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He's semi-inverse method for Camassa-Holm equation and simplified modified Camassa-Holm equation

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Abstract

In this Letter, we study Camassa-Holm equation and Simplified Modified Camassa-Holm equation by using the well-known He's Semi-inverse Method. The solitary solutions are obtained using the Ritz method. In fact, the He's Semi-inverse Method is a promising method to various systems of linear and nonlinear equations.

Keywords: *Hes Semi-inverse Method, Solitary solution, Soliton equation.*

1 Introduction

Many important phenomena and dynamic processes in physics, mechanics, chemistry and biology can be represented by nonlinear partial differential equations. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models.
The study of exact solutions of nonlinear partial differential equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models. Recently, many kinds of powerful methods have been proposed to find exact solutions of nonlinear partial differential equations, e.g., the tanh-method [1], the homogeneous balance method [2], homotopy analysis method [3, 4, 5, 6, 7, 8, 9], the $F-$expansion method [10], three-wave method [11, 12, 13], extended homoclinic test approach [14, 15, 16], the $(\frac{G'}{G})-$expansion method [17] and the exp-function method [18, 19, 20, 21, 22].
The outline of the present paper is as follows. In Sec. 2, we have a brief review on Camassa and Holm Equations.we introduce the He's Semi-inverse Method in Sec. 3. In Secs. 4 and 5 we apply the He's Semi-inverse Method on the Camassa-Holm equation and Simplified Modified Camassa-Holm equation, respectively. In those sections we obtain new and exact solutions for our equations. The paper is concluded in Sec. 6.

2 Camassa and Holm equations

Camassa and Holm [23] derived a completely integrable wave equation (CH equation) for water waves

$$u_t + 2\,\alpha\, u_x - u_{xxt} + b\, u\, u_x = 2\, u_x\, u_{xx} + u\, u_{xxx}, \tag{1}$$

by retaining two terms that are usually neglected in the small amplitude, shallow water limit. Tian and Song [24] investigated a modified CamassaHolm equation (MCH equation)

$$u_t + 2\,\alpha\, u_x - u_{xxt} + b\, u^n\, u_x = 2\, u_x\, u_{xx} + u\, u_{xxx}, \tag{2}$$

and obtained new peaked solitary wave solutions. In addition, Boyd [25] investigated that if the solitary wave varies slowly with $\xi = x - c\,t$, then the two extra terms on the right-hand side of (1) will be small and the soliton is given to lowest order by the solutions of

$$u_t + 2\,\alpha\,u_x - u_{xxt} + b\,u\,u_x = 0. \tag{3}$$

In view of (3), Wazwaz [26] investigated a modified form of Camassa-Holm equation, which is simplified from MCH equation and given by

$$u_t + 2\,\alpha\,u_x - u_{xxt} + b\,u^n\,u_x = 0. \tag{4}$$

In this paper, we only consider$n = 2$,

$$u_t + 2\,\alpha\,u_x - u_{xxt} + b\,u^2\,u_x = 0. \tag{5}$$

and for simplicity we call (5) simplified MCH equation. (For more details see [27])

3 Description of He's semi-inverse method

We suppose that the given nonlinear partial differential equation for to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0, \tag{6}$$

where P is a polynomial in its arguments. The essence of He's semi-inverse method can be presented in the following steps:

Step 1. Seek solitary wave solutions of (6) by taking $u(x,t) = U(\xi), \xi = x - c\,t$ and transform (6) to the ordinary differential equation

$$U(u, u^{'}, u^{''}, ...) = 0, \tag{7}$$

where prime denotes the derivative with respect to ξ.

Step 2. If possible, integrate (7) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3. According to Hes semi-inverse method, we construct the following trial-functional

$$J(U) = \int L\,d\xi, \tag{8}$$

where L is an unknown function of U and its derivatives.
There exist alternative approaches to the construction of the trial-functionals, see Refs. [28, 29].

Step 4. By the Ritz method, we can obtain different forms of solitary wave solutions, in the form

$$U(\xi) = p\,sech^n(q\,\xi), \tag{9}$$

where P and q are constants to be further determined.
Substituting (9) into (8) and making J stationary with respect to P and q results in

$$\frac{\partial J}{\partial p} = 0, \tag{10}$$

$$\frac{\partial J}{\partial q} = 0. \tag{11}$$

Solving simultaneously (10) and (11) we obtain and . Hence, the solitary wave solution (9) is well determined.

4 New and exact solutions to the Camassa-Holm equation

In order to seek exact solutions of Camassa-Holm equation,

$$u_t + 2\,\alpha\,u_x - u_{xxt} + b\,u\,u_x = 0. \tag{12}$$

We suppose

$$u(x,t) = u(\xi) \quad , \quad \xi = x - w\,t, \tag{13}$$

where w is complex constants to be determined later. Substituting (13) into (12), we have

$$-w\,u^{'} + 2\,\alpha\,u^{'} + w\,u^{'''} + b\,u\,u^{'} = 0, \tag{14}$$

or equivalently

$$(2\,\alpha - w)\,u^{'} + w\,u^{'''} + b\,u\,u^{'} = 0, \tag{15}$$

where prime denotes the differential with respect to ξ. Integrating (15) with respect to ξ and taking the integration constant as zero yields

$$(2\,\alpha - w)\,u + w\,u^{''} + \frac{b}{2}\,u^2 = 0, \tag{16}$$

According to Ref. [28], By He's semi-inverse method [29], we can arrive at the following variational formulation:

$$J(\phi) = \int_0^{\infty}\left[-\frac{w}{2}\,(u^{'})^2 + \frac{(2\,\alpha - w)}{2}\,u^2 + \frac{b}{6}\,u^3\right]\,d\xi. \tag{17}$$

We assume the soliton solution in the following form

$$\phi(\xi) = p\,sech^2(q\,\xi), \tag{18}$$

where p, q are an unknown constant to be further determined.
By Substituting (18) into (17) we obtain

$$\begin{aligned} J &= \textstyle\int_0^{\infty}\left(2\,p^2 w\,q^2 + \frac{1}{6}\,b\,p^3\right) sech^6\,(q\xi)\;d\xi + \textstyle\int_0^{\infty}\left(-2\,p^2 w\,q^2 + p^2\,\alpha - \frac{1}{2}\,w\,p^2\right) sech^4\,(q\xi)\;d\xi \\ &= \frac{1}{q}\left(2\,p^2 w\,q^2 + \frac{1}{6}\,b\,p^3\right)\textstyle\int_0^{\infty} sech^6\,(\theta)\,d\theta + \frac{1}{q}\left(-2\,p^2 w\,q^2 + p^2\,\alpha - \frac{1}{2}\,w\,p^2\right)\textstyle\int_0^{\infty} sech^4\,(\theta)\,d\theta \\ &= \frac{8}{15\,q}\left(2\,p^2 w\,q^2 + \frac{1}{6}\,b\,p^3\right) + \frac{2}{3\,q}\left(-2\,p^2 w\,q^2 + p^2\,\alpha - \frac{1}{2}\,w\,p^2\right), \end{aligned} \tag{19}$$

For making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p} = \frac{2\,p}{15\,q}\,\left(-4\,wq^2 + 10\,\alpha - 5\,w + 2\,b\,p\right), \tag{20}$$

$$\frac{\partial J}{\partial q} = -\frac{p^2}{45\,q^2}\,\left(12\,w\,q^2 + 30\,\alpha - 15\,w + 4\,b\,p\right), \tag{21}$$

or simplifying

$$-4\,wq^2 + 10\,\alpha - 5\,w + 2\,b\,p = 0, \tag{22}$$

$$12\,w\,q^2 + 30\,\alpha - 15\,w + 4\,b\,p = 0, \tag{23}$$

From (22) and (23), we can easily obtain the following relations:

$$p = -\frac{3\,(w - 2\,\alpha)}{b} \quad , \quad q = \pm\frac{1}{2}\,\frac{\sqrt{w\,(w - 2\,\alpha)}}{w}, \tag{24}$$

Therefore by (18), the solitary wave solutions can be approximated as

$$u(\xi) = -\frac{3\,(w - 2\,\alpha)}{b}\,sech^2\left[\pm\frac{1}{2}\,\frac{\sqrt{w\,(w - 2\,\alpha)}}{w}\,(x - w\,t)\right], \tag{25}$$

for some arbitrary constants w.

5 New and exact solutions to the simplified MCH equation

In order to seek exact solutions of simplified MCH equation,

$$u_t + 2\,\alpha\,u_x - u_{xxt} + b\,u^2\,u_x = 0. \tag{26}$$

We suppose

$$u(x,t) = u(\xi) \quad , \quad \xi = k\,x - w\,t, \tag{27}$$

where w is complex constants to be determined later. Substituting (27) into (26), we have

$$-w\,u^{'} + 2\,\alpha\,u^{'} + w\,u^{'''} + b\,u^2\,u^{'} = 0, \tag{28}$$

or equivalently

$$(2\,\alpha - w)\,u^{'} + w\,u^{'''} + b\,u^2\,u^{'} = 0, \tag{29}$$

where prime denotes the differential with respect to ξ. Integrating (29) with respect to ξ and taking the integration constant as zero yields

$$(2\,\alpha - w)\,u + w\,u^{''} + \frac{b}{3}\,u^3 = 0, \tag{30}$$

According to Ref. [28], By He's semi-inverse method [29], we can arrive at the following variational formulation:

$$J(\phi) = \int_0^\infty \left[-\frac{w}{2}\,(u^{'})^2 + \frac{(2\,\alpha - w)}{2}\,u^2 + \frac{b}{12}\,u^4\right]\,d\xi. \tag{31}$$

We assume the soliton solution in the following form

$$\phi(\xi) = p\,sech(q\,\xi), \tag{32}$$

where p, q are an unknown constant to be further determined.
By Substituting (32) into (31) we obtain

$$\begin{aligned} J &= \textstyle\int_0^\infty \left(\frac{1}{2}\,p^2 w\,q^2 + \frac{1}{12}\,b\,p^4\right) sech^4\,(q\xi)\;d\xi + \int_0^\infty \left(-\frac{1}{2}\,p^2 w\,q^2 + p^2\,\alpha - \frac{1}{2}\,w\,p^2\right) sech^2\,(q\xi)\;d\xi \\ &= \frac{1}{q}\left(\frac{1}{2}\,p^2 w\,q^2 + \frac{1}{12}\,b\,p^4\right) \textstyle\int_0^\infty sech^4\,(\theta)\,d\theta + \frac{1}{q}\left(-\frac{1}{2}\,p^2 w\,q^2 + p^2\,\alpha - \frac{1}{2}\,w\,p^2\right)) \int_0^\infty sech^2\,(\theta)\,d\theta \\ &= \frac{2}{3\,q}\left(\frac{1}{2}\,p^2 w\,q^2 + \frac{1}{12}\,b\,p^4\right) + \frac{1}{q}\left(-\frac{1}{2}\,p^2 w\,q^2 + p^2\,\alpha - \frac{1}{2}\,w\,p^2\right), \end{aligned} \tag{33}$$

For making J stationary with respect to p and q results in

$$\frac{\partial J}{\partial p} = \frac{p}{9\,q}\,\left(-3\,wq^2 + 18\,\alpha - 9\,w + 2\,b\,p^2\right), \tag{34}$$

$$\frac{\partial J}{\partial q} = -\frac{p^2}{18\,q^2}\,\left(3\,w\,q^2 + 18\,\alpha - 9\,w + b\,p^2\right), \tag{35}$$

or simplifying

$$-3\,wq^2 + 18\,\alpha - 9\,w + 2\,b\,p^2 = 0, \tag{36}$$

$$3\,w\,q^2 + 18\,\alpha - 9\,w + b\,p^2 = 0, \tag{37}$$

From (36) and (37), we can easily obtain the following relations:

$$p = \pm\frac{\sqrt{6\,b\,(w - 2\,\alpha)}}{b} \quad , \quad q = \pm\,\frac{\sqrt{w\,(w - 2\,\alpha)}}{w}, \tag{38}$$

Therefore by (32), the solitary wave solutions can be approximated as

$$u(\xi) = \pm\frac{\sqrt{6\,b\,(w - 2\,\alpha)}}{b}\,sech\left[\pm\,\frac{\sqrt{w\,(w - 2\,\alpha)}}{w}\,(x - w\,t)\right], \tag{39}$$

for some arbitrary constants w.

6 Conclusion

In summary, by means of the modification of He's semi-inverse method, we obtained new and exact solutions for the Camassa Holm equation and Simplified Modified Camassa-Holm equation. He's semi-inverse method is a very dominant instrument to find the solitary solutions for various nonlinear equations.

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