \right) R_2' + a \left\{ \begin{aligned} & -2R_3 \left(M' + \frac{2M}{r} \right) \\ & + R_2 \left(g' + \frac{2g}{r} \right) \end{aligned} \right\} \right] \\ &+ acR_1'' - 2aMR_2' + \frac{2aR_3(1+M^2)}{c} + \left(\frac{ac}{r} - R_e P_r \right) R_1' \\ &- a \left(M' + \frac{2M}{r} \right) R_2 = -E_c P_r \left(RM' + \frac{cR}{r^2} \nu \right) \end{aligned} \quad (49)$$

On comparing the coefficients of ν on both side of the equation (49), we find

$$R_3'' + \left(\frac{5}{r} - \frac{R_e P_r}{ac} \right) R_3' + \left(\frac{4}{r^2} \right) R_3 = 0 \quad (50)$$

$$\begin{aligned} & R_2'' + \left(\frac{3}{r} - \frac{R_e P_r}{ac} \right) R_2' + \frac{R_2}{r^2} \\ &= 4aMR_3' + 2aR_3 \left(M' + \frac{4M}{r} \right) - E_c P_r \left(\frac{c}{r^2} R \right) \end{aligned} \quad (51)$$

$$\begin{aligned} & R_1'' + \left(\frac{1}{r} - \frac{R_e P_r}{ac} \right) R_1' = \frac{2MR_2'}{c} + \frac{1}{c} \left(M' + \frac{2M}{r} \right) R_2 \\ &- \left(\frac{2(1+M^2)}{c^2} \right) R_3 - \frac{E_c P_r M'}{ac} R \end{aligned} \quad (52)$$

We use computer technology to solve equation (50). We apply computer algebra system (CAS) software Mathematica and find

$$\begin{aligned} R_3(r) &= \frac{a^2 c^2}{R_e^2 P_r^2} \left(\frac{1}{r^2} - \frac{R_e P_r}{ac} \right) \left(\frac{2}{r} \right) + \frac{R_e^2 P_r^2}{2a^2 c^2} E_2 \\ &+ E_3 \text{MeijerG}[\{\{\}, \{1\}\}, \{-2, 2\}, \{\}, \frac{-R_e P_r}{ac} r] \end{aligned} \quad (53)$$

Equation (51) is a non-homogeneous variable coefficient differential equation. Its homogeneous part is of the same form as of equation (50) whose solution is obtainable through CAS software. We further observe that the solution of equation (51) depends upon the value of the unknown function $R(r)$. Therefore, we set

$$R(r) = \frac{4ar^2 M}{cE_c P_r} R_3 + \frac{2ar^2}{cE_c P_r} \left(M' + \frac{4M}{r} \right) R_3 \tag{54}$$

And thus equation (51) becomes

$$R_2'' + \left(\frac{3}{r} - \frac{R_e P_r}{ac} \right) R_2' + \frac{R_2}{r^2} = 0 \tag{55}$$

The solution of the equation (55), using CAS software Mathematica is

$$R_2(r) = \frac{ac}{R_e P_r} \left(\frac{1}{r} - \frac{R_e P_r}{ac} \right) E_4 + E_5 \text{MeijerG}\left\{ \left\{ \left\{ \right\}, \{1\} \right\}, \left\{ \{-2, 2\}, \left\{ \right\} \right\}, \frac{-R_e P_r}{ac} r \right) \tag{56}$$

The solution of equation (52) is straightforward through reduction of order method

$$R_1(r) = \int \left\{ \frac{e^{\left(\frac{R_e P_r}{ac} \right) r}}{r} \int r e^{-\left(\frac{R_e P_r}{ac} \right) r} Z_1(r) dr \right\} dr + E_6 \int \left\{ \frac{e^{\left(\frac{R_e P_r}{ac} \right) r}}{r} \right\} dr + E_7 \tag{57}$$

Where

$$Z_1(r) = \frac{2MR_2'}{c} + \frac{1}{c} \left(M' + \frac{2M}{r} \right) R_2 - \left(\frac{2(1+M^2)}{c^2} \right) R_3 - \frac{E_c P_r M'}{ac} R \tag{58}$$

And $E_i, i \in \{1, 2, \dots, 7\}$ are constants. We find viscosity from equation (46) using equation (54), the temperature distribution T from equation (48) by supplying equation (53) and equations (55-58). We find the pressure p using equation (44) in equation (11) and velocity from equation (5) using equation (38) and equation (9).

4. Conclusion

We have used coordinate transformations technique on non-dimensional equations to determine a class of exact solutions for plane steady motion of incompressible fluids of variable viscosity in the presence of body force. For this class we have characterized the streamlines by equation $\theta = f(r) + g(r)v(\psi)$, in polar coordinates r, θ and took the function v of the stream function ψ as $v(\psi) = a\psi + b$ with constants $a \neq 0$ and b . We have determined exact solutions considering two cases based on velocity field shape for a suitable component of body force. The first velocity field shape demands $g(r) = \frac{c}{r}$ but leaves $f(r)$ arbitrary and provides

viscosity as function of temperature. Where as the second velocity field shape requires $g(r) = \frac{-1}{C_0 r^2 + C_1}$, $f(r) = n_1 g(r) + n_2$ and

presents viscosity and temperature. In both the cases, we can create infinite set of expressions for streamlines, the pressure, viscosity function, generalized energy function, temperature distribution and velocity vector field in the presence of body force. The symbols $c \neq 0, C_0 \neq 0, C_1, n_1$ and n_2 were constants.

We have applied the computer algebra system (CAS) software Mathematica to determine the solution of ordinary differential equations appearing due to the energy equation. Utilizing the CAS software, we can easily draw the streamlines pattern, find the effect of various parameters on the streamlines and discuss the flow characteristic.

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