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Generalized Synchronization of Bidirectionally Coupled Chaotic Systems

Mohammad Ali Khan[1], Swarup Poria[2]

[1]Department of Mathematics, Garhbeta Ramsundar Vidyabhaban,
West Bengal, India
E-mail: mdmaths@gmail.com
[2]Department of Applied Mathematics, University of Calcutta, Kolkata,
West Bengal, India
E-mail: swarup_p@yahoo.com

Abstract

In this paper, we develope the theory for generalized synchronization of bidirectionally coupled chaotic systems. The proposed theory is discussed taking Unified chaotic system and Rossler chaotic system as an example. Numerical simulation results are presented to show the feasibility and effectiveness of the approach. This synchronization method may be useful for sending secrete message and to understand synchronization of many biological systems, electronic circuits, chemical systems and potential applications to laser dynamics.

Keywords: *Bidirectionally coupled, Generalized synchronization (GS), Rossler system, Synchronization error and Unified chaotic system.*

1 Introduction

Since the pioneer work by Pecora and Carroll [1], chaos synchronization has received much attention because of its fundamental importance in non-linear dynamics and potential applications to laser dynamics, electronic circuits, chemical and biological systems and secure communication. Many chaos synchronization and control methods [2] have been developed, such as backstepping design method [3], impulsive control method [4], invariant manifold method [5], adaptive control method [6], active control method [7], synchronization in unidirectionally coupled systems [8] and bidirectionally coupled systems [9].

A pair of dynamical systems

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{Y} &= g(Y) \end{aligned} \tag{1}$$

are said to be unidirectionally coupled if

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{Y} &= g(Y) + h(X,Y) \end{aligned} \tag{2}$$

where $h(X,Y)$ is a nontrivial function of X and Y.

Physically, this means that in part of the phase space, the behavior of one system has no influence on the behavior of the other. If coupling is not unidirectional then it must be bidirectional. Systems are bidirectionally coupled if

$$\begin{aligned} \dot{X} &= f(X) + k(X,Y) \\ \dot{Y} &= g(Y) + h(X,Y) \end{aligned} \tag{3}$$

where $h(X,Y)$ and $k(X,Y)$ are nontrivial function of X and Y.

Two dynamical systems are called synchronized if the distance between the corresponding states of the systems converges to zero as time goes to infinity. This type of synchronization is known as identical synchronization [1]. However in the coupled chaotic systems identical synchronization is a fairly restrictive concept and often difficult to achieve except under ideal conditions. Recently a more elaborate form of synchronization called generalized synchronization (GS) was proposed by Kocarev and Parlitz [10]. They formulated a condition for the occurence of GS for the following systems

$$\begin{aligned} \dot{X} &= f(X) && \text{driving system} \\ \dot{Y} &= g(Y,U) = g(Y,h(X)) && \text{driven system} \end{aligned} \tag{4}$$

where $X \in R^n, Y \in R^m$ and $U(t) = (u_1(t), u_2(t), ...u_k(t))$ with $u_j = h_j(X(t,x_0))$. Here the variable u_j are introduced to include explicitly the case that a function $U = h(X)$ of X is used for driving the response system. According to Kocarev and Parlitz [10] the system (4) possess the property of GS between X and Y if there exists a transformation $H : R^n \to R^m$, a manifold $M = \{(X,Y) : Y = H(X)\}$, and a subset $B = B_X \times B_Y \subseteq R^n \times R^m$ with $M \subseteq B$ such that all trajectories of (4) with initial conditions in the basin B approach M as time t goes to infinity. If H equals to the identity transformation, this definition of generalized synchronization coincides with the usual definition of synchronization e.g. identical synchronization. Generalized synchronization (GS), which is defined by a time-independent nonlinear functional relation $Y = \Phi(X)$ between the states X and Y of two systems. Experimental detection and characterization of GS from observed data is a challenging problem, especially in biology; e.g., for study on nonlinear interdependence observed in binding of different features in cognitive process and epilepsies in the brain. In unidirectionally coupled system, a way to detect GS is to make an identical copy Y' of the response system Y driven by the common signal from the driver system X, then investigate whether orbits of both Y and Y' coincide after transient. Rulkov et.al. [11] discussed generalized synchronization of chaos in unidirectionally coupled chaotic systems. Hramov et.al. [12] proposed GS by a modified system approach. They investigated the physical reasons leading to GS appearance in unidirectionally coupled chaotic systems. In 2005 Hramov et.al. [13] explained the peculiarity of the GS onset in the unidirectionally coupled Rossler oscillators. Applications of GS may be more practical than those of identical synchronization because parameter mismatches and distortions always exist in the physical world. Yang and Chua [14] proposed a method for obtaining GS of two coupled chaotic systems via linear transformations. They also consider unidirectional coupled chaotic systems. Poria [15] discussed generalized chaos synchronization of two Lorenz dynamical systems via linear transformation considering unidirectional coupling. Recently in 2012 Khan et. al. [16] have proposed generalized anti-synchronization of different chaotic systems. There are very few results about synchronization of bidirectionally coupled chaotic systems. But most of the natural system are bidirectionally coupled. Therefore the study of bidirectionally coupled systems are necessary. In this paper, we develop the theory of generalized synchronization (GS) of bidirectionally coupled chaotic systems. We discuss the theory considering two bidirectionally coupled chaotic systems. Finally simulation results are presented and discussed.

2 Generalized Synchronization of Bidirectionally Coupled Chaotic Systems Design

Generalized synchronization is characterized by a constant matrix. Consider the following chaotic systems

$$\begin{aligned} \dot{X} &= f(X, Y, t) \leftarrow \quad \text{driving system} \\ \dot{Y} &= g(X, Y, t) \leftarrow \quad \text{driven system} \end{aligned} \tag{5}$$

where $X=(x_1,x_2,\dots\dots\dots,x_n)$ and $Y=(y_1,y_2,\dots\dots\dots\dots,y_n)$. If there exists a constant matrix α such that $\lim_{t\to\infty}\|X-\alpha Y\|=0$ then we call that two systems are in a state of generalized synchronization, where

$$\alpha=\begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix}.$$

Theorem 2.1

Consider the bidirectionally coupled chaotic systems in the form of

$$\dot{X} = AX + h(X,t) + u_1 \tag{6}$$

$$\dot{Y} = AY + g(Y,t) + u_2 \tag{7}$$

The generalized synchronization between the system (6) and (7) will occur for an invertible matrix α, if the following conditions are satisfied

(i). $u_1 = \alpha g(Y,t) + (\alpha A - A\alpha)Y$

(ii). $u_2 = \alpha^{-1} h(X,t) + \alpha^{-1} BK(X - \alpha Y)$

(iii). the real parts of all the eigen values of (A-BK) are negative.

Proof: Define the error $E = X - \alpha Y$ between the systems (6) and (7). Then we obtain $\dot{E} = AX - \alpha AY + h(X,t) - \alpha g(Y,t) + u_1 - \alpha u_2$.

Let $u_1 = \alpha g(Y,t) + (\alpha A - A\alpha)Y$ and $u_2 = \alpha^{-1} h(X,t) + \alpha^{-1} BK(X - \alpha Y)$. Therefore

$$\dot{E} = (A - BK)E \tag{8}$$

For a feasible control, the feedback gain K must be selected in such a way that all eigen values of $(A - BK)$ have negative real parts. Then the system (8) is asymptotically stable at the origin, which implies that (6) and (7) are in the state of generalized synchronization. i.e $\lim_{t\to\infty} \|X - \alpha Y\| = 0$.

3 Generalized Synchronization of Two Coupled Chaotic Unified Systems

In this section, we consider the well-known Unified chaotic system given by

$$\begin{aligned} \dot{x} &= (25a + 10)(y - x) \\ \dot{y} &= (28 - 35a)x - xz + (29a - 1)y \\ \dot{z} &= xy - \frac{8 + a}{3} z \end{aligned} \tag{9}$$

where $a \in [0,1]$. For a =0, 0.8, 1 the system (9) represents the Lorenz chaotic system, Lu chaotic system and Chen chaotic system respectively. Practically, Unified chaotic system is chaotic for any $a \in [0,1]$. The system (9) can be written as $\dot{X} = AX + h(X,t)$

where

$$A=\begin{bmatrix} -(25a+10) & 0 & 0 \\ (28-35a) & -1 & 0 \\ 0 & 0 & \dfrac{-(8+a)}{3} \end{bmatrix}, \text{h(x,t)}=\left[(25a+10)y, \quad -xz+29ay, \quad xy\right]^T \quad (10)$$

Following Theorem 1, the driving system and driven system can be constructed as the form of equation (6) and (7) in the following way

$$\dot{x}_1=(25a+10)(x_2-x_1+\alpha_{11}y_2)+\alpha_{12}\{(28-35a)y_1+(54a+9)y_2-y_1y_3\}+\alpha_{13}(y_1y_2+\frac{74a+22}{3}y_3)$$

$$\dot{x}_2=(28-35a)x_1-x_1x_3+(29a-1)x_2+\alpha_{21}\{(25a+10)y_2-(25a+9)y_1\}+\alpha_{22}\{29ay_2-y_1y_3+(28-35a)y_1\}$$
$$+\alpha_{23}(y_1y_2-\frac{5+a}{3}y_2)+\alpha_{11}(35a-28)y_1+\alpha_{12}(35a-28)y_2+\alpha_{13}(28-35a)y_3$$

$$\dot{x}_3=x_1x_2-\frac{8+a}{3}x_3+\alpha_{31}\{(25a+10)y_2-\frac{74a+22}{3}y_1\}+\alpha_{32}\{\frac{88a+5}{3}y_2+(28-35a)y_1-y_1y_3\}+\alpha_{33}y_1y_2$$

(11)

$$\text{where} \quad \alpha=\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \text{ and } \quad g(y,t)=\begin{bmatrix} (25a+10)y_2 \\ -y_1y_3+29ay_2 \\ y_1y_2 \end{bmatrix} \quad (12)$$

and

$$\dot{y}_1=(25a+10)(y_2-y_1)+\left[\sum_{i=1}^{3}k_i(x_i-\alpha_{i1}y_1-\alpha_{i2}y_2-\alpha_{i3}y_3)\right](\sum_{i=1}^{3}b_i\beta_{1i})$$

$$\dot{y}_2=(28-35a)y_1-y_1y_3+(29a-1)y_2+\left[\sum_{i=1}^{3}k_i(x_i-\alpha_{i1}y_1-\alpha_{i2}y_2-\alpha_{i3}y_3)\right](\sum_{i=1}^{3}b_i\beta_{2i})$$

$$\dot{y}_3=y_1y_2-\frac{8+a}{3}y_3+\left[\sum_{i=1}^{3}k_i(x_i-\alpha_{i1}y_1-\alpha_{i2}y_2-\alpha_{i3}y_3)\right](\sum_{i=1}^{3}b_i\beta_{3i})$$

(13)

$$\text{where} \quad \alpha^{-1}=\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}, B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ and } K=\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}^T \quad (14)$$

$$\text{Choose A=}\begin{bmatrix} -(25a+10) & 0 & 0 \\ 28-35a & -1 & 0 \\ 0 & 0 & -\dfrac{8+a}{3} \end{bmatrix}, \text{ B=}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{K=}\begin{bmatrix} 14 \\ 11 \\ 2 \end{bmatrix}^T \text{ and } \alpha=\begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & 9 \\ 1 & 0 & 0 \end{bmatrix}$$

Then the error system of (8) can be written as

$$\begin{aligned}\dot{e}_1 &= -(25a+24)e_1 + (25a-1)e_2 - 2e_3 \\ \dot{e}_2 &= (14-35a)e_1 - 12e_2 - 2e_3 \\ \dot{e}_3 &= -14e_1 - 11e_2 - \frac{14+a}{3}e_3\end{aligned} \tag{15}$$

where

$$\begin{aligned}e_1 &= x_1 - \alpha_{11}y_1 - \alpha_{12}y_2 - \alpha_{13}y_3 \\ e_2 &= x_2 - \alpha_{21}y_1 - \alpha_{22}y_2 - \alpha_{23}y_3 \\ e_3 &= x_3 - \alpha_{31}y_1 - \alpha_{32}y_2 - \alpha_{33}y_3\end{aligned} \tag{16}$$

Now we give numerical simulation results to show the effectiveness of the proposed theory. Here Fourth-order Runge-Kutta method is used to solve the system (15) with time step being equal to .001. The initial synchronization error are taken as $(e_1(0), e_2(0), e_3(0)) = (20, 2, -10)$, then the corresponding numerical simulation results are shown in Fig.1(a-c). Fig.1(a), Fig.1(b) and Fig.1(c) shows that the time evolution of the synchronization error $e = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T$ tends to zero as time goes to infinity for $a = 0, a = 0.8$ and for $a = 1$. Therefore generalized synchronization between the systems (11) and (13) are achieved.

Theorem 3.1

Consider the driving system in the form of

$$\dot{X} = AX + h(X,t) + k_1(X,Y) \tag{17}$$

where $X \in R^n$, A is an $n \times n$ constant matrix. Assume that the driven system coupled with (17) is as follows

$$\dot{Y} = AY + g(Y,t) + k_2(X,Y) + u_2 \tag{18}$$

Then for an invertible matrix α, generalized synchronization between the systems (17) and (18) will occur if the following conditions are satisfied.

(i). $u_2 = \alpha^{-1}[h(X,t) + k_1(X,Y)] + g(Y,t) + k_2(X,Y) - AY + \alpha^{-1}A\alpha Y$

(ii). The real parts of all the eigen values of A are negative.

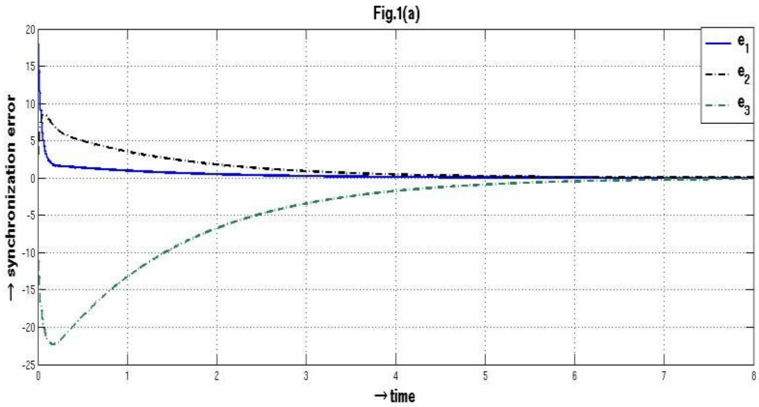

Fig.1(a): Generalized synchronization error of bidirectional coupled unified chaotic system for a=0.0 i.e for the Lorenz chaotic system.

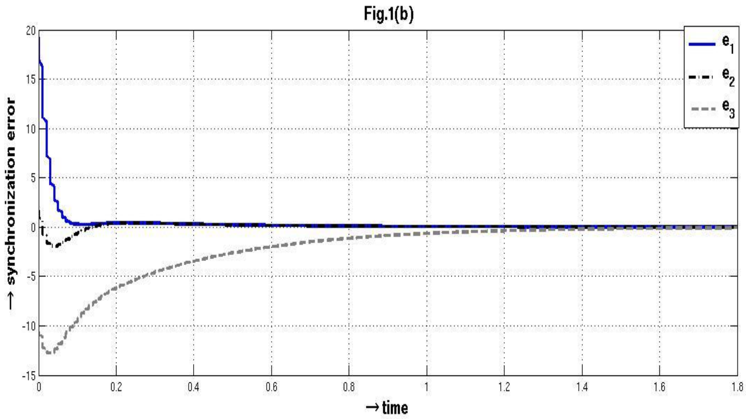

Fig.1(b): Generalized synchronization error of bidirectional coupled unified chaotic system for a=0.8 i.e for the Lu chaotic system

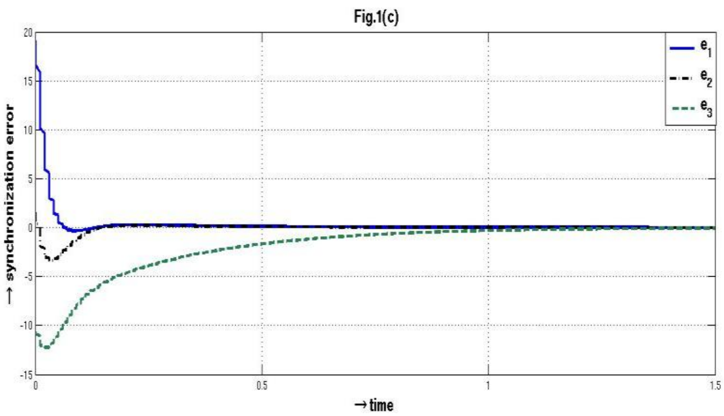

Fig.1(c): Generalized synchronization error of bidirectional coupled unified chaotic system for a=1.0 i.e for the Chen chaotic system.

Proof:

Define the error $E = X - \alpha Y$ between the system (17) and (18) then we obtain

$$\dot{E} = AX + h(X,t) + k_1(X,Y) - \alpha(AY + g(Y,t) + k_2(X,Y) + u_2) \tag{19}$$

Let $u_2 = \alpha^{-1}[h(X,t) + k_1(X,Y)] - g(Y,t) - k_2(X,Y) - AY + \alpha^{-1}A\alpha Y$

Therefore the equation (19) becomes

$$\dot{E} = AE \tag{20}$$

For a feasible control, we select A such that all eigen values of A, if any have negative real parts then the system (19) is asymptotically stable at the origin, which implies that (17) and (18) are in the state of generalized synchronization.

4 Generalized Synchronization of Coupled Rossler Systems

In the second example, we study generalized synchronization of two chaotic Rossler systems to discuss the above theory. Otto Rossler [17], proposed the following system which is known as Rossler system.

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + ax_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \end{aligned} \tag{21}$$

where a, b and c are three positive parameters. This system contains only one non-linear term x_1x_3 and is even simpler than Lorenz system which has two non-linear term. In the culinary spirit of the pastry map and the backer's map Otto Rossler found inspiration in a tuffy –putting machine. By pondering its action, he found the above set of three differential equations. The system (21) can be written as $\dot{X} = AX + h(X,t)$

where $$A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad h(X,t) = \begin{bmatrix} x_1 \\ (a+1)x_2 \\ b + x_1x_3 \end{bmatrix} \tag{22}$$

Choosing $k_1(X,Y) = \begin{bmatrix} y_1 - x_1 \\ (a+1)(y_2 - x_2) \\ y_1y_3 - x_1x_3 \end{bmatrix}$, $k_2(X,Y) = \begin{bmatrix} x_1 \\ (a+1)x_2 \\ b + x_1x_3 \end{bmatrix}$ and according to Theorem 2, the driving system and driven system can be constructed as

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 - x_3 + y_1\\ \dot{x}_2 &= x_1 - x_2 + (a+1)y_2\\ \dot{x}_3 &= -cx_3 + b + y_1 y_3\end{aligned} \tag{23}$$

and

$$\begin{aligned}\dot{y}_1 &= \beta_{11}\left[y_1 - (\sum\alpha_{1i}y_i + \sum\alpha_{2i}y_i + \sum\alpha_{3i}y_i)\right] + \beta_{12}\left[(a+1)y_2 + (\sum\alpha_{1i}y_i - \sum\alpha_{2i}y_i)\right] + \beta_{13}\left[(b + y_1y_3) - c(\sum\alpha_{3i}y_i)\right]\\ \dot{y}_2 &= \beta_{21}\left[y_1 - (\sum\alpha_{1i}y_i + \sum\alpha_{2i}y_i + \sum\alpha_{3i}y_i)\right] + \beta_{22}\left[(a+1)y_2 + (\sum\alpha_{1i}y_i - \sum\alpha_{2i}y_i)\right] + \beta_{23}\left[(b + y_1y_3) - c(\sum\alpha_{3i}y_i)\right]\\ \dot{y}_3 &= \beta_{31}\left[y_1 - (\sum\alpha_{1i}y_i + \sum\alpha_{2i}y_i + \sum\alpha_{3i}y_i)\right] + \beta_{32}\left[(a+1)y_2 + (\sum\alpha_{1i}y_i - \sum\alpha_{2i}y_i)\right] + \beta_{33}\left[(b + y_1y_3) - c(\sum\alpha_{3i}y_i)\right]\end{aligned} \tag{24}$$

Where

$$\alpha = \begin{bmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{bmatrix}, \alpha^{-1} = \begin{bmatrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{bmatrix} \text{ and } g(Y,t) = \begin{bmatrix} y_1\\ (a+1)y_2\\ b + y_1y_3\end{bmatrix} \tag{25}$$

Therefore the error dynamical system can be written as

$$\begin{aligned}\dot{e}_1 &= -e_1 - e_2 - e_3\\ \dot{e}_2 &= e_1 - e_2\\ \dot{e}_3 &= -ce_3\end{aligned} \tag{26}$$

where

$$\begin{aligned}e_1 &= x_1 - \alpha_{11}y_1 - \alpha_{12}y_2 - \alpha_{13}y_3\\ e_2 &= x_2 - \alpha_{21}y_1 - \alpha_{22}y_2 - \alpha_{23}y_3\\ e_3 &= x_3 - \alpha_{31}y_1 - \alpha_{32}y_2 - \alpha_{33}y_3\end{aligned} \tag{27}$$

Now we give numerical simulation results to discuss the matter. Here Fourth order Runge-Kutta method is used to solve the system (26). Let $\alpha = \begin{bmatrix}1 & 3 & 5\\ 2 & 7 & 9\\ 1 & 0 & 0\end{bmatrix}$ and the parameter of Rossler system are chosen as $a = b = 0.2$ and $c = 5$. The initial states for the synchronization error system are given by $(e_1(0), e_2(0), e_3(0)) = (20, 2, -10)$, then the corresponding numerical simulation results are shown in Fig.2. Figure shows the time evolution of the synchronization error $e = \begin{bmatrix}e_1 & e_2 & e_3\end{bmatrix}^T$ tends to zero as time goes to infinity for the Rossler chaotic system.

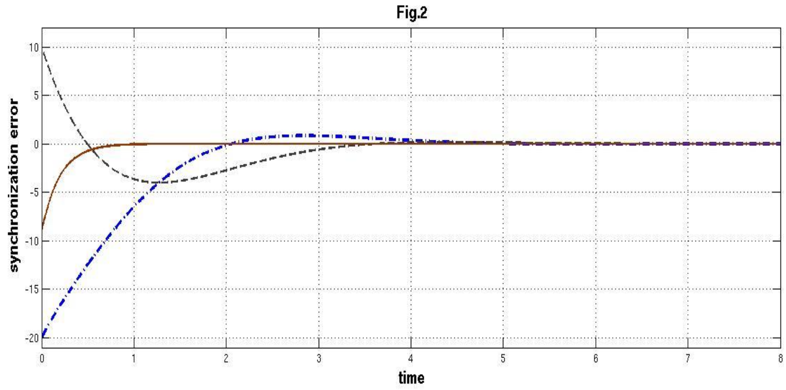

Fig.2: Generalized synchronization error of bidirectional coupled Rossler chaotic system.

5 Conclusion

We have proposed a scheme for generalized synchronization of bidirectionally coupled chaotic systems. We discuss the theories considering coupled Unified chaotic system and Rossler system. Our numerical simulation results show the efficiency of the proposed scheme. This scheme may be useful for sending secrete message and to understand synchronization of many biological systems *e.g.* for study of nonlinear interdependence observed in binding of different features in cognitive process and epilepsies in the brain.

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