Transverse vibration analysis of FGM plates with in-plane exponentially non-homogeneous material

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Abstract

In this research, free vibration of rectangular functionally graded (FG) plates with in-plane exponentially non-homogeneous material is investigated. Young's modulus and mass density are assumed to vary between a metal-rich and a ceramic-rich zone along one in-plane direction of the plate.
The governing differential equation is derived for the case, and a truncated Taylor series expansion technique is utilized to calculate natural frequencies. A Levy-type solution is obtained for plates having two simply supported edges parallel with the material gradient direction. Results for normalized natural frequency are compared with the 4th order Runge-Kutta method, and when possible with exact solution, showing an accurate agreement. Furthermore, a comprehensive parametric study is carried out to determine the effects of different boundary conditions, aspect ratios, and material variations on the free vibration of FGM plates.

Keywords: *FGM plate, in-plane inhomogeneity, free vibration, semi-analytical solution*

Nomenclature

c_1, c_2 non-homogeneity parameters of module of elasticity and density, respectively.
C, F, S Clamped, free, and simply supported edges of the plate.
$D(x), E(x), \rho(x)$ flexural rigidity, module of elasticity and density of the material.
D_0, E_0, ρ_0 reference values of $D(x), E(x)$ and $\rho(x)$ at $x = 0$, respectively.
N number of terms in the truncated Taylor series expansion.
$X(x)$ displacement (shape) function along x.
$\bar{x}, \bar{y}, \bar{w}$ non-dimensional x and y direction, and deflection of the plate along z direction, respectively.
ω, β natural frequency and normalized natural frequency, respectively.
a, b, λ, h Length along x and y direction, aspect ratio, and thickness of the plate, respectively.

1 Introduction

Composite materials are manufactured based on different industrial needs to optimize the response to external loads and reduce the residual and thermal stresses at desired regions of structures. Functionally graded materials (FGMs) are a class of composites with spatially continuous variation of mechanical properties along one or more directions. This has been achieved by gradually changing the composition of the constituent materials, usually ceramics and metals. Therefore, dealing with interfacial stress concentrations can be avoided. Due to their applicability, FGMs have garnered the attention of many researchers in different structures such as beams [1–3] and plates [4–6].

Vibration analysis of structures with directionally – but not through the thickness – FGM structures is of significance importance. This has been addressed by in [7–9] for axially graded Euler-Bernoulli and Timoshenko beams. It has also been fully discussed for radially FGM circular plates by Sahraee [10], Shariyat and Alipour [11] and Hosseini-Hashemi et al. [12,13]. However, information about rectangular plates with in-plane inhomogeneity is very limited.

Fundamental frequencies of FGM rectangular plates with in-plane material inhomogeneity was studied by Liu et al. [14] using the Fourier series expansion and an integration technique. Uymaz et al. [15] presented natural frequencies for classical and higher order plates by the Ritz method. Both studies are limited to assuming a power form of the in-plane direction for changes in material properties. Therefore, due to lack of a wide range of data for in-plane inhomogeneous plates, it is needed to thoroughly investigate this problem from other aspects.

In this paper, we investigated vibration characteristics of a rectangular FGM plate with in-plane exponentially non-homogeneous material. Section 2, introduces the theory and methodology of the problem. The equation of motion for the exponentially FGM plate with in-plane exponentially non-homogeneous material is also derived in this section. Results are provided, and the accuracy of the method is investigated in section 3. Thereafter, a parametric study is performed to analyze the effects of different parameters including boundary conditions, material inhomogeneity and aspect ratios on the frequency of the plate. Finally, the authors draw a conclusion based on the most important results.

2 Theory and methodology

In this section the theory and boundary conditions of the problem is explained.

2.1 Finite Taylor series expansion technique

The present method is a semi-analytical technique based on the truncated Taylor series expansion, with a recursive formulation resulting in reduction of time calculation. Based on the truncated Taylor series expansion, a function $f(x)$ can be approximately replaced by Eq. (1) near a point $x = x_0$:

$$f(x) = \sum_{k=0}^{N}(x - x_0)^k F(k), \tag{1}$$

where $F(k)$ is referred to the k$^{\text{th}}$ order differential transform of the function $f(x)$ and is defined as:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f}{dy^k}\right]_{x=x_0} . \tag{2}$$

Accordingly, the inverse transform of Eq. (2) is expressed as Eq. (1), where uppercase and lowercase letters indicate the transformed function $F(k)$ and the original function $f(x)$, respectively. Equation (1) is approximately true, if the terms $\sum_{k=N+1}^{\infty}(x - x_0)^k F(k)$ become negligible. The appropriate value of N in Eq. (1) depends on the rate of convergence of the series and the desired accuracy.

Table 1 presents fundamental relations derived directly from the definition of the Taylor series expansion [16,17].

Table 1: Basic rules for transforming of functions by Taylor series expansion

Original Function	Transformed Function
$f(x) = \alpha u(x) \pm \beta v(x)$	$F(k) = \alpha U(k) \pm \beta V(k)$
$f(x) = u(x)v(x)$	$F(k) = \sum_{n=0}^{k} U(n)V(k-n)$
$f(x) = \frac{d^m u(x)}{dx^m}$	$F(k) = (k+1)(k+2)\ldots(k+m)U(k+m)$
$f(x) = x^m$	$\delta(k-m)$

$f(x) = exp(\lambda x)$	$\frac{\lambda^k}{k!}$

2.2 Plate constitutive equation

The constitutive equation of motion of a plate can be written as:

$$D(\nabla^4 W) + 2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}(\nabla^2 W) + 2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}(\nabla^2 W) + (\nabla^2 D)(\nabla^2 W) - (1-\nu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 W}{\partial x^2}\right) + \rho h\frac{\partial^2 W}{\partial t^2} = f(x,y,t), \tag{3}$$

in which $D = \frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity, and h, E, ρ, ν are thickness, Young's modulus, density and Poisson's ratio of the material, respectively. For harmonic free vibration of the plate one can set $W(x,y,t) = w(x,y)\sin(\omega t)$, where ω is the natural frequency. Therefore for free vibration analysis, Eq. (3) can be reduced to:

$$D(\nabla^4 w) + 2\frac{\partial D}{\partial x}\frac{\partial}{\partial x}(\nabla^2 w) + 2\frac{\partial D}{\partial y}\frac{\partial}{\partial y}(\nabla^2 w) + (\nabla^2 D)(\nabla^2 w) - (1-\nu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 D}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) - \rho h\omega^2 w = 0. \tag{4}$$

In this study, a plate with in-plane non-homogeneous material was investigated. Changing along only one direction was taken into account in order to consider limitations of construction. Without loss of generality, we assumed E and ρ change along the x direction (see Fig. 1), and based on exponential functions as:

$$E = E(x) = E_0 e^{c_1 x}, \quad \rho = \rho(x) = \rho_0 e^{c_2 x}. \tag{5}$$

Here, E_0 and ρ_0 are reference values of E, ρ at $x = 0$. A metal-rich or ceramic-rich region can be obtained by properly setting values of E_0, ρ_0, c_1 and c_2 in Eq. (5). This method is adopted and fully discussed by Li et al. [18] for beams. Therefore flexural rigidity, D, is also a function of x, with constant D_0 as a reference value at $x = 0$ as:

$$D = D_0 e^{c_1 x}, \quad D_0 = \frac{E_0 h^3}{12(1-\nu)^2} \ . \tag{6}$$

Fig. 1: Changes of material properties along one in-plane direction

Variations of ν are usually negligible, and it can be considered as constant [19]; hence for an FGM plate with in-plane inhomogeneity, Eq. (4) can be simplified as:

$$D(x)(\nabla^4 w) + 2c_1 D(x)\frac{\partial}{\partial x}(\nabla^2 w) + {c_1}^2 D(x)(\nabla^2 w) - (1-\nu){c_1}^2 D(x)\frac{\partial^2 w}{\partial y^2} - \rho(x)h\omega^2 w = 0. \tag{7}$$

It should be mentioned that the neutral plane location is different from the mid plane location in plates with material changes along the thickness [20,21], the thing which is not true for plates with in-plane changes.

2.3 Normalization

Introducing the following non-dimensional parameters

$$\bar{x} = x/a, \ \bar{y} = y/b, \ \bar{w} = w/h, \quad \lambda = a/b, \tag{8}$$

Eq. (7) can be rewritten as:

$$D_0 e^{c_1 a\bar{x}}\left(\frac{h}{a^4}\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+2\frac{h}{a^2b^2}\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{y}^2}+\frac{h}{b^4}\frac{\partial^4\bar{w}}{\partial\bar{y}^4}\right)+2D_0c_1e^{c_1a\bar{x}}\left(\frac{h}{a^3}\frac{\partial^3\bar{w}}{\partial\bar{x}^3}+\frac{h}{ab^2}\frac{\partial^3\bar{w}}{\partial\bar{x}\partial\bar{y}^2}\right)+ D_0{c_1}^2e^{c_1a\bar{x}}\left(\frac{h}{a^2}\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\frac{h}{b^2}\frac{\partial^2\bar{w}}{\partial\bar{y}^2}\right)-(1-\nu)D_0{c_1}^2e^{c_1a\bar{x}}\frac{h}{b^2}\frac{\partial^2\bar{w}}{\partial\bar{y}^2}-\rho_0e^{c_2a\bar{x}}h^2\omega^2\bar{w}=0 \quad (9)$$

where λ, a and b are aspect ratio, length and width of the plate, respectively. The governing equation for free vibration of FG plates with in-plane material inhomogeneity can be simplified to a non-dimensional form as:

$$\left(\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+2\lambda^2\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{y}^2}+\lambda^4\frac{\partial^4\bar{w}}{\partial\bar{y}^4}\right)+2c_1a\left(\frac{\partial^3\bar{w}}{\partial\bar{x}^3}+\lambda^2\frac{\partial^3\bar{w}}{\partial\bar{x}\partial\bar{y}^2}\right)+a^2\left({c_1}^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\nu\lambda^2\frac{\partial^2\bar{w}}{\partial\bar{y}^2}\right)- \beta^2\bar{w}=0, \quad (10)$$

where

$$\beta^2=\frac{12(1-\nu)^2a^4\rho_0\omega^2}{E_0h^2}.$$

Considering two opposite edges, $\bar{y}=0$ and $\bar{y}=1$, simply supported (Levy-type solution), one can write $\bar{w}(\bar{x},\bar{y})=X(\bar{x})sin(m\pi\bar{y})$.
In the remaining of the paper the bar $(\bar{\ })$ symbol will not be shown for the sake of simplicity. Equation (10) can then be simplified as:

$$X^{(4)}+A_1X'''+A_2X''+A_3X'+\left[A_4-\beta^2e^{-ax(c_1-c_2)}\right]X=0, \quad (11)$$

where

$$A_1=2c_1a\ ,\ A_2=a^2{c_1}^2-2(m\pi\lambda)^2,$$
$$A_3=-2c_1a(m\pi\lambda)^2\ ,\ A_4=(m\pi\lambda)^4-\nu(c_1am\pi\lambda)^2.$$

Applying the differential transform method presented in subsection 2.1 to Eq. (11) gives the following relation.

$$\left(\prod_{i=1}^{4}(k+i)\right)X(k+4)+A_1.\left(\prod_{i=1}^{3}(k+i)\right)X(k+3)+A_2.\left(\prod_{i=1}^{2}(k+i)\right)X(k+2)+ A_3.(k+1)X(k+1)+A_4X(k)-\beta^2\sum_{n=0}^{k}\frac{[a(c_2-c_1)]^n}{n!}X(k-n)=0 \quad (12)$$

Therefore, a recursive formula can be obtained by rearranging the above equation.

$$X(k+4)=-\frac{A_1}{(k+4)}X(k+3)-\frac{A_2}{(k+4)(k+3)}X(k+2)-\frac{A_3}{(k+4)(k+3)(k+2)}X(k+1)- \frac{A_4}{\prod_{i=1}^{4}(k+i)}X(k)+\frac{\beta^2}{\prod_{i=1}^{4}(k+i)}\sum_{n=0}^{k}\frac{[a(c_2-c_1)]^n}{n!}X(k-n) \quad (13)$$

Solving the problem depends on the recursive formula of Eq. (13) for $k=0$ or $X(4)$. Two of the four terms $X(0)$, $X(1)$, $X(2)$ and $X(3)$ are given by boundary conditions, and the two remaining terms form a matrix. Setting the determinant of the coefficients to zero gives the natural frequency, β.

3 Results and discussion

In this section, validation of the model and results are presented.

3.1 Validation

For the situation where non-homogeneity parameters (c_1, c_2) in Eq. (13) are set to zero, analytical solutions are available. In Table 2, a comparison was made between the DTM, 4th order Runge-Kutta and those from analytical solution by Leissa [22].

Table 2: Natural frequencies of homogeneous square plates

B.Cs.	method	mode

		1st	2nd	3rd
SCSC	Exact	28.9509	54.7431	69.3270
	DTM	28.9509	54.7430	69.3270
	RK4	28.9695	54.7531	69.7485
SCSS	Exact	23.6463	51.6743	58.6464
	DTM	23.6463	51.6743	58.6463
	RK4	23.6530	51.6790	58.8883
SSSS	Exact	19.7392	49.3480	49.3480
	DTM	19.7392	49.3480	49.3480
	RK4	19.7415	49.3503	49.4791
SCSF	Exact	12.6874	33.0651	41.7019
	DTM	12.6874	33.0651	41.7019
	RK4	12.6875	33.0926	41.7220
SFSS	Exact	11.6845	27.7563	41.1967
	DTM	11.6845	27.7563	41.1966
	RK4	11.6845	27.7687	41.1966
SFSF	Exact	9.6314	16.1348	36.7256
	DTM	9.6313	16.1348	36.7256
	RK4	9.6313	16.1351	36.7648

It can be seen from Table 2 that the present method is precise in calculating frequencies, while RK4 demonstrates some small variations. Moreover, it is worth mentioning that obtaining more accurate results from the RK4 took a few days, while DTM results were easily achievable in a few hours on the same computer.

The present method is semi-analytical, and results are dependent on the number of terms, *N*, in finite Taylor series expansion of Eq. (1). Here, Fig. 2 and Fig. 3 are presented to check the convergence of solutions. They consider three different values for c_1 and c_2 for FG plates with SCSC and SFSF boundary conditions, respectively. Convergence diagrams for other boundary conditions are the same and have not been presented.

In addition, They show that for larger values of non-homogeneity parameters, c_1 and c_2, more terms should be considered to obtain desired accuracy. For instance if the first natural frequency is considered, $N = 20$ is a proper value for $(c_1, c_2) = (0,0)$ --Fig. 2 -a and Fig. 3-a -- but it is not sufficient when $(c_1, c_2) = (1, 0.5)$, see Fig. 2-b and Fig. 3-b. Moreover as it is expected, obtaining higher natural frequencies needs consideration of higher values of N.

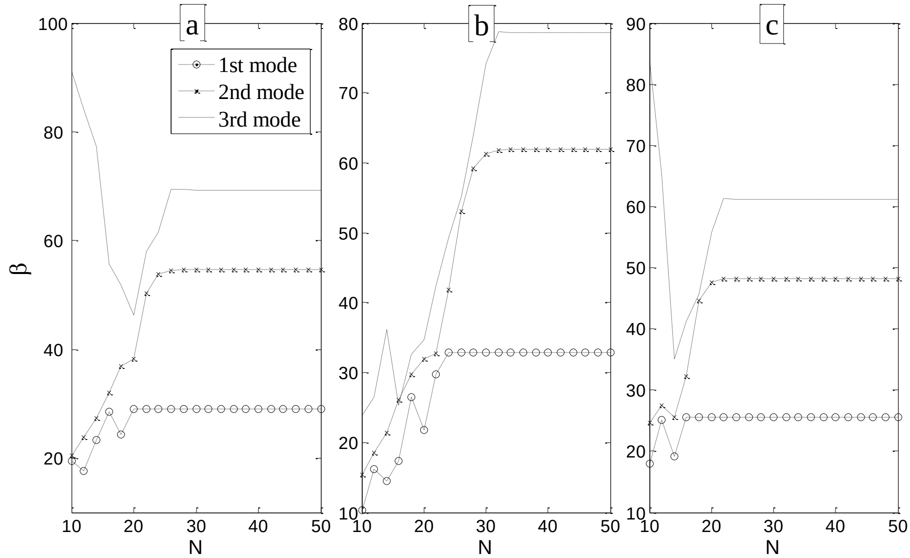

Fig. 2: Convergence of the first three natural frequencies vs. N, number of terms, for FGM SCSC plate for (c_1, c_2)= a: (0,0), b: (1, 0.5), c: (-1, -0.5)

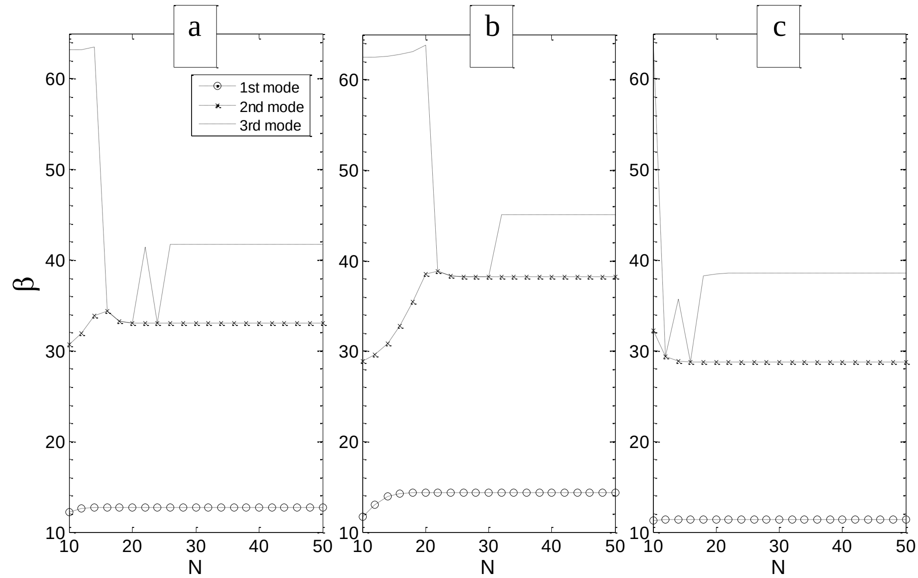

Fig. 3: Convergence of the first three natural frequencies vs. N, number of terms, for FGM SFSF plate for (c_1, c_2)= a: (0,0), b: (1, 0.5), c: (-1, -0.5)

3.2 Parametric studies

In Table 3 to Table 8, the first three natural frequencies of FGM plates are presented for SSSS, SCSC, SFSF, SCSS, SFSS, and SCSF boundary conditions for three different aspect ratios and with non-zero values of c_1 and c_2. Changes in natural frequency versus aspect ratio, in-plane non-homogeneous parameters and different boundary conditions are presented for the first three vibrational modes of the plate in these tables.

Table 3: 1st three frequency parameters of the FGM SSSS plate

	$\lambda = 1$			$\lambda = 1.5$			$\lambda = 2$		
(c_1, c_2)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)
ω_1	19.739	22.425	17.465	32.076	36.399	28.347	49.348	55.870	43.512
ω_2	49.348	55.947	43.572	61.685	69.959	54.484	78.957	89.571	69.758
ω_3	49.348	55.947	43.572	98.696	111.198	86.601	128.305	145.402	113.239

Table 4: 1st three frequency parameters of the FGM SCSC plate

	$\lambda = 1$			$\lambda = 1.5$			$\lambda = 2$		
(c_1, c_2)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)
ω_1	28.951	32.897	25.620	39.089	44.339	34.531	54.743	61.987	48.276
ω_2	54.743	61.987	48.276	79.525	90.176	70.229	94.585	107.223	83.505
ω_3	69.327	78.641	61.246	102.216	115.358	89.841	154.776	175.370	136.578

Table 5: 1st three frequency parameters of the FGM SFSF plate

	$\lambda = 1$			$\lambda = 1.5$			$\lambda = 2$		
(c_1, c_2)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)
ω_1	9.631	10.980	8.551	21.095	23.932	18.638	37.958	42.010	32.718
ω_2	16.135	18.419	14.344	25.311	28.800	22.429	40.866	47.702	37.151
ω_3	36.726	41.716	32.488	45.646	51.643	40.220	60.104	67.840	52.834

Table 6: 1st three frequency parameters of the FGM SCSS plate

	$\lambda = 1$			$\lambda = 1.5$			$\lambda = 2$		
(c_1, c_2)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)
ω_1	23.646	26.543	21.175	35.051	39.822	30.929	51.674	58.865	45.298
ω_2	51.674	58.865	45.298	69.913	78.986	61.973	86.134	97.669	76.069
ω_3	58.646	65.939	52.241	100.270	113.874	87.455	140.846	159.367	124.473

Table 7: 1st three frequency parameters of the FGM SFSS plate

	$\lambda = 1$			$\lambda = 1.5$			$\lambda = 2$		
(c_1, c_2)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)
ω_1	11.685	13.121	10.539	22.582	24.935	20.592	39.075	42.145	36.495
ω_2	27.756	31.735	24.415	38.252	44.171	33.096	54.195	62.551	46.741
ω_3	41.197	44.670	37.889	72.718	83.234	63.490	88.553	101.554	77.102

Table 8: 1st three frequency parameters of the FGM SCSF plate

	$\lambda = 1$			$\lambda = 1.5$			$\lambda = 2$		
(c_1, c_2)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)	(0,0)	(1, 0.5)	(-1, -0.5)
ω_1	12.687	14.569	11.199	22.958	26.829	19.771	39.246	47.069	32.955
ω_2	33.065	36.921	29.777	42.516	47.185	38.265	57.508	63.887	51.507
ω_3	41.702	49.495	35.103	82.289	92.003	72.271	97.028	108.494	86.628

4 Conclusion

In this research, the formulation for plates with exponential gradient of the material and having levy-type boundary conditions was derived. A semi-analytical solution, truncated Taylor series expansion technique, was employed to obtain approximate but highly accurate solution for free vibration of the FGM rectangular plate. A number of comparisons were made with the fourth order Runge-Kutta method and those available in the literature, showing an excellent agreement. Finally, the dependency of the free vibration of plates on in-plane FG properties, boundary conditions and aspect ratios were investigated. All semi-analytical results presented in this paper may serve to validate other analytical and numerical methods. In addition and in the light of results the following conclusions can be drawn:

- Natural frequency is directly related to the value of non-homogeneous parameters, c_1 and c_2, and increases with their increment;
- Larger number of terms, N, in the finite Taylor series expansion should be taken for higher values of non-homogeneous parameters, c_1 and c_2, which is also directly proportioned to the desired vibrational modes;
- Natural frequency has a direct relation with aspect ratio for all in-plane non-homogeneous parameters.

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