

# Application of the extended $\exp(-\varphi(\xi))$ -expansion method to the nonlinear conformable time-fractional partial differential equations

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## Abstract

This paper investigates the new exact solutions of the three nonlinear time fractional partial differential equations namely the nonlinear time fractional Clannish Random Walker's Parabolic (CRWP) equation, the nonlinear time fractional modified Kawahara equation, and the nonlinear time fractional BBM-Burger equation by utilizing an extended form of  $\exp(-\varphi(\xi))$ -expansion method in the sense of conformable fractional derivative. As outcomes, some new exact solutions are obtained and signified by hyperbolic function solutions, trigonometric function solutions, and rational function solutions. Some solutions have been plotted by MATLAB software to show the physical significance of our studied equations. In the point of view of our executed method and generated results, we may conclude that extended  $\exp(-\varphi(\xi))$ -expansion method is more efficient than  $\exp(-\varphi(\xi))$ -expansion method to extract the new exact solutions for solving any types of integer and fractional differential equations arising in mathematical physics.

**Keywords:** Conformable Fractional Derivative; Extended  $\exp(-\Phi(\Xi))$ -Expansion Method; Exact Solution; Time Fractional CRWP Equation; Time Fractional BBM-Burger Equation; Time Fractional Modified Kawahara Equation.

## 1. Introduction

Nowadays, fractional partial differential equations (FPDEs) have received considerable attention owing to their various physical aspects in different fields, such as physics, applied mathematics, mathematical biology, engineering, fluid mechanics, plasma physics, optical fibers, neural physics, solid state physics, viscoelasticity, electromagnetism, electrochemistry, signal processing, chaos, the finance and fractal dynamics and etc. [1–3].

Over the past few decades, due to the advent of computational facilities, many powerful symbolic computer software such as Maple, Mathematica, and MATLAB have provided a platform in which researchers [5–34] can obtain new exact solution of well-known FPDEs that arises in applied sciences by numerous robust influential methods, such as Sub-equation method [4,5], Improved sub-equation method [6], [7], Exp-function method [8], First integral method [9],  $(G'/G)$ -expansion method [10–12], Improved  $(G'/G)$ -expansion method [13], [14], [27],  $(G'/G, 1/G)$ -expansion method [14], Improved  $(1/G')$ -expansion method [14], Modified simple equation method [15], [16], Modified Kudryashov method [17–19], The generalized Kudryashov method [20], [21], Exponential rational function method [22],  $\exp(-\varphi(\xi))$ -expansion method [23–25], Extension  $\exp(-\varphi(\xi))$ -expansion method [26], [27], Sine-Gordon equation expansion method [29,30], Extended sin-Gordon equation method [31–33], and so on [34]. Exact solutions for FDEs are used in mathematical modeling of physical phenomena and become one of the furthestmost exciting active areas of research investigation for mathematicians, physicists, and engineers.

Many scholars [23–25] have employed the  $\exp(-\varphi(\xi))$  expansion method to look for new types of traveling wave solutions of the nonlinear partial differential equations (PDEs) arising in above discussed fields. In the  $\exp(-\varphi(\xi))$  expansion method,  $u(\xi) = \sum_{i=0}^n a_i (\exp(-\varphi(\xi)))^i$  is considered as the exact solutions of the nonlinear PDEs with the aid of an auxiliary equation which is defined by  $\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda$ , where  $\mu, \lambda \in \mathbb{R}$ .

Recently, authors [26], [27] have executed the extended  $\exp(-\varphi(\xi))$ - expansion method to solve numerous nonlinear PDEs and space and time fractional PDES and obtained some new solutions rather than the  $\exp(-\varphi(\xi))$ - expansion method. For this purpose, they considered the solution type  $u(\xi) = \sum_{i=0}^n a_i (\exp(-\varphi(\xi)))^i + \sum_{i=1}^n b_i (\exp(-\varphi(\xi)))^{-i}$  through an auxiliary ordinary differential equation  $\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda$ .

To date, as far as authors knowledge, no scholars did not further investigate the exact solutions of the nonlinear time fractional Clannish Random Walker's Parabolic (CRWP) equation, the nonlinear time fractional modified Kawahara equation and the nonlinear time fractional BBM-Burger equation through the extended  $\exp(-\varphi(\xi))$ -expansion method with conformable fractional derivative sense.

This article aims to adopt the extended  $\exp(-\varphi(\xi))$ -expansion method for constructing the exact solutions for the nonlinear time fractional Clannish Random Walker's Parabolic (CRWP) equation, the nonlinear time fractional modified Kawahara equation and the nonlinear time fractional BBM-Burger equation with the aid of conformable fractional derivative sense.

The rest of this paper is organized as follows. In section 2, the description of the conformable fractional derivative and method are discussed. In section 3, As applications of this method, to construct the exact solitary wave solutions of the nonlinear conformable time fractional equations. In section 4, we provide some graphical illustrations among the derived solutions. Finally, we briefly conclude our generated solutions and executed method in section 5.

## 2. Preliminaries and methods

### 2.1. Definition and some features of conformable fractional derivative

The conformable fractional derivative with a limit operator which was initially introduced by Khalil et al. [35]. The definition of the conformable fractional derivative with the limit operator is as follows:

Definition 1.  $f: (0, \infty) \rightarrow \mathbb{R}$ , then, the conformable fractional derivative of  $f$  order  $\alpha$  is defined as

$$D_t^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \text{ for all } t > 0, 0 < \alpha \leq 1.$$

Later, Abdeljawad [36] has also offered chain rule, exponential functions, Gronwalls inequality, integration by parts, Taylor power series expansions and Laplace transform for conformable derivative in fractional versions. The definition of conformable fractional derivative can easily overcome the difficulties of exiting modified Riemann-Liouville derivative definition [37]. The conformable fractional derivative satisfies some workable features which are mentioned in the following theorems [18,19,31,35,36]:

Theorem 1. Let  $\alpha \in (0,1]$ , and  $f = f(t)$ ,  $g = g(t)$  be  $\alpha$ -conformable differentiable at a point  $t > 0$ , then:

- i)  $D_t^\alpha (af + bg) = aD_t^\alpha f + bD_t^\alpha g$ , for all  $a, b \in \mathbb{R}$ .
- ii)  $D_t^\alpha (t^\mu) = \mu t^{\mu-\alpha}$ , for all  $\mu \in \mathbb{R}$ .
- iii)  $D_t^\alpha (fg) = gD_t^\alpha (f) + fD_t^\alpha (g)$ .
- iv)  $D_t^\alpha \left(\frac{f}{g}\right) = \frac{gD_t^\alpha (f) - fD_t^\alpha (g)}{g^2}$ .

Furthermore, if  $f$  is differentiable, then  $D_t^\alpha (f(t)) = t^{1-\alpha} \frac{df}{dt}$ .

**Theorem 2:** Let  $f: (0, \infty) \rightarrow \mathbb{R}$  be a function such that  $f$  is differentiable and  $\alpha$ -conformable differentiable. Also, let  $g$  be a differentiable function defined in the range of  $f$ . Then

$$D_t^\alpha (f \circ g)(t) = t^{1-\alpha} g'(t)^{\alpha-1} D_t^\alpha (f(t))_{t=g(t)}.$$

where prime denotes the classical derivatives with respect to  $t$ .

### 2.2. Outline of the extended $\exp(-\varphi(\xi))$ - expansion technique

Let us consider general nonlinear FPDEs in the form

$$P(u, D_t^\alpha u, u_{x_1}, \dots, u_{x_m}, D_t^{2\alpha} u, u_{x_1 x_2}, \dots) = 0, 0 < \alpha \leq 1. \quad (1)$$

Where  $u = u(t, x_1, \dots, x_m)$ , is an unknown function,  $D_t^\alpha u$  and  $D_t^{2\alpha} u$  are the modified Riemann-Liouville derivatives of  $u$  with respect to  $t$ .  $P$  is a polynomial in  $u = u(t, x_1, \dots, x_m)$  and its various partial derivatives, in which the nonlinear terms and highest order derivatives are involved.

The main steps of this method are detailed in the article (see details Kumar and Kaplan [26]).

**Step 1:** Combine the real variables  $x_1, x_2, \dots, x_m$  and  $t$  by a compound variable  $\xi$

$$u = u(t, x_1, \dots, x_m) = u(\xi), \xi = k_1 x_1 + \dots + k_m x_m \pm \frac{vt^\alpha}{\alpha}, \quad (2)$$

where,  $k_1, k_2, k_3, \dots, k_m$  and  $v$  are arbitrary constants. By applying the traveling wave transformation of Eq. (2) into Eq. (1), we have an ordinary differential equation (ODE) for  $u = u(\xi)$

$$Q(u, u', u'', u''', \dots) = 0. \quad (3)$$

where  $Q$  is a polynomial of  $u$  and, its derivatives and the superscripts indicate the ordinary derivatives with respect to  $\xi$ . If possible, we should integrate Eq. (3) term by term one or more times.

**Step 2:** Assume the traveling wave solution of Eq. (3) can be manifested as follows:

$$u(\xi) = \sum_{i=0}^n a_i \left(\exp(-\varphi(\xi))\right)^i + \sum_{i=1}^n b_i \left(\exp(-\varphi(\xi))\right)^{-i}, \quad (4)$$

where the coefficients  $a_i (0 \leq i \leq n, n \in \mathbb{N})$  and  $b_i (1 \leq i \leq n, n \in \mathbb{N})$  are constants to be determined and either  $a_n$  or  $b_n$  may be zero but both  $a_n$  and  $b_n$  cannot be zero simultaneously. The positive integer  $n$  can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (3). Moreover, we define the degree of  $u(\xi)$  as  $D(u(\xi)) = n$ , which gives rise to a degree of another expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n + q).$$

Where  $\varphi = \varphi(\xi)$  satisfies the following ordinary differential equation:

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda, \quad (5)$$

**Step 3:** By determining the value of parameter  $n$ , we substitute this value into Eq. (4) along with Eq. (5). Then we put the value of  $u(\xi)$  and its derivatives into Eq.(3) and collecting all the terms of the same power  $\exp(-\varphi(\xi))^i, i = 0, \pm 1, \pm 2 \dots \pm n$  and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple to get the values of  $a_i$ 's,  $b_i$ 's,  $k_1, k_2, k_3, \dots, k_n, v$  and constant of integration.

As we know, Eq. (5) has the following general solutions [26]:

*Type-I (Hyperbolic function solutions):* When  $\mu \neq 0, \Delta = \lambda^2 - 4\mu > 0$

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu} \tanh\{\sqrt{\lambda^2 - 4\mu}(\xi + E)\} - \lambda}{2\mu}\right),$$

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu} \coth\{\sqrt{\lambda^2 - 4\mu}(\xi + E)\} - \lambda}{2\mu}\right),$$

*Type-II (Trigonometric function solutions):* When  $\mu \neq 0, \Delta = \lambda^2 - 4\mu < 0$

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2} \tan\{\sqrt{4\mu - \lambda^2}(\xi + E)\} - \lambda}{2\mu}\right),$$

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2} \cot\{\sqrt{4\mu - \lambda^2}(\xi + E)\} - \lambda}{2\mu}\right),$$

*Type-III (Exponential function solutions):* When  $\mu = 0, \lambda \neq 0, \lambda^2 - 4\mu > 0$

$$\varphi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi + E)) - 1}\right),$$

*Type-IV (Rational function solutions):* When  $\mu \neq 0, \lambda \neq 0, \lambda^2 - 4\mu = 0$

$$\varphi(\xi) = \ln\left(-\frac{2(\lambda(\xi + E) + 2)}{\lambda^2(\xi + E)}\right),$$

*Type-V: (Other solutions):* When  $\mu = 0, \lambda = 0, \lambda^2 - 4\mu = 0$

$$\varphi(\xi) = \ln(\xi + E),$$

where  $E$  is the integrating constant.

**Step 4:** Substituting the values of  $a_i$ 's,  $b_i$ 's,  $k_1, k_2, k_3, \dots, k_n, v$  and constant value along with the general solutions of Eq. (5) into Eq. (4). Then we find the complete solution of the general nonlinear FPDEs of Eq. (1).

### 3. Applications of the suggested technique

In this section, we will adopt the technique described in Section 2 to seek the exact solutions for the nonlinear time fractional Clannish Random Walker's Parabolic (CRWP) equation, the nonlinear time fractional modified Kawahara equation and the nonlinear time fractional BBM-Burger equation.

#### 3.1. Time fractional clannish random walker's parabolic (CRWP) equation

In this sub-section, we consider the time fractional Clannish Random Walker's Parabolic (CRWP) equation [38].

$$D_t^\alpha u - u_x + 2uu_x + u_{xx} = 0, t > 0, x \in \mathbb{R}. \quad (6)$$

where,  $\alpha$  is a parameter describing the order of the fractional time derivative and  $0 < \alpha \leq 1$ .

In the past, there are a few articles about this equation. In 2013, Hasan Bulut and Bülent Kılıç [38] applied the Kudryashov method of Eq. (6) and generated the hyperbolic function solutions. Recently, Odabasi and Misirli [39] employed the modified trial equation method and secured several types of new solutions such as periodic function solutions, rational function solutions, and single kink solutions. Very

recently, Ozkan Gunar and his teammates [40] utilized two reliable and efficient methods namely  $(G'/G)$ -expansion method and  $(G'/G, 1/G)$ -expansion method. As outcomes, some new soliton solutions are derived which have been involved as hyperbolic function solutions, trigonometric function solutions, and rational function solutions.

Based on the conformable complex fractional transformation like as section 2, we perform the following transformation:

$$u(x, t) = u(\xi), \xi = kx - \frac{vt^\alpha}{\alpha}. \quad (7)$$

where  $k$  and  $v$  are nonzero constants.

Then Eq. (7) can be reduced to the following ODE is in the form

$$-(k + v)u' + 2kuu' + k^2u'' = 0. \quad (8)$$

Integrating once w.r.t.  $\xi$  and we obtain an ODE from Eq. (8)

$$-(k + v)u + ku^2 + k^2u' + C = 0. \quad (9)$$

where primes denote differentiation with respect to  $\xi$  and  $C$  is the integration constant. By balancing the highest order derivative term  $u'$  with the nonlinear term  $u^2$  in (9), gives  $n = 1$ . Therefore, the extended  $\exp(-\varphi(\xi))$ -expansion method allows us to use the solution in the following form:

$$u(\xi) = a_0 + a_1 \exp(-\varphi(\xi)) + b_1 \exp(-\varphi(\xi))^{-1}. \quad (10)$$

where either  $a_1$  or  $b_1$  may be zero but both  $a_1$  and  $b_1$  cannot be zero simultaneously and  $\varphi(\xi)$  satisfies the Eq. (5).

By substituting Eq. (10) into Eq. (9), we obtain a polynomial in  $\exp(-\varphi(\xi))$ . Setting the coefficients of the powers of  $\exp(-\varphi(\xi))$  to zero, we obtain the following system of algebraic equations:

$$e^{-2\varphi(\xi)} : -k^2a_1 + ka_1^2 = 0, \quad (11a)$$

$$e^{-\varphi(\xi)} : -k^2\lambda a_1 + 2ka_0a_1 - ka_1 - va_1 = 0, \quad (11b)$$

$$e^0 : -k^2\mu a_1 + k^2b_1 + ka_0^2 + 2ka_1b_1 - ka_0 - va_0 + C = 0, \quad (11c)$$

$$e^{\varphi(\xi)} : k^2\lambda b_1 + 2ka_0b_1 - kb_1 - vb_1 = 0, \quad (11d)$$

$$e^{2\varphi(\xi)} : k^2\mu b_1 + kb_1^2 = 0. \quad (11e)$$

Solving the above system of Eq. (11a – 11e) for  $a_0, a_1, b_1, v$  and  $C$ , we obtain the following set of values with the aid of symbolic computer software Maple.

$$\text{Set-1: } C = k^3\mu - k^2\lambda a_0 + ka_0^2, v = -k^2\lambda + 2ka_0 - k, a_0 = a_0, a_1 = k \text{ and } b_1 = 0.$$

$$\text{Set-2: } C = k^3\mu + k^2\lambda a_0 + ka_0^2, v = k^2\lambda + 2ka_0 - k, a_0 = a_0, a_1 = 0 \text{ and } b_1 = -k\mu.$$

The above sets are discussed for different types of solution.

Type-I (Hyperbolic function solutions): When  $\mu \neq 0, \lambda^2 - 4\mu > 0$

Substituting the values of set-1 and set-2 into Eq. (10) along with the hyperbolic function solutions of Eq. (5), we get the following families of hyperbolic function solutions.

$$u_1(x, t) = a_0 - \frac{2k\mu}{\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) + \lambda}, \quad (12)$$

$$u_2(x, t) = a_0 - \frac{2k\mu}{\sqrt{\lambda^2 - 4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) + \lambda}, \quad (13)$$

$$u_3(x, t) = a_0 + \frac{1}{2}k \left( \sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) + \lambda \right), \quad (14)$$

$$u_4(x, t) = a_0 + \frac{1}{2}k \left( \sqrt{\lambda^2 - 4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) + \lambda \right), \quad (15)$$

Type-II (Trigonometric function solutions): When  $\mu \neq 0, \lambda^2 - 4\mu < 0$

Substituting the values of set-1 and set-2 into Eq. (10) along with the trigonometric function solutions of Eq. (5), we obtain the following families of trigonometric function solutions.

$$u_5(x, t) = a_0 + \frac{2k\mu}{\sqrt{4\mu - \lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) - \lambda}, \tag{16}$$

$$u_6(x, t) = a_0 + \frac{2k\mu}{\sqrt{4\mu - \lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) - \lambda}, \tag{17}$$

$$u_7(x, t) = a_0 - \frac{1}{2}k \left( \sqrt{4\mu - \lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) - \lambda \right), \tag{18}$$

$$u_8(x, t) = a_0 - \frac{1}{2}k \left( \sqrt{4\mu - \lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)\right) - \lambda \right), \tag{19}$$

Type-III (Rational function solutions): When  $\mu \neq 0, \lambda \neq 0, \lambda^2 - 4\mu = 0$

Substituting the values of set-1 and set-2 into Eq. (10) along with the rational function solutions of Eq. (5), we find the following families of rational function solutions.

$$u_9(x, t) = a_0 + \frac{1}{2} \left( \frac{k\lambda^2 \left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)}{\lambda \left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right) + 2} \right), \tag{20}$$

$$u_{10}(x, t) = a_0 + \frac{2k\mu \left(\lambda \left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right) + 2\right)}{\lambda^2 \left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right) + 2}, \tag{21}$$

Type-IV (Exponential function solutions): When  $\mu = 0, \lambda \neq 0, \lambda^2 - 4\mu > 0$

Substituting the values of set-1 and set-2 into Eq. (10) along with the exponential function solutions of Eq. (5), we produce the following families of exponential function solutions.

$$u_{11}(x, t) = a_0 + \frac{k\lambda}{e^{\lambda \left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)} - 1}, \tag{22}$$

$$u_{12}(x, t) = a_0 - \frac{k\mu}{\lambda} \left( e^{\lambda \left(kx - \frac{(-k^2\lambda - 2ka_0 - k)t^\alpha}{\alpha} + E\right)} - 1 \right), \tag{23}$$

By comparing our generated solutions with the authors [38], [39] results, we can see that our solutions are new in the sense of conformable fractional derivative.

### 3.2. Time fractional modified Kawahara equation

In this sub-section, we consider the nonlinear time fractional modified Kawahara equation [41].

$$D_t^\alpha u + u^2 u_x + \varepsilon u_{xx} + \delta u_{xxx} = 0, t > 0, x \in \mathbb{R}. \tag{24}$$

Where  $\alpha$  is a parameter describing the order of the fractional time derivative and  $0 < \alpha \leq 1$ . The nonlinear time fractional modified Kawahara equation was studied by different researchers, and it has been solved using various analytical and numerical approaches. For instance, Atangana et al. [40] studied the numerical solutions of time fractional modified nonlinear Kawahara equation using the homotopy decomposition and the Sumudu transform methods. Guner and Hasan [41] solved the time fractional modified nonlinear Kawahara equation using another analytical method namely fractional exp-function method and secured some exact soliton solutions.

For our purpose, we introduce the same transformations as Eq. (7). Moreover, the same procedure we have the ODE from Eq. (24),

$$-vu' + ku^2u' + \varepsilon k^2u'' + \delta k^3u''' = 0, \tag{25}$$

By once integration with respect to, we obtain

$$-vu + \frac{k}{3}u^3 + \varepsilon k^2u' + \delta k^3u'' + C = 0. \tag{26}$$

where primes denote differentiation with respect to  $\xi$  and  $C$  is the integration constant. By balancing the highest order derivative term  $u''$  with the nonlinear term  $u^3$  in (26), gives  $n = 1$ . Therefore, extended  $exp(-\varphi(\xi))$ -expansion method allows us to use the solution in the following form:

$$u(\xi) = a_0 + a_1 exp(-\varphi(\xi)) + b_1 exp(-\varphi(\xi))^{-1}, \tag{27}$$

where either  $a_1$  or  $b_1$  may be zero but both  $a_1$  and  $b_1$  cannot be zero simultaneously and  $\varphi(\xi)$  satisfies the Eq. (5).

By substituting Eq. (27) into Eq. (26), we obtain a polynomial in  $exp(-\varphi(\xi))$ . Setting the coefficients of the powers of  $exp(-\varphi(\xi))$  to zero, we obtain the following system of algebraic equations:

$$e^{-3\varphi(\xi)} : 2\delta k^3 a_1 + \frac{1}{3} k a_1^2 = 0, \quad (28a)$$

$$e^{-2\varphi(\xi)} : 3\delta k^3 \lambda a_1 - k^2 \varepsilon a_1 + k a_0 a_1^2 = 0, \quad (28b)$$

$$e^{-\varphi(\xi)} : k^3 \delta \lambda^2 a_1 + 2k^3 \mu \delta a_1 - k^2 \varepsilon \lambda a_1 + k a_0^2 a_1 + k a_1^2 b_1 - v a_1 = 0, \quad (28c)$$

$$e^0 : k^3 \delta \lambda b_1 - k^2 \mu \varepsilon a_1 + \delta k^3 \mu \lambda a_1 + \frac{1}{3} k a_0^3 + k^2 \varepsilon b_1 + 2k a_0 a_1 b_1 - v a_0 + C = 0, \quad (28d)$$

$$e^{\varphi(\xi)} : k^3 \delta \lambda^2 b_1 + 2k^3 \mu \delta b_1 + k^2 \varepsilon \lambda b_1 + k a_0^2 b_1 + k a_1 b_1^2 - v b_1 = 0, \quad (28e)$$

$$e^{2\varphi(\xi)} : 3k^3 \mu \delta \lambda b_1 + k^2 \mu \varepsilon b_1 + k a_0 b_1^2 = 0, \quad (28f)$$

$$e^{3\varphi(\xi)} : 2\delta k^3 \mu^2 b_1 + \frac{1}{3} k b_1^3 = 0. \quad (28g)$$

Solving the above system of Eq. (28a – 28g) for  $a_0, a_1, b_1, v$ , and  $C$ , we obtain the following set of values with the aid of symbolic computer software Maple, Mathematica or MATLAB.

$$\text{Set-1: } C = \pm \frac{1}{54} \left( \frac{\varepsilon k \sqrt{-6\delta} (-9k^2 \delta^2 \lambda^2 + 36k^2 \mu \delta^2 + \varepsilon^2)}{\delta^2} \right), a_0 = \pm \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}}, a_1 = \pm \sqrt{-6\delta} k, b_1 = 0, \text{ and } v = \frac{k}{6} \left( \frac{-3k^2 \delta^2 \lambda^2 + 12k^2 \mu \delta^2 - \varepsilon^2}{\delta} \right)$$

$$\text{Set-2: } C = \mp \frac{1}{54} \left( \frac{\varepsilon k \sqrt{-6\delta} (-9k^2 \delta^2 \lambda^2 + 36k^2 \mu \delta^2 + \varepsilon^2)}{\delta^2} \right), a_0 = \mp \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}}, a_1 = 0, b_1 = \mp \sqrt{-6\delta} \mu k, \text{ and } v = \frac{k}{6} \left( \frac{-3k^2 \delta^2 \lambda^2 + 12k^2 \mu \delta^2 - \varepsilon^2}{\delta} \right)$$

The above sets are discussed for different types of solutions.

Substituting the values of set-1 and set-2 into Eq. (27) along with the hyperbolic function solutions of Eq. (5), we obtain the following families of hyperbolic function solutions.

$$u_{1,2}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} + \frac{2k\mu\sqrt{-6\delta}}{\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + E)\right) + \lambda} \right), \quad (29)$$

$$u_{3,4}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} + \frac{2k\mu\sqrt{-6\delta}}{\sqrt{\lambda^2 - 4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + E)\right) + \lambda} \right), \quad (30)$$

$$u_{5,6}(x, t) = \pm \left( \frac{3k\delta\lambda + \varepsilon}{\sqrt{-6\delta}} + \frac{1}{2} k \sqrt{-6\delta} \left( \sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + E)\right) + \lambda \right) \right), \quad (31)$$

$$u_{7,8}(x, t) = \pm \left( \frac{3k\delta\lambda + \varepsilon}{\sqrt{-6\delta}} + \frac{1}{2} k \sqrt{-6\delta} \left( \sqrt{\lambda^2 - 4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + E)\right) + \lambda \right) \right), \quad (32)$$

$$\text{For Eq. (29)-Eq. (32), } \xi = kx - \frac{k}{6} \left( \frac{-3k^2 \delta^2 \lambda^2 + 12k^2 \mu \delta^2 - \varepsilon^2}{\delta} \right) \frac{t^\alpha}{\alpha}.$$

Substituting the values of set-1 and set-2 into Eq. (27) along with the trigonometric function solutions of Eq. (5), we extract the following families of trigonometric function solutions.

$$u_{9,10}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} + \frac{2k\mu\sqrt{-6\delta}}{\sqrt{4\mu - \lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}(\xi + E)\right) - \lambda} \right), \quad (33)$$

$$u_{11,12}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} + \frac{2k\mu\sqrt{-6\delta}}{\sqrt{4\mu - \lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}(\xi + E)\right) - \lambda} \right), \quad (34)$$

$$u_{13,14}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} - \frac{1}{2} k \sqrt{-6\delta} \sqrt{4\mu - \lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}(\xi + E)\right) - \lambda \right), \quad (35)$$

$$u_{15,16}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} - \frac{1}{2} k \sqrt{-6\delta} \sqrt{4\mu - \lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}(\xi + E)\right) - \lambda \right), \quad (36)$$

$$\text{For Eq. (33) to Eq. (36), } \xi = kx - \frac{k}{6} \left( \frac{-3k^2 \delta^2 \lambda^2 + 12k^2 \mu \delta^2 - \varepsilon^2}{\delta} \right) \frac{t^\alpha}{\alpha}.$$

Substituting the values of set-1 and set-2 into Eq. (27) along with the rational function solutions of Eq. (5), we explore the following families of rational function solutions.

$$u_{17,18}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} + \frac{1}{2} \frac{\sqrt{-6\delta} k \lambda^2 (\xi + E)}{\lambda(\xi + E) + 2} \right), \quad (37)$$

$$u_{19,20}(x, t) = \pm \left( \frac{3k\delta\lambda + \varepsilon}{\sqrt{-6\delta}} + \frac{2\sqrt{-6\delta} \mu k (\lambda(\xi + E) + 2)}{\lambda^2 (\xi + E)} \right), \quad (38)$$

$$\text{where } \xi = kx - \frac{k}{6} \left( \frac{-3k^2\delta^2\lambda^2 + 12k^2\mu\delta^2 - \varepsilon^2}{\delta} \right) \frac{t^\alpha}{\alpha}.$$

Substituting the values of set-1 and set-2 into Eq. (27) along with the exponential function solutions of Eq. (5), we find the following families of exponential function solutions.

$$u_{21,22}(x, t) = \pm \left( \frac{3k\delta\lambda - \varepsilon}{\sqrt{-6\delta}} + \frac{\sqrt{-6\delta}k\lambda}{e^{\lambda(\xi+E)} - 1} \right), \quad (39)$$

$$u_{23,24}(x, t) = \pm \left( \frac{3k\delta\lambda + \varepsilon}{\sqrt{-6\delta}} \right), \quad (40)$$

$$\text{where } \xi = kx - \frac{k}{6} \left( \frac{-3k^2\delta^2\lambda^2 - \varepsilon^2}{\delta} \right) \frac{t^\alpha}{\alpha}.$$

Substituting the values of set-1 and set-2 into Eq. (27) along with the other solutions of Eq. (5), we get the following solutions.

$$u_{25,26}(x, t) = \pm \left( \frac{\sqrt{-6\delta}k}{(\xi+E)} \right), \quad (41)$$

$$u_{27,28}(x, t) = \pm \left( \frac{\varepsilon}{\sqrt{-6\delta}} \right), \quad (42)$$

$$\text{where } \xi = kx + \frac{k}{6} \left( \frac{\varepsilon^2}{\delta} \right) \frac{t^\alpha}{\alpha}.$$

Comparing our results with Atangana et al. [40], Guner and Hasan [41] results, it can be seen that the produced results are new and different in the sense of conformable fractional derivative.

### 3.3. Time fractional BBM-Burger equation

In this sub-section, we consider the time the nonlinear time fractional BBM-Burger equation [46].

$$D_t^\alpha u - u_{xxt} + u_x + \left( \frac{1}{2} u^2 \right)_x = 0, t > 0, x \in \mathbb{R}. \quad (43)$$

where  $\alpha$  is a parameter describing the order of the fractional time derivative and  $0 < \alpha \leq 1$ .

S. Kumar and D. Kumar [43] applied new fractional homotopy analysis transform method to time fractional BBM-Burger equation and received the series solution of the Eq. (43). Song and Zhang [44] and Fakhari et al. [45] derived a different type of solutions of the fractional BBM-Burger equation by using homotopy analysis method. Shakeel et al. [46] executed an analytical method which is called fractional novel ( $G'/G$ )-expansion method and attained some new explicit exact solutions.

For our purpose, we introduce the same transformations as Eq. (7). Moreover, the same procedure we have the ODE from Eq. (43),

$$(k - v)u' + k^2 v u''' + k \left( \frac{u^2}{2} \right)' = 0. \quad (44)$$

By once integration with respect to, we obtain

$$(k - v)u + \frac{1}{2} k u^2 + k^2 v u'' + C = 0. \quad (45)$$

where primes denote differentiation with respect to  $\xi$  and  $C$  is the integration constant. By balancing the highest order derivative term  $u''$  with the nonlinear term  $u^2$  in Eq. (45), gives  $n = 2$ . Therefore, extended  $\exp(-\varphi(\xi))$ -expansion method allows us to use the solution in the following form:

$$u(\xi) = a_0 + a_1 \exp(-\varphi(\xi)) + a_2 \exp(-\varphi(\xi))^2 + b_1 \exp(-\varphi(\xi))^{-1} + b_2 \exp(-\varphi(\xi))^{-2}, \quad (46)$$

where either  $a_2$  or  $b_2$  may be zero but both  $a_2$  and  $b_2$  cannot be zero simultaneously and  $\varphi(\xi)$  satisfies the Eq. (5).

Now by substituting Eq. (46) into Eq. (45), we obtain a polynomial in  $\exp(-\varphi(\xi))$ . Setting the coefficients of the powers of  $\exp(-\varphi(\xi))$  to zero and we obtained a system of algebraic equation. Solving this system of algebraic equations using Maple for the set of values of  $a_0, a_1, a_2, b_1, b_2, v, k$ , and  $C$ .

$$\text{Set-1: } C = -\frac{1}{2} \frac{k^4 v^2 \lambda^4 - 8k^4 \mu v^2 \lambda^2 + 16k^4 \mu^2 v^2 - k^2 + 2kv - v^2}{k}, a_0 = -\frac{((\lambda^2 + 8\mu)k^2 - 1)v + k}{k}, a_1 = -12vk\lambda, a_2 = 12kv, b_1 = 0, \text{ and } b_2 = 0.$$

$$\text{Set-2: } C = -\frac{1}{2} \frac{k^4 v^2 \lambda^4 - 8k^4 \mu v^2 \lambda^2 + 16k^4 \mu^2 v^2 - k^2 + 2kv - v^2}{k}, a_0 = -\frac{((\lambda^2 + 8\mu)k^2 - 1)v + k}{k}, a_1 = 0, a_2 = 0, b_1 = -12vk\mu\lambda, \text{ and } b_2 = -12vk\mu^2.$$

The above sets are discussed for different types of solution.

Substituting the values of set-1 and set-2 into Eq. (46) along with the hyperbolic function solutions of Eq. (5), we get the following families of hyperbolic function solutions.

$$u_1(x, t) = -\frac{((\lambda^2 + 8\mu)k^2 - 1)v + k}{k} + \frac{24vk\lambda\mu}{\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + E)\right) + \lambda} - \frac{48vk\mu^2}{\left(\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}(\xi + E)\right) + \lambda\right)^2}, \quad (47)$$

$$u_2(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} + \frac{24vk\lambda\mu}{\sqrt{\lambda^2-4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+E)\right) + \lambda} - \frac{48vk\mu^2}{\left(\sqrt{\lambda^2-4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+E)\right) + \lambda\right)^2}, \quad (48)$$

$$u_3(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} + 6vk\lambda\left(\sqrt{\lambda^2-4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+E)\right) + \lambda\right) - 3vk\left(\sqrt{\lambda^2-4\mu} \tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+E)\right) + \lambda\right)^2, \quad (49)$$

$$u_4(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} + 6vk\lambda\left(\sqrt{\lambda^2-4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+E)\right) + \lambda\right) - 3vk\left(\sqrt{\lambda^2-4\mu} \coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+E)\right) + \lambda\right)^2, \quad (50)$$

Substituting the values of set-1 and set-2 into Eq. (46) along with the trigonometric function solutions of Eq. (5), we find the following families of trigonometric function solutions.

$$u_5(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - \frac{24vk\lambda\mu}{\sqrt{4\mu-\lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda} - \frac{48vk\mu^2}{\left(\sqrt{4\mu-\lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda\right)^2}, \quad (51)$$

$$u_6(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - \frac{24vk\lambda\mu}{\sqrt{4\mu-\lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda} - \frac{48vk\mu^2}{\left(\sqrt{4\mu-\lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda\right)^2}, \quad (52)$$

$$u_7(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - 6vk\lambda\left(\sqrt{4\mu-\lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda\right) - 3vk\left(\sqrt{4\mu-\lambda^2} \tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda\right)^2, \quad (53)$$

$$u_8(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - 6vk\lambda\left(\sqrt{4\mu-\lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda\right) - 3vk\left(\sqrt{4\mu-\lambda^2} \cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+E)\right) - \lambda\right)^2, \quad (54)$$

Substituting the values of set-1 and set-2 into Eq. (46) along with the rational function solutions of Eq. (5), we get the following families of rational function solutions.

$$u_9(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} + \frac{6vk\lambda^2(\xi+E)}{\lambda(\xi+E)+2} - \frac{3vk\lambda^4(\xi+E)^2}{(\lambda(\xi+E)+2)^2}, \quad (55)$$

$$u_{10}(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} + \frac{24vk\mu(\lambda(\xi+E)+2)}{\lambda(\xi+E)} - \frac{48vk\mu^2(\lambda(\xi+E)+2)^2}{\lambda^4(\xi+E)^2}, \quad (56)$$

Substituting the values of set-1 and set-2 into Eq. (46) along with the exponential function solutions of Eq. (5), we derive the following families of exponential function solutions.

$$u_{11}(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - \frac{12vk\lambda^2}{e^{\lambda(\xi+E)}-1} - \frac{12vk\lambda^2}{(e^{\lambda(\xi+E)}-1)^2}, \quad (57)$$

$$u_{12}(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - 12vk\mu(e^{\lambda(\xi+E)} - 1) - \frac{12vk\lambda^2(e^{\lambda(\xi+E)}-1)^2}{\lambda^2}, \quad (58)$$

Substituting the values of set-1 and set-2 into Eq. (46) along with the other solutions of Eq. (5), we get the following solutions.

$$u_{13}(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - \frac{12vk\lambda}{\xi+E} - \frac{12vk}{(\xi+E)^2}, \quad (59)$$

$$u_{14}(x, t) = -\frac{((\lambda^2+8\mu)k^2-1)v+k}{k} - 12vk\mu\lambda(\xi+E) - 12vk\mu^2(\xi+E)^2, \quad (60)$$

For Eq. (47) to Eq. (60),  $\xi = kx - \frac{vt^\alpha}{\alpha}$ .

If we compare our solutions with the solutions appeared in the literature before [46], we can see that our solutions are new in the conformable fractional derivative sense.

## 4. Graphical illustration of the obtained solutions

Graphical illustration is the proper way to understand the real physical significance of any real-world problems. In this section, with the aid of MATLAB software, we have shown the graphical representation of some results in the sense of conformable fractional derivative by choosing different fractional values. By assigning suitable values to the unknown parameters in order to visualize the real mechanism of the derived solutions. Among them, some acquired solutions have been plotted the studied equations which are shown in Figs. 1 – 10.

### 4.1. Time fractional clannish random walker's parabolic (CRWP) equation

In order to visualize the produced solutions mechanism, the kink, periodic, singular periodic, and bright shaped profile have been observed in Figs. 1 – 4. Among them, the kink shaped profile of the solutions (13), and (14) are shown in Figs. 1 – 2 for various values of  $\alpha$ . We observed that when the fractional derivative order  $\alpha$  increased, the shape is closer to the known kink wave as the velocity of the propagation wave decreases. The kink profile keeps its height for various values of  $\alpha$ . On the other hand, it should also be mentioned that the solutions (16) and (26) indicates the periodic and bright soliton profile, which are shown in Figs. 3 – 4, respectively.

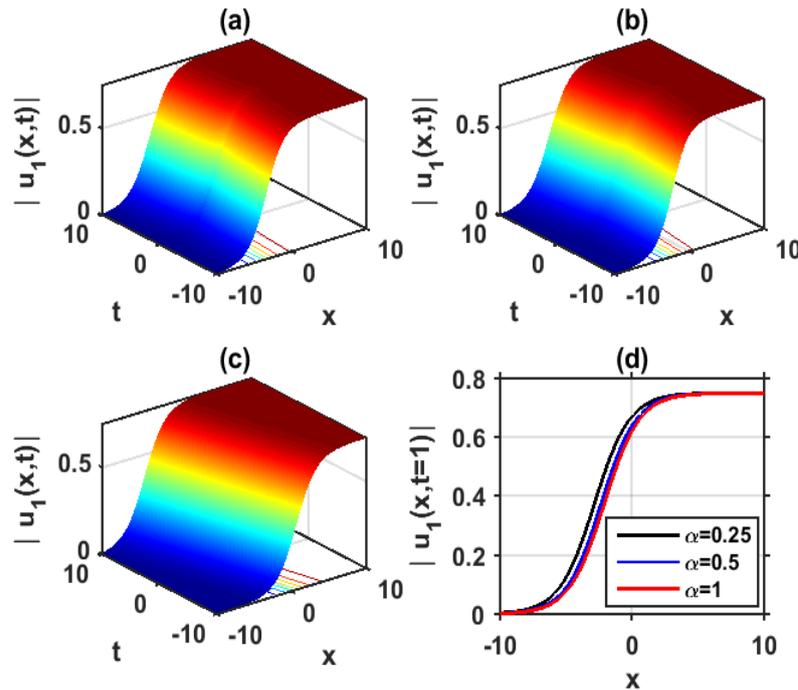
### 4.2. Time fractional modified Kawahara equation

The dynamics of the dark, periodic, and bright soliton solutions (31), (34), and (39) are shown in Figs. 5 – 7 for various values of  $\alpha$ . Remaining of the produced solutions signify the same physical tendency.

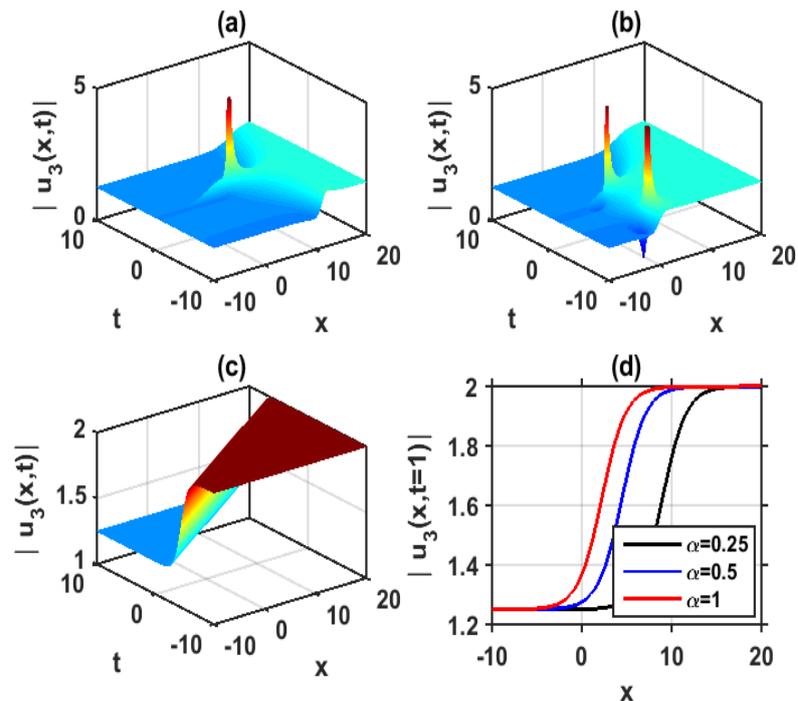
### 4.3. Time fractional BBM-Burger equation

The dynamics of the bright, periodic singular, and bright singular soliton solutions (47), (51), and (56) are shown in Figs. 8 – 10 for various values of  $\alpha$ . Remaining of the derived solutions shapes signify the similar physical tendency which we described as earlier.

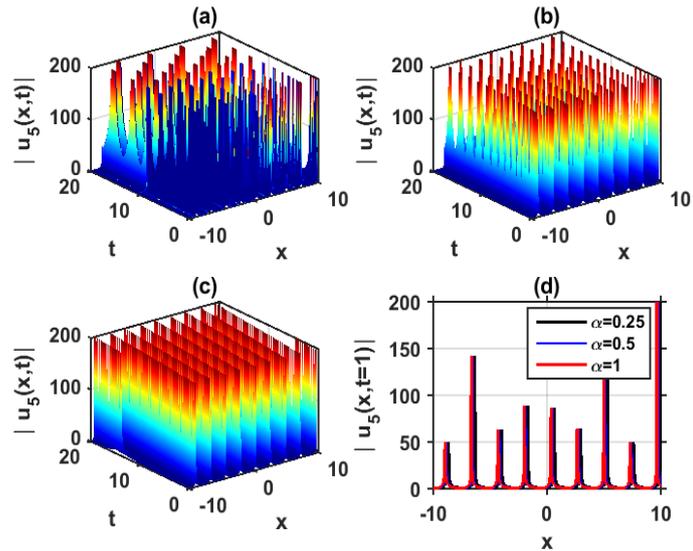
It is noteworthy that the acquired solutions in this article have potential physical meaning for the underlying equations. In addition to the physical meaning, these solutions can be applied to identify the accuracy of numerical results and to help in the study of stability analysis.



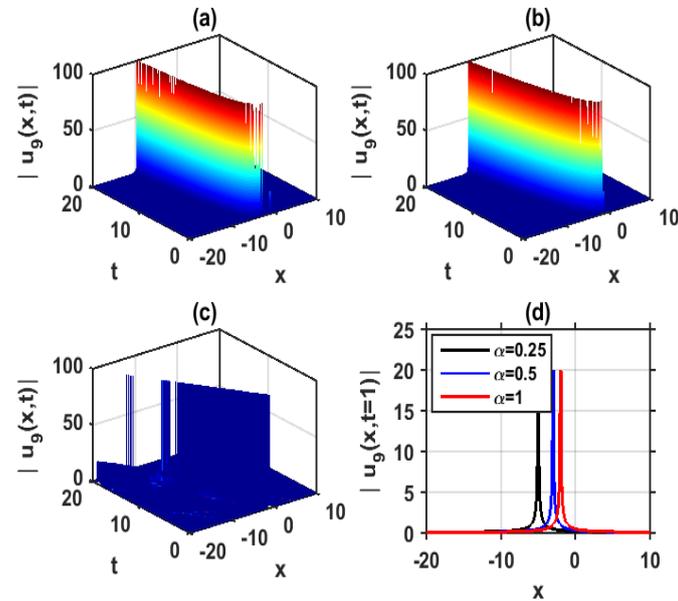
**Fig. 1:** Three-Dimensional (3D) Plot of the Solution (12) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, When  $a_1 = 1, \lambda = 2.5, \mu = 1, k = 0.5, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



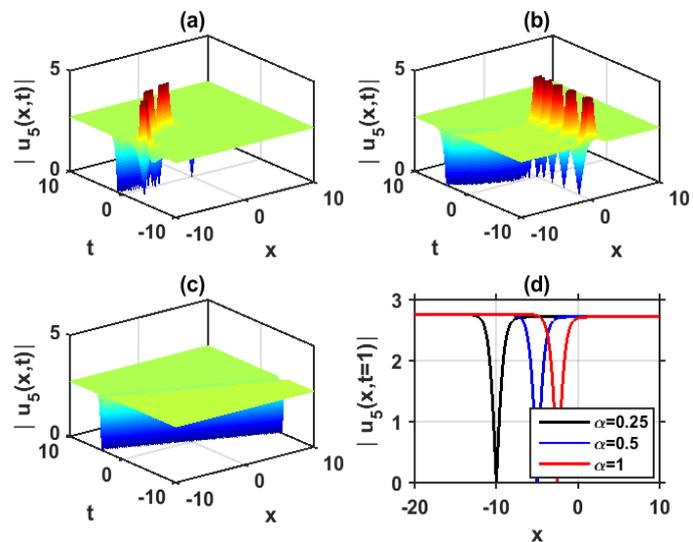
**Fig. 2:** Three-Dimensional (3D) Plot of Solution (14) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, When  $a_1 = 1, \lambda = 2.5, \mu = 1, k = 0.5, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



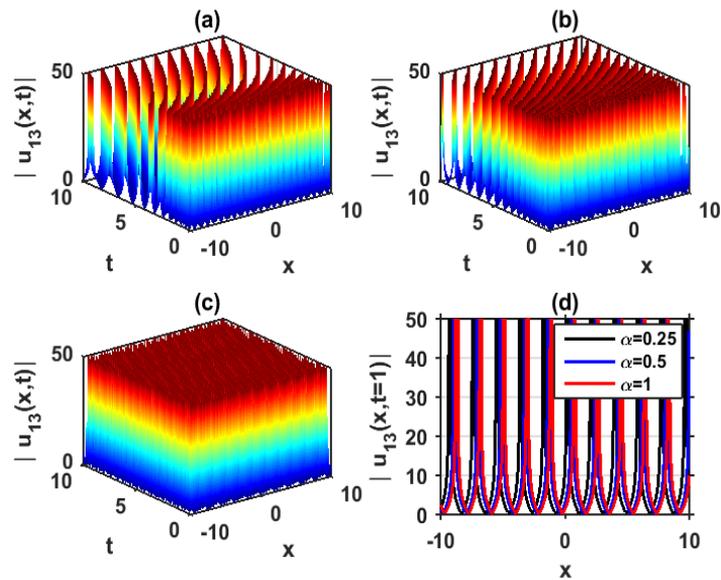
**Fig. 3:** Three-Dimensional (3D) Plot of Solution (16) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, When  $a_1 = 1, \lambda = 1, \mu = 2.5, k = 0.9, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



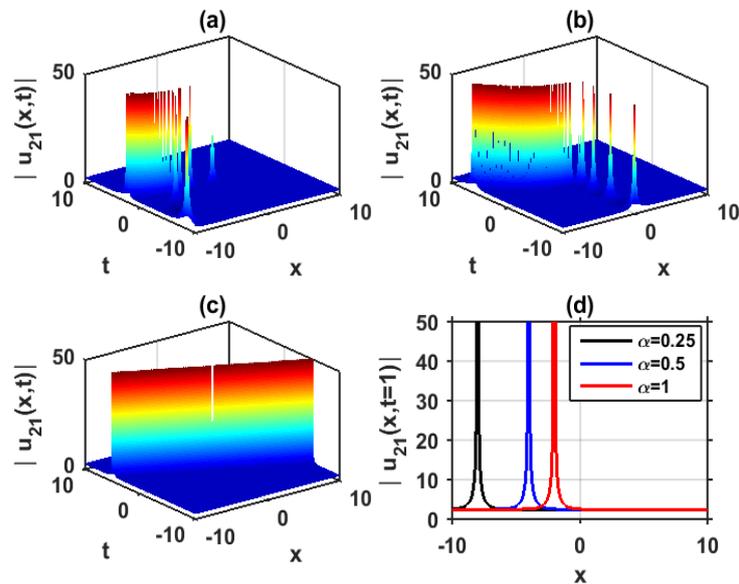
**Fig. 4:** Three-Dimensional (3D) Plot of Solution (20) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, When  $a_1 = 1, \lambda = 2, \mu = 1, k = 1, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



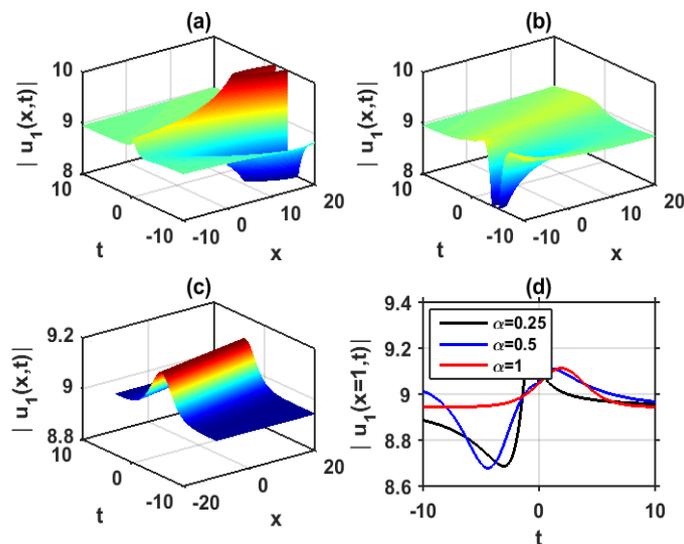
**Fig. 5:** Three-Dimensional (3D) Plot of Solution (31) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, when  $\lambda = 3, \mu = 1, \varepsilon = 0.05, \delta = 1, k = 1, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



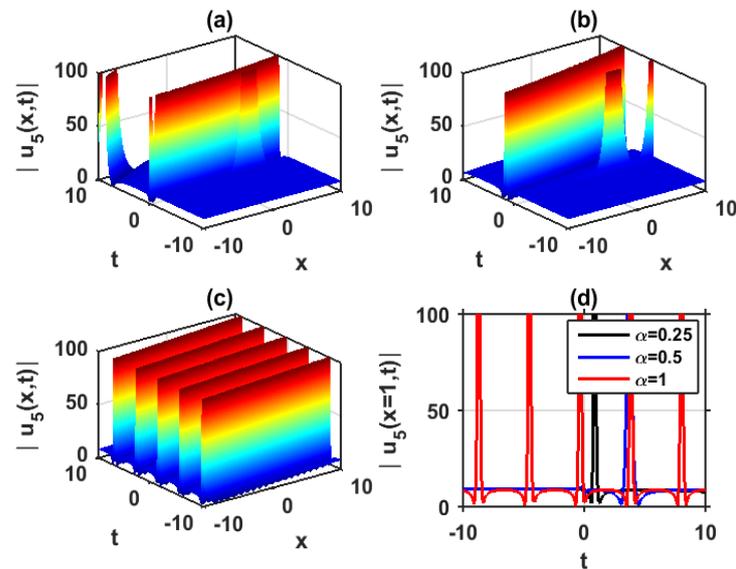
**Fig. 6:** Three-Dimensional (3D) Plot of Solution (34) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, when  $\lambda = 1, \mu = 3, \varepsilon = 0.05, \delta = 1, k = 1, E = 0$ . (d) Variation In Two-Dimensional (2D) Line Plot for Different Fractional Values.



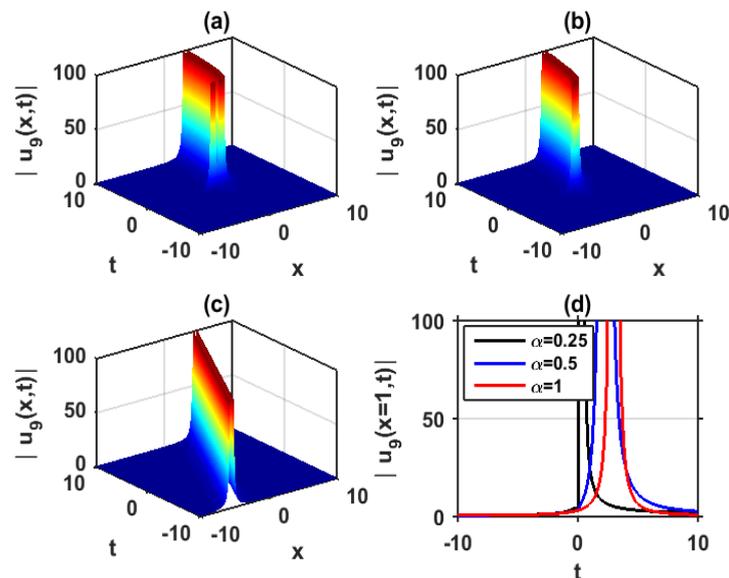
**Fig. 7:** Three-Dimensional (3D) Plot of Solution (39) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, when  $\lambda = 2, \mu = 1, \varepsilon = 0.05, \delta = 1, k = 1, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



**Fig. 8:** Three-Dimensional (3D) Plot Solution (47) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, when  $\lambda = 2.5, \mu = 1, k = 0.05, v = 0.5, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



**Fig. 9:** Three-Dimensional (3D) Plot of Solution (51) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, when  $\lambda = 1, \mu = 2.5, k = 0.05, v = 0.5, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.



**Fig. 10:** Three-Dimensional (3D) Plot of Solution (56) with (a)  $\alpha = 0.25$ , (b)  $\alpha = 0.5$ , and (c)  $\alpha = 1$  Respectively, when  $\lambda = 2.5, \mu = 0, k = 0.05, v = 1, E = 0$ . (d) Variation in Two-Dimensional (2D) Line Plot for Different Fractional Values.

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