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Research paper

Electromagnetic field in a rectangular cavity: an example of second quantization

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Abstract

We consider the case of electromagnetic field inside a rectangular cavity with conducting walls as a form of a system described by classical mechanics equations. We pass these equations through the Lagrangian formalism to obtain the Hamiltonian formulation. Finally we apply canonical quantization to end up with a quantum theory of the electromagnetic field. Since classical electrodynamics can be interpreted as the quantum theory of a one photon system, then the above quantization is taken as the "quantization of the quantum theory of the electromagnetic field" or simply second quantization.

Keywords: *Electromagnetic Field; Rectangular Cavity; Second Quantization.*

1. Introduction

In a double slit experiment a beam of monochromatic light of monochromatic light of frequency ν falls on a screen with two slits. Behind the screen a photographic plate registers the light. The wave nature of light determines the interaction pattern that appears on the photographic plate [1].When the light is dimmed to an extent that the particle nature of light becomes apparent, that is the absorption of light occurs at scarcely spaced but well defined positions on the photographic plate. If one waits long enough for a very large number of such absorption to have taken place, then the pattern appearing on the photographic plate is still the same interference pattern as observed for an intense beam [2].Then, when the light is dimmed, we may assume that there is only a single photon at an instance of time. The absorption of the photon is considered as a measurement. However, no definite prediction about the position of absorption on the screen can be made, but the probability density for the absorption at a given by the intensity of the electromagnetic field of the free photon at that given position at time of absorption [3].In this way, we see that the Maxwell's equations of the classical electrodynamics can be interpreted as the Schrodinger equation governing the time evolution of the electromagnetic field considered as a wave function for a one-photon system [4]. The six component wave function Ψ is an assemble of the electric field $\vec{E}$ and the magnetic field $\vec{B}$

$$\Psi = \begin{pmatrix} \vec{E} \\ c\vec{B} \end{pmatrix} \tag{1.1}$$

where c is the speed of light in vacuum. The two Maxwell equations in vacuum are

$$\vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0 \tag{1.2}$$

impose constraints on the possible wave function Ψ : only wave functions whose components $\vec{E}$ and $\vec{B}$ fields satisfy equation (1.2) are physically allowed. The remaining two Maxwell's equations in vacuum,

$$\frac{\partial\vec{E}}{\partial t} = c^2\vec{\nabla}\times\vec{B}, \qquad \frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\vec{E} \tag{1.3}$$

can then be considered as a first order time evolution equation for Ψ

$$\frac{\partial\Psi}{\partial t} = D\Psi \tag{1.4}$$

where D is the linear operator [5]. Equation (1.4) can be interpreted as Schrödinger equation governing the time evolution of the wave function Ψ . We note that equation (1.4) preserves the constraint equation (1.2): for the electric field $\vec{E}$ and for the magnetic field $\vec{B}$. This implies that, if Ψ solves equation (1.4) and satisfies the constraint equation (1.2) at time $t=0$, if then it satisfies equation (1.2) at all times t . We note, unlike the Schrodinger equation for massive particles, equation (1.4) is first order not only in time but also in space.

Then, the Maxwell's equations are relativistic equations for a photon the same way as the Dirac equation for the electron, are first order both in time and space [6].
In section 2 we solve the time evolution equations of the electromagnetic field inside a rectangular cavity with conducting wall, and then express the corresponding Lagrangian and Hamiltonian formulation. In section 3, we apply the canonical quantization for the system in section 2 to end up with the quantum theory of the electromagnetic field or simply "quantization of classical electrodynamics".

2. Electromagnetic Field in the cavity

We consider a rectangular cavity C of length l, width w and height h represented in a Cartesian coordinate system as follows

$$C=\{(x,y,z),\quad 0\le x\le l,\quad 0\le y\le w,\quad 0\le z\le h\} \tag{2.1}$$

We assume that the cavity has ideally conducting walls, which translates into the following boundary condition for $\vec{E}$ and $\vec{B}$:

$$\vec{E}_{\parallel}\Big|_{\partial C}=0 \qquad\qquad \vec{B}_{\perp}\Big|_{\partial C}=0 \tag{2.2}$$

where $\vec{E}_{\parallel}$ denotes the component of the $\vec{E}$-field parallel to the boundary ∂C of C, while $\vec{B}_{\perp}$ denotes the component of $\vec{B}$-field perpendicular to ∂C. We consider the initial value problem: given the fields $\vec{E}_0(\vec{r})$ and $\vec{B}_0(\vec{r})$ subject to the constraint equation (1.2) and the boundary conditions (2.2) and which satisfies the initial conditions

$$\vec{E}(\vec{r},0)=\vec{E}_0(\vec{r}) \qquad\qquad \vec{B}(\vec{r},0)=\vec{B}_0(\vec{r}) \tag{2.3}$$

We note that Maxwell's equation (1.3) can be decoupled by applying the curl on equations (1.3) separately and then using the constraint equation (1.2) to find separate wave equations for $\vec{E}$ and $\vec{B}$, defined as [7]

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0; \qquad\qquad \nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0 \tag{2.4}$$

Equation (2.4) can be solved using separation of variables, for instance the first component $\vec{E}_1$ of $\vec{E}$ is expressed as

$$\vec{E}_1(\vec{r},t)=U_1(x)V_1(y)W_1(z)T_1(t) \tag{2.5}$$

substituting in the wave equation (2.4) and diving by $\vec{E}_1$, we obtain

$$\frac{1}{U_1}\frac{d^2U_1}{dx^2}+\frac{1}{V_1}\frac{d^2V_1}{dy^2}+\frac{1}{W_1}\frac{d^2W_1}{dz^2}-\frac{1}{c^2}\frac{1}{T_1}\frac{d^2T_1}{dt^2}(t)=0 \tag{2.6}$$

If (2.5) is to hold for arbitrary values of x, y, z and t, then each of the summands is constant and the constants add up to zero. These constants can be positive, zero or negative, but for conservation of boundary conditions one can conclude that

$$\frac{1}{U_1}\frac{d^2U_1}{dx^2}=-k_1^2;\qquad \frac{1}{V_1}\frac{d^2V_1}{dy^2}=-k_2^2;\qquad \frac{1}{W_1}\frac{d^2W_1}{dz^2}=-k_3^2;\qquad \frac{1}{T_1}\frac{d^2T_1}{dt^2}(t)=-k_0^2 \tag{2.7}$$

where k_0, k_1, k_2 and k_3 are positive constants subject to

$$k_0=c\left|\vec{k}\right|=c\sqrt{k_1^2+k_2^2+k_3^2} \tag{2.8}$$

where we have considered k_1, as components k_2 and k_3 of a wave vector $\vec{k}$ [6].The boundary condition from (2.2) for $\vec{E}_1$ becomes

$$\vec{E}_1\Big|_{y=0}=\vec{E}_1\Big|_{y=w}=\vec{E}_1\Big|_{z=0}=\vec{E}_1\Big|_{z=h}=0 \tag{2.9}$$

which is equivalent to requiring that

$$V_1(0)=V_1(w)=W_1(0)=W_1(h)=0 \tag{2.10}$$

Using (2.7) combined with the condition (2.10), the constants k_2 and k_3 can be expressed as

$$k_2=\frac{m_1\pi}{w}y, \qquad\qquad k_3=\frac{p_1\pi}{h}y \tag{2.11}$$

Then for some positive integers m_1 and p_1, we have

$$V_1(y)=\sin\left(\frac{m_1\pi}{w}y\right), \qquad\qquad W_1(z)=\sin\left(\frac{p_1\pi}{h}z\right) \tag{2.12}$$

Proceeding in the same way for $\vec{E}_2$ and $\vec{E}_3$, we obtain

$$\vec{E} = \begin{pmatrix} T_1(t)U_1(x)\sin\left(\frac{m_1\pi}{w}y\right)\sin\left(\frac{p_1\pi}{h}z\right) \\ T_2(t)\sin\left(\frac{n_2\pi}{l}x\right)V_2(y)\sin\left(\frac{p_2\pi}{h}z\right) \\ T_3(t)\sin\left(\frac{n_3\pi}{l}x\right)\sin\left(\frac{m_3\pi}{w}y\right)W_3(z) \end{pmatrix} \quad (2.13)$$

We impose the constraint equation $\vec{\nabla}\cdot\vec{E}$ on (2.13), to obtain

$$T_1(t)U_1'(x)\sin\left(\frac{m_1\pi}{w}y\right)\sin\left(\frac{p_1\pi}{h}z\right)+T_2(t)\sin\left(\frac{n_2\pi}{l}x\right)V_2'(y)\sin\left(\frac{p_2\pi}{h}z\right)+$$
$$T_3(t)\sin\left(\frac{n_3\pi}{l}x\right)\sin\left(\frac{m_3\pi}{w}y\right)W_3'(z) \quad (2.14)$$

Keeping two of the three variables in (2.14) fixed and letting the third to run, it is easy to show that these equations can be satisfied only if, for instance $n_2 = n_3 = n$ and $U'(x)$ is proportional to $\sin\left(\frac{n\pi}{l}x\right)$, and similarly for y and z . Absorbing the proportionality constant in the time dependent factors, we can therefore assume that

$$U_1(x)=\cos\left(\frac{n\pi}{l}x\right), \qquad V_2(y)=\cos\left(\frac{m\pi}{w}y\right), \qquad W_3(z)=\cos\left(\frac{p\pi}{h}z\right) \quad (2.15)$$

Then equation (2.14) reduces to

$$\left(\frac{n\pi}{l}T_1(t)+\frac{m\pi}{w}T_2(t)+\frac{p\pi}{h}T_3(t)\right)\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{w}y\right)\sin\left(\frac{p\pi}{h}z\right)=0 \quad (2.16)$$

Equation (2.16) is satisfied for all x , y and z , if and only if

$$\vec{k}\cdot\vec{T}(t)=0 \qquad \vec{k}=\left(\vec{k}_1,\vec{k}_2,\vec{k}_3\right)=\left(\frac{n\pi}{l},\frac{m\pi}{w},\frac{p\pi}{h}\right) \quad (2.17)$$

for all t. We introduce a the vector $\vec{K}$ that denotes the set of possible vectors $\vec{k}$

$$\vec{K} := \left\{\left(\frac{n\pi}{l},\frac{m\pi}{w},\frac{p\pi}{h}\right)\middle|\, n, m, p \in \mathrm{N}\right\} \quad (2.18)$$

We consider the variables $\alpha_{\vec{k}}$, $\beta_{\vec{k}}$, for every $\vec{k}\in\vec{K}$ as the generalized coordinates of the system and introduce two units vectors $\vec{u}\left(\vec{k}\right)$ and $\vec{v}\left(\vec{k}\right)$ with the property $\left(\vec{u}\left(\vec{k}\right),\vec{v}\left(\vec{k}\right),\frac{1}{\left|\vec{k}\right|}\vec{k}\right)$ that forms a positively oriented, orthonormal basis.Then the most general solution of equation (2.17) can be written as

$$T(t)=\alpha_{\vec{k}}(t)\vec{u}\left(\vec{k}\right)+\beta_{\vec{k}}v_{\vec{k}}\left(\vec{k}\right) \quad (2.19)$$

We substitute equations (2.13) and (2.19) in (2.13) to obtain the general expression for the $\vec{E}$ -field as

$$\vec{E}(\vec{r},t)=\alpha_{\vec{k}}(t)\vec{e}_{\vec{k},u}(\vec{r})+\beta_{\vec{k}}(t)\vec{e}_{\vec{k},v}(\vec{r}) \quad (2.20)$$

where we have introduced the vector-valued functions $e_{\vec{k},u}(\vec{r})$ and $e_{\vec{k},v}(\vec{r})$ defined by

$$\vec{e}_{\vec{k},u}(\vec{r})=\begin{pmatrix} u\left(\vec{k}\right)_1\cos(k_1x)\sin(k_2y)\sin(k_3z) \\ u\left(\vec{k}\right)_2\sin(k_1x)\cos(k_2y)\sin(k_3z) \\ u\left(\vec{k}\right)_3\sin(k_1x)\sin(k_2y)\cos(k_3z) \end{pmatrix}, \quad \vec{e}_{\vec{k},v}(\vec{r})=\begin{pmatrix} v\left(\vec{k}\right)_1\cos(k_1x)\sin(k_2y)\sin(k_3z) \\ v\left(\vec{k}\right)_2\sin(k_1x)\cos(k_2y)\sin(k_3z) \\ v\left(\vec{k}\right)_3\sin(k_1x)\sin(k_2y)\cos(k_3z) \end{pmatrix} \quad (2.21)$$

The same chain of arguments can be applied to the $\vec{B}$ -field to obtain

$$\vec{B}(\vec{r},t)=\frac{1}{c}\gamma_{\vec{k}}(t)\vec{f}_{\vec{k},u}(\vec{r})+\frac{1}{c}\delta_{\vec{k}}(t)\vec{f}_{\vec{k},v}(\vec{r}) \quad (2.22)$$

where

$$\vec{f}_{\vec{k},u}(\vec{r})=\begin{pmatrix} u(\vec{k})_1\sin(k_1x)\cos(k_2y)\cos(k_3z)\\ u(\vec{k})_2\cos(k_1x)\sin(k_2y)\cos(k_3z)\\ u(\vec{k})_3\cos(k_1x)\cos(k_2y)\sin(k_3z)\end{pmatrix},\quad \vec{f}_{\vec{k},v}(\vec{r})=\begin{pmatrix} v(\vec{k})_1\sin(k_1x)\cos(k_2y)\cos(k_3z)\\ v(\vec{k})_2\cos(k_1x)\sin(k_2y)\cos(k_3z)\\ v(\vec{k})_3\cos(k_1x)\cos(k_2y)\sin(k_3z)\end{pmatrix}\quad(2.23)$$

and the factor $\frac{1}{c}$ in (2.22) is introduced so that the functions $\gamma_{\vec{k}}$ and $\delta_{\vec{k}}$ are of the same dimension as the functions $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$. It is easy to verify that

$$\begin{aligned}\vec{\nabla}\times\vec{e}_{\vec{k},u}&=\frac{k_0}{c}\vec{f}_{\vec{k},v}, &\quad \vec{\nabla}\times\vec{e}_{\vec{k},v}&=-\frac{k_0}{c}\vec{f}_{\vec{k},u}\\ \vec{\nabla}\times\vec{f}_{\vec{k},u}&=-\frac{k_0}{c}e_{\vec{k},v} &\quad \vec{\nabla}\times\vec{f}_{\vec{k},v}&=\frac{k_0}{c}e_{\vec{k},u}\end{aligned}\quad(2.24)$$

We substitute equations (2.21) and (2.22) for in (1.3) and use (2.24), we obtain

$$\begin{aligned}\dot{\alpha}_{\vec{k}}(t)\vec{e}_{\vec{k},u}(\vec{r})+\dot{\beta}_{\vec{k}}(t)\vec{e}_{\vec{k},v}(\vec{r})&=k_0'\delta_{\vec{k}'}(t)\vec{e}_{\vec{k}',u}(\vec{r})-k_0'\gamma_{\vec{k}'}(t)\vec{e}_{\vec{k}',v}(\vec{r})\\ \dot{\gamma}_{\vec{k}'}(t)\vec{f}_{\vec{k}',u}(\vec{r})+\dot{\delta}_{\vec{k}'}(t)\vec{f}_{\vec{k}',v}(\vec{r})&=k_0\beta_{\vec{k}}(t)\vec{f}_{\vec{k},u}(\vec{r})-k_0\alpha_{\vec{k}}(t)\vec{f}_{\vec{k},v}(\vec{r})\end{aligned}\quad(2.25)$$

where $\vec{k}$ and $\vec{k}'$ are taken as the wave vectors for the $\vec{E}$ and $\vec{B}$ respectively. If $\vec{k}=\vec{k}'$ for all x y, z and t, equation (2.25) is satisfied if and only if the following equations hold

$$\begin{aligned}\dot{\alpha}_{\vec{k}}(t)&=k_0\delta_{\vec{k}}(t) &\quad \dot{\beta}_{\vec{k}}(t)&=-k_0\gamma_{\vec{k}}(t)\\ \dot{\gamma}_{\vec{k}}(t)&=k_0\beta_{\vec{k}}(t) &\quad \dot{\delta}_{\vec{k}}(t)&=-k_0\alpha_{\vec{k}}(t)\end{aligned}\quad(2.26)$$

this satisfies the Maxwell's equations in the cavity if and only if

$$\ddot{\alpha}_{\vec{k}}(t)=-k_0^2\alpha_{\vec{k}}(t),\qquad \ddot{\beta}_{\vec{k}}(t)=-k_0^2\beta_{\vec{k}}(t)\quad(2.27)$$

for every $\vec{k}\in\vec{K}$ .Substituting(2.20) and (2.22) in (1.3) and apply (2.24), the electromagnetic field can be expanded in terms of the generalized coordinates and their velocities $\dot{\alpha}_{\vec{k}}$, $\dot{\beta}_{\vec{k}}$, $\vec{k}\in\vec{K}$, as

$$\Psi(r,t)=\begin{pmatrix}\vec{E}(r,t)\\ c\vec{B}(r,t)\end{pmatrix}=\sum_{\vec{k}\in K}\begin{pmatrix}\alpha_{\vec{k}}(t)\vec{e}_{\vec{k},u}(\vec{r})+\beta_{\vec{k}}(t)\vec{e}_{\vec{k},v}(\vec{r})\\ \frac{1}{k_0}\dot{\alpha}_{\vec{k}}(t)\vec{f}_{\vec{k},v}(\vec{r})-\frac{1}{k_0}\dot{\beta}_{\vec{k}}(t)\vec{f}_{\vec{k},u}(\vec{r})\end{pmatrix}\quad(2.28)$$

Then the problem has been reduced to an infinite system of decoupled harmonic oscillators.
We proceed into Lagrangian formalism by using the Euler–Lagrange equations to write the Lagrangian of the dynamical system as [8]

$$\mathrm{L}\equiv\sum_{\vec{k}\in K}c_{\vec{k}}\left(\dot{\alpha}_{\vec{k}}^2-k_0^2\alpha_{\vec{k}}^2\right)+d_{\vec{k}}\left(\dot{\beta}_{\vec{k}}^2-k_0^2\beta_{\vec{k}}^2\right)\quad(2.29)$$

where $c_{\vec{k}}$, $d_{\vec{k}}$ are non-zero constants. We calculate the momenta $\pi_{\vec{k}}$, $\omega_{\vec{k}}$ conjugate to $\alpha_{\vec{k}}$, $\beta_{\vec{k}}$ as

$$\pi_{\vec{k}}=\frac{\partial L}{\partial\dot{\alpha}_{\vec{k}}}=2c_{\vec{k}}\dot{\alpha}_{\vec{k}},\qquad \omega_{\vec{k}}=\frac{\partial L}{\partial\dot{\beta}_{\vec{k}}}=2d_{\vec{k}}\dot{\beta}_{\vec{k}}\quad(2.30)$$

and the corresponding Hamiltonian becomes

$$H\equiv\sum_{\vec{k}\in K}\left(\frac{1}{4c_{\vec{k}}}\pi_{\vec{k}}^2+c_{\vec{k}}k_0^2\alpha_{\vec{k}}^2+\frac{1}{4d_{\vec{k}}}\omega_{\vec{k}}^2+d_{\vec{k}}k_o^2\beta_{\vec{k}}^2\right)\quad(2.31)$$

The total energy of the system is easily obtained using equations (2.20), (2.22), (2.26) and (2.30)and using the orthonomality of the functions $\vec{e}_{\vec{k},u}$, $\vec{e}_{\vec{k},v}$, $\vec{f}_{\vec{k},u}$, $\vec{f}_{\vec{k},v}$ as

$$E\equiv\frac{lwh\varepsilon_0}{16}\left(\left\langle\vec{E}\middle|\vec{E}\right\rangle_R+c^2\left\langle\vec{B}\middle|\vec{B}\right\rangle_R\right)=\frac{lwh\varepsilon_0}{16}\sum_{\vec{k}\in K}\left(\alpha_{\vec{k}}^2+\beta_{\vec{k}}^2+\frac{\omega_{\vec{k}}^2}{4d_{\vec{k}}^2k_o^2}+\frac{\pi_{\vec{k}}^2}{4c_{\vec{k}}^2k_o^2}\right)\quad(2.32)$$

Since the Hamiltonian (2.31) is associated with the total energy of the system (2.32), we chose $c_{\vec{k}}=d_{\vec{k}}=\frac{1}{2k_0^2}$ and rewrite (2.31) as [8]

$$H \equiv \sum_{\vec{k}\in K}\left(\frac{k_0^2}{2}\left(\pi_{\vec{k}}^2+\omega_{\vec{k}}^2\right)+\frac{1}{2}\left(\alpha_{\vec{k}}^2+\beta_{\vec{k}}^2\right)\right), \qquad (2.33)$$

then (2.32) becomes

$$E \equiv \frac{lwh\varepsilon_0}{16}H \qquad (2.34)$$

The electric and magnetic field can be expressed in terms of canonical coordinates $\alpha_{\vec{k}}, \beta_{\vec{k}}$ and momenta $\pi_{\vec{k}}, \omega_{\vec{k}}$, then (2.28) becomes

$$\Psi(r,t)=\begin{pmatrix}\vec{E}(r,t)\\ c\vec{B}(r,t)\end{pmatrix}=\sum_{\vec{k}\in K}\begin{pmatrix}\alpha_{\vec{k}}(t)\vec{e}_{\vec{k},u}(\vec{r})+\beta_{\vec{k}}(t)\vec{e}_{\vec{k},v}(\vec{r})\\ k_0\left(\pi_{\vec{k}}(t)\vec{f}_{\vec{k},v}(\vec{r})-\omega_{\vec{k}}\vec{f}_{\vec{k},u}(\vec{r})\right)\end{pmatrix} \qquad (2.35)$$

These completes the Hamiltonian formalism of the problem, we proceed to its quantization. In the light of the interpretation of classical electrodynamics as the quantum theory of a one photon system, we consider the above quantization as the the first quantized theory.

3. Quantization of the Electromagnetic field

To quantize the Hamiltonian (2.33), we assume that for a Hilbert state S of the system and for every dynamical variable $\mathbf{A}$ of the classical field, there is a self adjoint operator $\hat{\mathbf{A}}$ acting on S . In particular, we have operators $\hat{\alpha}_{\vec{k}}, \hat{\beta}_{\vec{k}}, \hat{\pi}_{\vec{k}}, \hat{\omega}_{\vec{k}}$ for all $\vec{k}\in\vec{K}$, and these operators are assumed to obey the canonical commutation relation [9]

$$\left[\hat{\alpha}_{\vec{k}},\hat{\pi}_{\vec{k}'}\right]=\left[\hat{\beta}_{\vec{k}},\hat{\omega}_{\vec{k}'}\right]=\begin{cases}i\hbar & if\ \vec{k}=\vec{k}'\\ 0 & if\ \vec{k}\neq\vec{k}'\end{cases} \qquad (3.1)$$

By factorization of equations (2.33), we define lowering and raising operators

$$\begin{aligned}\hat{a}_{\vec{k}}&=\frac{1}{\sqrt{2\hbar k_0}}\left(\hat{\alpha}_{\vec{k}}+ik_0\hat{\pi}_{\vec{k}}\right), & \hat{b}_{\vec{k}}&=\frac{1}{\sqrt{2\hbar k_0}}\left(\hat{\beta}_{\vec{k}}+ik_0\omega_{\vec{k}}\right)\\ \hat{a}_{\vec{k}}^{\dagger}&=\frac{1}{\sqrt{2\hbar k_0}}\left(\hat{\alpha}_{\vec{k}}+ik_0\hat{\pi}_{\vec{k}}\right) & \hat{b}_{\vec{k}}&=\frac{1}{\sqrt{2\hbar k_0}}\left(\hat{\beta}_{\vec{k}}-ik_0\omega_{\vec{k}}\right)\end{aligned} \qquad (3.2)$$

where the constants in front of the brackets have been chosen such that (Wysin, 2011)

$$\left[\hat{a}_{\vec{k}},\hat{a}_{\vec{k}'}^{\dagger}\right]=\left[\hat{b}_{\vec{k}},\hat{b}_{\vec{k}'}^{\dagger}\right]=\begin{cases}i\hbar & if\ \vec{k}=\vec{k}'\\ 0 & if\ \vec{k}\neq\vec{k}'\end{cases} \qquad (3.3)$$

Solving (3.2) for the operators corresponding to the canonical coordinates, we find

$$\begin{aligned}\hat{\alpha}_{\vec{k}}&=\frac{\sqrt{\hbar k_0}}{2}\left(a_{\vec{k}}+a_{\vec{k}}^{\dagger}\right), & \hat{\beta}_{\vec{k}}&=\frac{\sqrt{\hbar k_0}}{2}\left(b_{\vec{k}}+b_{\vec{k}}^{\dagger}\right)\\ \hat{\pi}_{\vec{k}}&=-i\sqrt{\frac{\hbar}{2k_0}}\left(a_{\vec{k}}-a_{\vec{k}}^{\dagger}\right) & \hat{\omega}_{\vec{k}}&=-i\sqrt{\frac{\hbar}{2k_0}}\left(b_{\vec{k}}-b_{\vec{k}}^{\dagger}\right)\end{aligned} \qquad (3.4)$$

The straight forward way to write the Hamiltonian operator for the system is simply to put hats on the classical equation (2.33) and apply (3.2), to obtain

$$\hat{H}\equiv\sum_{\vec{k}\in K}\hbar k_0\left(\hat{a}_{\vec{k}}^{\dagger}a_{\vec{k}}+\hat{b}_{\vec{k}}^{\dagger}\hat{b}_{\vec{k}}+1\right) \qquad (3.5a)$$

However, this cannot work, for instance

$$\hat{H}|0\rangle=\left(\sum_{\vec{k}\in K}\hbar k_0\right)|0\rangle \qquad (3.5b)$$

which imply $\hat{H}$ is not defined on the vacuum state. Also, it is straight forward to verify that the same problem occurs for all states $|n;m\rangle$. This problem may stem from the ground state energies $\hbar k_0/2$ of all the infinitely many oscillators, which add up to an infinite vacuum energy [9]. In quantum mechanics, two Hamiltonians defined only up to the addition of a constant, generate the same dynamics [3]. Then, what is relevant is only the energy difference between the two states, and any constant added to the Hamiltonian cancels out in the calculation of such differences. This suggest that we can redefine the vacuum energy in (3.5b) to be equal to zero and re-write (3.5a) as a well defined Hamiltonian operator

$$H \equiv \sum_{\vec{k}\in K} \hbar k_0 \left(\hat{a}_{\vec{k}}^{\dagger} a_{\vec{k}} + \hat{b}_{\vec{k}}^{\dagger} \hat{b}_{\vec{k}} \right) \tag{3.6}$$

Since the states $|n; m\rangle$ are eigenstates of the Hamiltonian H , then

$$H|n;m\rangle \equiv \left(\sum_{\vec{k}\in K} \hbar k_0 \left(n\left(\vec{k}\right) + m\left(\vec{k}\right) \right) \right) |n;m\rangle \tag{3.7}$$

and we can express all the time dependent operators in the Heisenberg picture. For instance, we have

$$a_{\vec{k}}(t)|n;m\rangle = e^{\frac{i}{\hbar}Ht} a_{\vec{k}} e^{-\frac{i}{\hbar}Ht} |n;m\rangle = e^{-ik_0 t} a_{\vec{k}} |n;m\rangle \tag{3.8}$$

and applying the same argument for $b_{\vec{k}}$, it follows that

$$\begin{aligned} a_{\vec{k}}(t) &= e^{-ik_0 t} a_{\vec{k}} \qquad & a_{\vec{k}}^{\dagger}(t) &= e^{-ik_0 t} a_{\vec{k}}^{\dagger} \\ b_{\vec{k}}(t) &= e^{-ik_0 t} b_{\vec{k}} \qquad & b_{\vec{k}}^{\dagger}(t) &= e^{-ik_0 t} b_{\vec{k}}^{\dagger} \end{aligned} \tag{3.9}$$

for all states $|n; m\rangle$ [6]. Substitute (3.9) in (3.4), the time evolution operators for the canonical coordinates become

$$\begin{aligned} \hat{\alpha}_{\vec{k}}(t) &= \frac{\sqrt{\hbar k_0}}{2}\left(e^{-ik_0 t} a_{\vec{k}} + e^{ik_0 t} a_{\vec{k}}^{\dagger} \right), \qquad & \hat{\beta}_{\vec{k}}(t) &= \frac{\sqrt{\hbar k_0}}{2}\left(e^{-ik_0 t} b_{\vec{k}} + e^{ik_0 t} b_{\vec{k}}^{\dagger} \right) \\ \hat{\pi}_{\vec{k}}(t) &= -i\sqrt{\frac{\hbar}{2k_0}}\left(e^{-ik_0 t} a_{\vec{k}} - e^{ik_0 t} a_{\vec{k}}^{\dagger} \right) \qquad & \hat{\omega}_{\vec{k}}(t) &= -i\sqrt{\frac{\hbar}{2k_0}}\left(e^{-ik_0 t} b_{\vec{k}} - e^{ik_0 t} b_{\vec{k}}^{\dagger} \right) \end{aligned} \tag{3.10}$$

We substitute (3.10) in (2.35), the explicit expression for the field operators in Heisenberg picture (or in Schrodinger picture by putting $(t = 0)$

$$\Psi(r,t) = \begin{pmatrix} \hat{\vec{E}}(r,\, t) \\ c\hat{\vec{B}}(r,\, t) \end{pmatrix} = \sum_{\vec{k}\in K} \frac{\sqrt{\hbar k_0}}{2} \begin{pmatrix} e^{-ik_0 t}\left(\vec{e}_{\vec{k},u}(\vec{r}) a_{\vec{k}} + \vec{e}_{\vec{k},v}(\vec{r}) b_{\vec{k}} \right) + e^{ik_0 t}\left(\vec{e}_{\vec{k},u}(\vec{r}) a_{\vec{k}}^{\dagger} + \vec{e}_{\vec{k},v}(\vec{r}) b_{\vec{k}}^{\dagger} \right) \\ ie^{-ik_0 t}\left(\vec{f}_{\vec{k},u}(\vec{r}) b_{\vec{k}} - \vec{f}_{\vec{k},v}(\vec{r}) a_{\vec{k}} \right) - ie^{ik_0 t}\left(\vec{f}_{\vec{k},u}(\vec{r}) b_{\vec{k}}^{\dagger} - \vec{f}_{\vec{k},v}(\vec{r}) a_{\vec{k}}^{\dagger} \right) \end{pmatrix} \tag{3.11}$$

which is the short hand for the six equations, each defining the operator corresponding to a component of the electromagnetic field.

4. Conclusion

With these explicit expressions for all the operators in the Heisenberg picture, we have obtained the second quantization of the electromagnetic field. We could work out expressions for commutators or vacuum expectation values of products at various points and times. In most instances, we would find that the corresponding sums over the $\vec{k} \in K$ do not converge as functions, at best they can be interpreted as Fourier series of the distributions. Even the field operators are therefore rather singular objects, sometimes referred to as operator-valued distributions. As for ordinary distributions, a good way of making sense of such operator-valued distributions is to smear them with suitable test functions. Such smeared operators will naturally arise in the next article under preparation. The well defined operators corresponding to the canonical variable are examples of such smeared field operators.

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