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Research Paper

Fuzzy spanners

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Abstract

Graphs have wide applications in variety of applications, especially network analysis. Spanners give a sparse approximation of any graph as well as complete graph, which has wide applications so far. Because of its applications, graph definitions generalized to fuzzy sets and fuzzy weighted graph defined and studied vastly in past and has applications in several fields. In this paper, we define spanners for fuzzy weighted graphs and well as points come from a fuzzy metric space. We propose some algorithm to compute such a spanner. Finally, we pose some open problem that can be considered for future works.

Keywords: *networks, graphs, fuzzy spanners, greedy algorithm, fuzzy weighted graphs.*

1. Introduction

A network on a set V of n points can be modeled as an undirected graph G with vertex set V of size n and an edge set E of size m where every edge $e = (u, v)$ has a weight $wt(e)$. A network is called a geometric or Euclidean network if the weight of the edge $e = (u, v)$ is the Euclidean distance $|uv|$ between its endpoints u and v.

There are several ways to measure the quality of a network; one is the dilation or stretch factor of the network. For each pair of points $u, v \in V$, the ratio of the length of the shortest path between u and v in G, denoted by $\mathbf{d}_G(u, v)$, to the distance between u and v is called the dilation between u and v in G. The length of a path is the summation of the length of all edges on the path. The maximum dilation between all pairs of vertices of G is called the dilation of G. Informally, in a network with low dilation, the amount of detour one might need to take to travel from one node to another node of the graph, when it is only allowed to travel along edges of the graph, is close to actual Euclidean distance between the nodes.

We can have this measure if vertices of the graph come from any metric space and the weight of each edge in the graph is the distance between its endpoints in the metric space.

Let $t \geq 1$ be a real number. A network G is called a t-spanner on vertices of G, if the dilation of G is at most t. In other words, a graph $G(V, E)$ is a t-spanner of V, if for each pair of vertices $u, v \in V$, there exists a path in G between u and v of weight at most $t \cdot |uv|$. We call such a path a *t-path* between u and v. Finally, a subgraph G' of a given graph G is a t-spanner for G if for each pair of points $u, v \in V$, there exists a path in G' of weight at most $t \times \mathbf{d}_G(u, v)$. Based on the definition of a t-spanner of V, a t-spanner of V is a t-spanner of the complete graph on V.

The main objective of studying spanners is to construct sparse t-spanners with t sufficiently close to 1. The sparseness measures are between the weight of the spanner, i.e. sum of weights of all of its edges; the number of edges and the maximum degree. Spanners have applications in several fields: from robotics, network topology design, distributed systems and design of parallel machines to metric space searching [13] and broadcasting in communication networks [11].

The spanner concept can be generalized to any fuzzy graph. The most natural generalization is to consider the weight of each edge to be a fuzzy number. Despite different definitions of fuzzy numbers used for edge weights, we can define spanners. The organization of the paper is as follows. In the next section, we define fuzzy spanner and then we focus on algorithms to compute such a spanners. The we conclude with some open problems for future works.

2. Fuzzy spanners

To add fuzziness to spanner concept, one can use several ways. One way is to replace the vertex set of the graph with a fuzzy point set. This means, each point belong to the vertex set based on a fuzzy relation. This can have applications in networks whose vertices act like a fuzzy element. Based on this, one need to define paths between pairs of points and then use the following definition to define a fuzzy spanner.

Another way, which we focus on here, is that, we have a normal point set as vetrices of the fuzzy graph, but instead of having real numbers as edge weights, we have fuzzy numbers. This kind of fuzzy weighted spanners defined and used vastly before, see for example, [5, 3, 9, 14]. There are also some other generalization, called intuitionistic fuzzy graph, which defined and used before, see [15, 8, 7]. Even more, there some definitions that allows fuzzy numbers from different fuzzy sets of numbers as edge weights, see [17].

Despite different generalization that might exist now or may propose in the future, one can use the following definition to define a t-spanner, if there exists a concept of path length in the model and also a way to compare path length to another number.

Definition 2.1 (Fuzzy t-spanner) *Let $G(V, E)$ be a fuzzy weighted graph and $t > 1$ be a real number. We call $G'(V, E')$ a t-spanner of G, $E' \subset E$, if for each pair $u, v \in V$, there is a path between u and v in G' with length at most t times the shortest path distance between u and v in G.*

Here, we consider fuzzy weighted graph that assigned a fuzzy number to each edge in the graph. This definition makes sense in applications like road network, where the distance between any two vertices depends on traffic, time and other parameters, so assigning a fuzzy number as edge weight is logical.

Note that finding the shortest path in the graph can be used to see if there is a t-path between each pairs of points. So far, there are several ways to define and compute shortest paths in a fuzzy weighted graph, see [14, **?**]. However, in any algorithm for solving the shortest path problem, one will face with various ranking methods between fuzzy numbers, which have been proposed in literature. There are several papers on designing a way to solve it. For example, Dubois and Prade [5] proposed a way based on "fuzzy min". As mentioned in the paper by Okada and Soper [14], this relation leads to the concept of a nondominated path or Pareto Optimal path, and an algorithm for solving fuzzy shortest path problems is derived on the basis of the multiple labeling method for a multicriteria shortest path problem. The problem with this way is that at the end of the algorithm for finding the shortest path between a pair of points, one can see a number of nondominated paths and this number can be high for a large scale network. In the same paper, they proposed a method to reduce the number of paths according to a possibility level. They proposed an algorithm to compute shortest paths, but they did not say anything about the time complexity of the algorithm. However, their algorithm is similar to Dijkstra's algorithm for computing the shortest path. There are some similar results in [10].

In 2004, Cornelis *et al.*[3] proposed a defuzzification method that not only reduced the complexity of the fuzzy shortest path problem to that of the crisp single-criterion labeling algorithms but also serve to give more insight into heuristic approaches.

Tajdin *et al.*[17] proposed an approach for adding various fuzzy numbers approximately using α-cuts. The approximation was based on fitting an appropriate model for the sum using α-cuts of the addition. This enables edges of the graph to have weights from different fuzzy systems. The following dynamic programming algorithm is for computing the shortest path in a network. They also presented an algorithm which is based on Floyds dynamic programming method to find a shortest path between every pair of nodes in the fuzzy graph. The time complexity of the algorithm shows the number of summation and comparisons that it does, so its time complexity is dependent on the time complexity of computing summation and comparing of two fuzzy numbers.

Recently, Biswas and Doja [15] found a way to apply classical Dijkstras algorithm to graphs with intuitionistic fuzzy number edge weights. They claimed that the method may play a major role in many application areas of computer science, communication network, transportation systems, and so forth. in particular to those networks for which the link weights (costs) are ill defined.

3. Algorithms for computing fuzzy spanners

There are several algorithms that given a set of points and a $t > 1$, compute a t-spanner of the points set. Most of the algorithms work in the geometric case, i.e., the point set comes from $\mathbb{R}^d$, see the book by Narasimhan and Smid [12]. Except theoretical results, the algorithms compared in practice based of different properties, like number of edges, maximum degree and weight of generated graph and also the running time of them, see [6, 13, 16].

The algorithms for computing spanners used several techniques, from checking shortest path between all pairs of points in the graph to the algorithms that work based on geometric properties of the vertices of the graph. For example to compute a t-spanner of graph G, the greedy algorithm [1] start from an empty edge set graph and check all edges of G in increasing order based on their weights. It add the edge to the new graph if there is no t-path in the graph between the pair of points. For details of the algorithm see Algorithm 3.1.

Algorithm 3.1: GREEDY

Input: $G(V, E)$ and $t > 1$.
Output: t-spanner $G' = (V, E')$.

Sort the edges in E by increasing weight; `/* ties are broken arbitrarily */`
$E' := \emptyset$;
$G' := (V, E')$;
foreach $(u, v) \in E'$ **do** `/* in sorted order */`
  if SHORTESTPATHLENGTH$(G', u, v) > t \cdot weight(u, v)$ **then**
    $E := E \cup \{(u, v)\}$;
  end
end
return $G' = (V, E')$;

The greedy algorithm needs two things: computing shortest path length and comparing the outcome with a number which guaranty the t-spannerness. So it can be applied for any weighted fuzzy graph, if it gives us these two possibilities. Based on the results on computing shortest path in different fuzzy weighted graphs and also the possibility to compare fuzzy numbers, the greedy algorithm can apply on them without any changes.

The time complexity of the algorithm, however, depend on the time complexity of computing the shortest path between any pair of points. More precisely, the greedy algorithm performs $\mathcal{O}(|E|)$ shortest path queries, so based on the time complexity of computing shortest path, one can get the time complexity of the greedy algorithm.

The greedy algorithm can apply on any points set that have any kind of distance between pair of points. Using the distances between pairs of points, pairs of points are sorted based on distances and then checked in the algorithm. For example, if we have a points from a fuzzy metric space, see [4], one can uses the greedy algorithm to construct the greedy spanner of the point set. Note that the input of the greedy algorithm is the complete graph on the pointy set, where the weight of each edge is the distance between its endpoints in the metric space.

However, the upper bound on the number of edges in the generated t-spanner depends on the fuzzy system. One major open problem is to find a non-clear bound on the number of edges of the t-spanner generated by the greedy algorithm. Note that this bound for points in the plane is linear which the best one can have because the generated graph should make a connected graph which need linear edges.

There are several other algorithm that work on geometric properties of the vertices of the graph. To apply these algorithm, like Θ-graph algorithm, for fuzzy point sets, one need to have some geometric properties on points which seems very unclear. It would be nice if one can find some kind of geometry on fuzzy point set, at least for some specific definition and then try to apply these algorithm on the point set.

Another algorithm which seems more suitable for fuzzy point set is the WSPD- algorithm. The well-separated pair decomposition (WSPD) was developed by Callahan and Kosaraju [2]. It decompose the set of all pairs of points to a set of pairs of subsets of the point set such that each pair of points appears in at least one pairs of sets. The pairs of sets are some far away compare to the distances between points in each set. So, for points from a fuzzy metric space, one might be able to use similar techniques to compute WSPD of the points sets. Using the WSPD, constructing the spanner is easy. It just add an edge between one pair of point from each pairs of sets in WSPD. Note that there is a parameter for computing WSPD, which will be dependent on t, when we want to use it to construct a t-spanner. For details of the algorithm, one can see [12].

4. Conclusion and future works

In this paper, we defined spanners for fuzzy points sets and edge weighted graphs. Since the spanners have wide applications, it is natural that its generalization to fuzzy points sets also find weveral application. We used a greedy algorithm to construct the spanner for fuzzy weighted graph as well as on points sets from a fuzzy metric space.

As future works, one can study finding the bound on the number of edges in the graph generated by the greedy algorithm. Also there are several theoretical results on greedy spanner that seems interesting to check in fuzzy structure, like containing the minimum spanning tree. Also, generalizing other algorithm on fuzzy point sets are very interesting and we expect it needs vast effort to overcome it.

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