International Journal of Engineering & Technology, 7 (2) (2018) 597-604

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET
doi: 10.14419/ijet.v7i2.9191
Research paper

Analysis of nonlinear oscillator arising in the microelectromechanical system by using the parameter expansion and equivalent linearization methods

D. V. Hieu *, N. T. K. Thoa, L. Q. Duy

Department of Mechanics, Thai Nguyen University of Technology, Thai Nguyen, Viet Nam
**Corresponding author E-mail: hieudv@tnut.edu.vn*

Abstract

In this paper, the nonlinear oscillator arising in the microbeam-based micro-electromechanical system (MEMS) is described. The motion equation of a microbeam is simplified into an ordinary differential equation by using the Galerkin method. The nonlinear ordinary differen-tial equation is solved by using two methods including the Parameter-Expansion and Equivalent Linearization Methods. To verify the accu-racy of the present methods, illustrative examples are provided and compared with other analytical, exact and numerical solutions.

Keywords: *Parameter Expansion Method; Equivalent Linearization Method; Nonlinear Oscillator; MEMS.*

1. Introduction

Micro-electromechanical systems (MEMS) is a process technology used to create tiny integrated devices or systems that combine mechanical and electrical components. They are fabricated using integrated circuit (IC) batch processing techniques and can range in size from a few micrometers to millimetres. These devices (or systems) have the ability to sense, control and actuate on the micro scale, and generate effects on the macro scale. MEMS are used in many engineering applications such as: microswitches, transistors, accelerometers, etc. While working, electrostatic actuation, large deflections and damping caused by different sources raises nonlinear behavior. Nonlinearity in MEMS may cause some difficulties in computations. Until now, several techniques have been used to find numerical solutions of this nonlinear problem, for example the shooting method [1], the differential quadrature method [2] and the Adomian decomposition method [3]. Although it is difficult to get analytic approximations for different phenomena in MEMS, there are some analytic techniques for nonlinear problems of MEMS such as perturbation techniques [4], the energy balance method (EBM) [5], the homotopy analysis method (HAM) [6] and the He's Variational Approach (VA) [7].

The Equivalent Linearization method (ELM) is one of the common approaches to approximate analysis of nonlinear dynamical systems. The original linearization for deterministic systems was proposed by Krylov and Bugoliubov [20]. Then Caughey [21] expanded the method for stochastic systems. To date, there have been some extended versions of the Equivalent Linearization method [22, 23]. It has been shown that the Equivalent Linearization method is presently the simplest tool widely used for analyzing nonlinear stochastic problems.

The Parameter-Expansion Method proposed by He [8, 10] is an effective approach to analytical investigation of nonlinear problems. This approach are investigated in different works [9, 11-17].

In this work, the Parameter-Expansion Method (PEM) and the Equivalent Linearization Method (ELM) are applied to analyse the nonlinear oscillator arising in the microbeam-based micro-electromechanical system. The analytical results achived by two methods are compared with the previous analytical results, the exact results and the numerical results.

2. Basic of parameter-expansion method

In order to introduce the PEM, we consider the general form of Duffing equation in the following form:

$$\ddot{u} + \varepsilon u + 1 \cdot N(u,t) = 0 \tag{1}$$

Where $N(u, t)$ includes the nonlinear term.

Expanding the solution u, ε as a coefficient of u, and 1 as a coefficient of $N(u, t)$, the series of p can be introduced as follows [8, 10]:

$$u = \sum_{i=0}^{n} p^i u_i(t) = u_0(t) + p u_1(t) + p^2 u_2(t) + \cdots \tag{2}$$

$$\varepsilon = \omega^2 + \sum_{i=1}^{n} p^i \omega_i = \omega^2 + p\omega_1 + p^2 \omega_2 + \cdots \tag{3}$$

$$1 = \sum_{i=1}^{n} p^i a_i = p a_1 + p^2 a_2 + \cdots \tag{4}$$

Substituting Eqs. (2)- (4) into Eq. (1) and equating the terms with the identical powers of p, we have:

$$p^0 : \ \ddot{u}_0 + \omega^2 u_0 = 0 \tag{5}$$

$$p^1 : \ \ddot{u}_1 + \omega^2 u_1 + \omega_1 u_0 + a_1 N(u_0, t) = 0 \tag{6}$$

Copyright © 2018 D. V. Hieu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Considering the initial conditions $u_0(0) = A$ and $\dot{u}_0(0) = 0$, the solution of Eq. (5) is $u_0 = A\cos(\omega t)$. Substituting u_0 into Eq. (6), we obtain:

$$p^1: \ \ddot{u}_1 + \omega^2 u_1 + \omega_1 A\cos(\omega t) + a_1 N(A\cos(\omega t), t) = 0 \tag{7}$$

For achieving the secular term, we use Fourier expansion series as follows:

$$N(A\cos(\omega t), t) = \sum_{k=0}^{\infty} b_{2k+1}\cos((2k+1)\omega t) \tag{8}$$

Substituting Eq. (8) into Eq. (7) yields:

$$p^1: \ \ddot{u}_1 + \omega^2 u_1 + (\omega_1 A + a_1 b_1)\cos(\omega t) = 0 \tag{9}$$

For avoiding secular term, we consider:

$$\omega_1 A + a_1 b_1 = 0 \tag{10}$$

Setting p=1 in Eqs. (3) and (4), we have:

$$\omega_1 = \varepsilon - \omega^2 \tag{11}$$

$$a_1 = 1 \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10), we will achieve the first-order approximation frequency of this oscillator (1). Note that, from Eqs. (4) and (12), we can find that a_i=0 for all i=1,2,3,4,…

3. Basic of equivalent linearization method

To introduce an overview of the Equivalent Linearization method, we consider the oscillation described by following nonlinear differential equation:

$$\ddot{X} + g(X) = 0, \quad X(0) = A, \dot{X}(0) = 0 \tag{13}$$

where $g(X)$ is a nonlinear function of X and A is the initial amplitude.
The idea of the Equivalent Linearization method is to replace the nonlinear term $g(X)$ in Eq. (13) by the linear term as follows [21]:

$$g(X) \to \alpha X \tag{14}$$

By this manner, the linearized equation of Eq. (13) is given by:

$$\ddot{X} + \alpha X = 0 \tag{15}$$

where the cofficient α of the linear term is determined by using the mean-square criterion [21-23]:

$$e^2(X) = [g(X) - \alpha X]^2 \to \underset{\alpha}{Min} \tag{16}$$

Thus, from:

$$\frac{\partial \langle e^2(X)\rangle}{\partial \alpha} = 0$$

yields:

$$\alpha = \frac{\langle g(X)X\rangle}{\langle X^2\rangle} \tag{17}$$

In Eq. (17), the symbol $\langle \bullet \rangle$ donotes the time-averaging operator in classical meaning [20]:

$$\langle f(t)\rangle = \lim_{T\to+\infty}\frac{1}{T}\int_0^T f(t)dt \tag{18}$$

For a ω-frequency function $f(\omega t)$, the averaging process is taken during one period T, i.e.

$$\langle f(\omega t)\rangle = \frac{1}{T}\int_0^T f(\omega t)dt = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau, \quad \tau = \omega t \tag{19}$$

Using the solution of the linearized equation (15), we can calculate averaging values in Eq. (17), and thus the frequency - amplitude relationship of the oscillation is obtained in closed form.

4. Modeling and formulation

Consider a fixed-fixed microbeam placed between two stationary electrodes with length l, width band thickness h, whose sketch is shown in Fig. 1 together with coordinate oxyz, where g_0 is the initial gap and v the electrostatic load. The equation of motion that governs the transverse deflection $w(x,t)$ is written as [5]:

$$\bar{E}I\frac{\partial^4 w}{\partial x^4} + \rho S\frac{\partial^2 w}{\partial t^2} = \left[\bar{N} + \frac{ES}{2l}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2 dx\right]\frac{\partial^2 w}{\partial x^2} + q(x,t) \tag{20}$$

Where $\bar{E}$ is the effective modulus, should be noted that, when a wide microbeam ($b \geq 5h$) is considered, $\bar{E}$ becomes the plate modulus $\bar{E} = E/(1-\nu^2)$ where E is the Young's modulus and ν is the Poisson ratio, while a narrow microbeam is considered, $\bar{E}$ simply becomes the Young's modulus E; $S=bh$ and $I=bh^3/12$ are the area and moment of inertia of the cross-section, respectively. $\bar{N}$ denotes the tensile or compressive axial load created by the mismatch of both thermal expansion coefficient and crystal lattice period between substrate and the thinfilm (microbeam). The second term on the right-hand side, $q(x,t)$, denotes the driving force per unit length, resulting from electrostatic excitation [20]:

$$q(x,t) = \frac{\varepsilon_v b v^2}{2}\left[\frac{1}{(g_0 - w)^2} - \frac{1}{(g_0 + w)^2}\right] \tag{21}$$

Where ε_v is the dielectric constant of the gap medium which is usually taken as $\varepsilon_v = 8.85\text{pF/m}$.
As fixed-fixed microbeams are of primary interest here, the boundary conditions are given by:

$$x = 0, \ l: \quad w = \frac{\partial w}{\partial x} = 0 \tag{22}$$

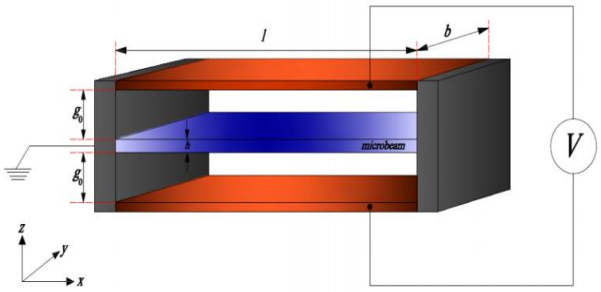


Fig. 1: Schematics of A Double-Sided Driven Clamped-Clamped Microbeam-Based Electromechanical Resonator.

For convenience, the following nondimensional variables and parameters are introduced:

$$\xi = \frac{x}{l}, W = \frac{w}{g_0}, \tau = t\sqrt{\frac{\bar{E}I}{\rho b h l^4}}, \alpha = 6\left(\frac{g_0}{h}\right)^2,$$
$$N = \frac{\bar{N}l^2}{\bar{E}I}, V^2 = \frac{24\varepsilon_v l^4 v^2}{\bar{E}h^3 g_0^3} \tag{23}$$

The normalized form of the governing equation is given by:

$$\frac{\partial^4 W}{\partial \xi^4} + \frac{\partial^2 W}{\partial \tau^2} = \left[\begin{array}{c} N \\ +\alpha\int_0^1 \left(\frac{\partial W}{\partial \xi}\right)^2 d\xi \end{array}\right] \frac{\partial^2 W}{\partial \xi^2} + \frac{V^2}{4}\left[\begin{array}{c} \frac{1}{(1-W)^2} \\ -\frac{1}{(1+W)^2} \end{array}\right] \tag{24}$$

And the corresponding nondimensional boundary conditions are:

$$\xi = 0, 1: \quad W = \frac{\partial W}{\partial \xi} = 0 \tag{25}$$

The deflection $W(\xi,\tau)$ in Eq. (24) is expressed as a sum of spatial shapes that, a priori, satisfy the imposed boundary conditions:

$$W(\xi,\tau) = \sum_{i=1}^{n} \phi_i(\xi) u_i(\tau) \tag{26}$$

Where n is the number of degrees of freedom, $\phi_i(\xi)$ is the ith eigenfunction of the beam and $u_i(\tau)$ is the ith time-dependent deflection parameter of the beam. Based on a single degree-of-freedom model of the beams (n=1), Eq. (24) can be solved with appropriate accuracy [18]. Hence, the solution is constructed by expressing the deflection function $W(\xi,\tau)$ as the product of two separate functions:

$$W(\xi,\tau) = \phi(\xi) u(\tau) \tag{27}$$

As most earlier work suggested, here the trial function is $\phi(\xi) = 16\xi^2(1-\xi)^2$ [19]. Obviously, Eq. (27) satisfies all the boundary conditions listed in Eq. (25). In order to avoid division by zero in the electrostatic force term, we multiply Eq. (24) by $(1-W)^2$. Substituting Eq. (27) into the resulting equation, multiplying by $\phi(\xi)$, and integrating the outcome from 0 to 1, we obtain:

$$(a_1 u^4 + a_2 u^2 + a_3)\ddot{u} + a_4 u + a_5 u^3 + a_6 u^5 + a_7 u^7 = 0 \tag{28}$$

where

$$\begin{aligned}
&a_1 = \int_0^1 \phi^6 d\xi;\ a_2 = -2\int_0^1 \phi^4 d\xi;\ a_3 = \int_0^1 \phi^2 d\xi; \\
&a_4 = \int_0^1 \left(\phi''\phi - N\phi'\phi - V^2\phi^2\right) d\xi; \\
&a_5 = -\int_0^1 \left(2\phi''\phi^3 - 2N\phi'\phi^3 + \alpha\phi'\phi\int_0^1 (\phi')^2 d\xi\right) d\xi; \\
&a_6 = \int_0^1 \left(\phi''\phi^5 - N\phi'\phi^5 + 2\alpha\phi'\phi^3\int_0^1 (\phi')^2 d\xi\right) d\xi; \\
&a_7 = -\int_0^1 \left(\alpha\phi'\phi^5\int_0^1 (\phi')^2 d\xi\right) d\xi.
\end{aligned} \tag{29}$$

Here a overdot (.) denotes differentiation with respect to the time variable τ, while a prime (') indicates the partial differentiation with respect to the coordinate variable ξ. Eq. (28) is a nonlinear ordinary differential equation. In the next sections, the parameter-expansion and equivalent linearization methods are utilized to study this nonlinear oscillator.

5. Application of the parameter-expansion method

To apply PEM, first we rewrite Eq. (28) in the form:

$$\ddot{u} + \frac{a_4}{a_3}u + 1 \cdot \frac{a_1 u^4 \ddot{u} + a_2 u^2 \ddot{u} + a_5 u^3 + a_6 u^5 + a_7 u^7}{a_3} = 0,$$
$$u(0) = A,\ \dot{u}(0) = 0 \tag{30}$$

Assume that the solution can be expressed as a power series in an artificial parameter p:

$$u = \sum_{i=0}^{n} p^i u_i(\tau) = u_0(\tau) + p u_1(\tau) + p^2 u_2(\tau) + \cdots \tag{31}$$

where p is a booking parameter. We assume that the coefficients a_4/a_3 and 1 on the left of Eq. (30) can be respectively expanded into a series of p:

$$\frac{a_4}{a_3} = \omega^2 + \sum_{i=1}^{n} p^i \omega_i = \omega^2 + p\omega_1 + p^2\omega_2 + \cdots \tag{32}$$

$$1 = \sum_{i=1}^{n} p^i b_i = p b_1 + p^2 b_2 + \cdots \tag{33}$$

Substituting Eqs. (31), (32) and (33) into Eq. (30) yields:

$$\begin{aligned}
&(\ddot{u}_0 + p\ddot{u}_1 + p^2\ddot{u}_2 + \cdots) + (\omega^2 + p\omega_1 \\
&+ p^2\omega_2 + \cdots)(u_0 + pu_1 + p^2 u_2 + \cdots) + \\
&+ (p b_1 + p^2 b_2 + \cdots) \times \left[\begin{array}{l}
\frac{a_1}{a_3}(u_0 + pu_1 \\
+ p^2 u_2 + \cdots)^4 (\ddot{u}_0 + p\ddot{u}_1 + p^2\ddot{u}_2 + \cdots) + \\
\frac{a_2}{a_3}(u_0 + pu_1 \\
+ p^2 u_2 + \cdots)^2 (\ddot{u}_0 + p\ddot{u}_1 + p^2\ddot{u}_2 + \cdots) + \\
\frac{a_5}{a_3}(u_0 + pu_1 + p^2 u_2 + \cdots)^3 \\
+ \frac{a_6}{a_3}(u_0 + pu_1 + p^2 u_2 + \cdots)^5 \\
+ \frac{a_7}{a_3}(u_0 + pu_1 + p^2 u_2 + \cdots)^7
\end{array}\right] = 0
\end{aligned} \tag{34}$$

Equating the terms with the identical powers of p yields:

$$p^0: \ \ddot{u}_0 + \omega^2 u_0 = 0, \quad u_0(0) = A, \dot{u}_0(0) = 0 \tag{35}$$

$$\begin{aligned}
p^1: \ &\ddot{u}_1 + \omega^2 u_1 + \omega_1 u_0 + b_1\frac{a_1}{a_3}u_0^4\ddot{u}_0 + b_1\frac{a_2}{a_3}u_0^2\ddot{u}_0 + b_1\frac{a_5}{a_3}u_0^3 \\
&+ b_1\frac{a_6}{a_3}u_0^5 + b_1\frac{a_7}{a_3}u_0^7 = 0,\ u_1(0) = 0
\end{aligned} \tag{36}$$

The solution of Eq. (35) can be easily obtained

$$u_0 = A\cos(\omega\tau) \tag{37}$$

Substituting the result (37) into Eq. (36), we have:

$$\begin{aligned}
&\ddot{u}_1 + \omega^2 u_1 + \omega_1 A\cos(\omega t) - b_1\frac{a_1}{a_3}u_0^4 A^5\omega^2\cos^5(\omega t) - \\
&b_1\frac{a_2}{a_3}A^3\omega^2\cos^3(\omega t) + b_1\frac{a_5}{a_3}A^3\cos^3(\omega t) + \\
&b_1\frac{a_5}{a_3}A^5\cos^5(\omega t) + b_1\frac{a_7}{a_3}b_1\frac{a_5}{a_3}A^7\cos^7(\omega t) = 0
\end{aligned} \tag{38}$$

Note that:

$$\cos^3(\omega\tau) = \frac{3\cos(\omega\tau) + \cos(3\omega\tau)}{4} \tag{39}$$

$$\cos^5(\omega\tau)=\frac{10\cos(\omega\tau)+5\cos(3\omega\tau)+\cos(5\omega\tau)}{16} \tag{40}$$

$$\cos^7(\omega\tau)=\frac{35\cos(\omega\tau)+21\cos(3\omega\tau)+7\cos(5\omega\tau)+\cos(7\omega\tau)}{64} \tag{41}$$

Substituting Eqs. (39), (40) and (41) into Eq. (38) yields:

$$\begin{aligned}\ddot{u}_1+\omega^2u_1+&\left(\begin{array}{l}\omega_1A-\frac{15}{32}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{9}{16}b_1\frac{a_2}{a_3}A^3\omega^2\\+\frac{3}{4}b_1\frac{a_5}{a_3}A^3+\frac{10}{16}b_1\frac{a_6}{a_3}A^5+\frac{35}{64}b_1\frac{a_7}{a_3}A^7\end{array}\right)\cos(\omega\tau)\\&+\left(\begin{array}{l}-\frac{5}{16}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{1}{4}b_1\frac{a_2}{a_3}A^3\omega^2+\\\frac{1}{4}b_1\frac{a_5}{a_3}A^3+\frac{5}{16}b_1\frac{a_5}{a_3}A^5+\frac{21}{64}b_1\frac{a_7}{a_3}A^5\end{array}\right)\cos(3\omega\tau)\\&+\left(-\frac{1}{16}b_1\frac{a_1}{a_3}A^5\omega^2+\frac{1}{16}b_1\frac{a_6}{a_3}A^5+\frac{7}{64}b_1\frac{a_7}{a_3}A^7\right)\cos(5\omega\tau)\\&+\frac{1}{64}b_1\frac{a_7}{a_3}A^7\cos(7\omega\tau)=0\end{aligned} \tag{42}$$

No secular terms in u_1 requires:

$$\begin{aligned}&\omega_1A-\frac{15}{32}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{9}{16}b_1\frac{a_2}{a_3}A^3\omega^2\\&+\frac{3}{4}b_1\frac{a_5}{a_3}A^3+\frac{10}{16}b_1\frac{a_6}{a_3}A^5+\frac{35}{64}b_1\frac{a_7}{a_3}A^7=0\end{aligned} \tag{43}$$

If the first-order approximation is sufficient, then we set $p=1$ and from Eqs. (32) and (33), we have:

$$\frac{a_4}{a_3}=\omega^2+\omega_1 \tag{44}$$

$$1=b_1 \tag{45}$$

From Eqs. (43), (44) and (45), we have the approximate frequency:

$$\omega_{PEM}=\sqrt{\frac{64a_4+48a_5A^2+40a_6A^4+35a_7A^6}{4\left(16a_3+10a_1A^4+12a_2A^2\right)}} \tag{46}$$

Substituting Eq. (46) into Eq. (36), and noting Eq. (43), yields:

$$\ddot{u}_1+\omega^2u_1=B_1\cos(3\omega\tau)+B_2\cos(5\omega\tau)+B_3\cos(7\omega\tau) \tag{47}$$

where:

$$B_1=-\left(\begin{array}{l}-\frac{5}{16}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{1}{4}b_1\frac{a_2}{a_3}A^3\omega^2\\+\frac{1}{4}b_1\frac{a_5}{a_3}A^3+\frac{5}{16}b_1\frac{a_5}{a_3}A^5+\frac{21}{64}b_1\frac{a_7}{a_3}A^5\end{array}\right) \tag{48}$$

$$B_2=-\left(\begin{array}{l}-\frac{1}{16}b_1\frac{a_1}{a_3}A^5\omega^2+\frac{1}{16}b_1\frac{a_6}{a_3}A^5\\+\frac{7}{64}b_1\frac{a_7}{a_3}A^7\end{array}\right) \tag{49}$$

$$B_3=-\frac{1}{64}b_1\frac{a_7}{a_3}A^7 \tag{50}$$

Solving Eq. (47), we obtain:

$$\begin{aligned}u_1(\tau)=&-(E_1+E_2+E_3)\cos(\omega\tau)+E_1\cos(3\omega\tau)+\\&E_2\cos(5\omega\tau)+E_3\cos(7\omega\tau)\end{aligned} \tag{51}$$

where:

$$E_1=-\frac{B_1}{8\omega^2} \tag{52}$$

$$E_2=-\frac{B_2}{24\omega^2} \tag{53}$$

$$E_3=-\frac{B_3}{48\omega^2} \tag{54}$$

Substituting Eqs. (48)- (50) into Eqs. (52)- (54), and then substituting the results into Eq. (51), we have:

$$\begin{aligned}u_1(\tau)=&-\left[\begin{array}{l}-\frac{5}{16}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{1}{4}b_1\frac{a_2}{a_3}A^3\omega^2\\+\frac{1}{4}b_1\frac{a_5}{a_3}A^3+\frac{5}{16}b_1\frac{a_5}{a_3}A^5+\frac{21}{64}b_1\frac{a_7}{a_3}A^5\\\hline8\omega^2\\-\frac{1}{16}b_1\frac{a_1}{a_3}A^5\omega^2+\frac{1}{16}b_1\frac{a_6}{a_3}A^5\\+\frac{7}{64}b_1\frac{a_7}{a_3}A^7\\\hline24\omega^2\end{array}+\frac{\frac{1}{64}b_1\frac{a_7}{a_3}A^7}{48\omega^2}\right]\cos(\omega\tau)-\\&\frac{-\frac{5}{16}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{1}{4}b_1\frac{a_2}{a_3}A^3\omega^2+\frac{1}{4}b_1\frac{a_5}{a_3}A^3+\frac{5}{16}b_1\frac{a_5}{a_3}A^5+\frac{21}{64}b_1\frac{a_7}{a_3}A^5}{8\omega^2}\cos(3\omega\tau)-\\&\frac{-\frac{1}{16}b_1\frac{a_1}{a_3}A^5\omega^2+\frac{1}{16}b_1\frac{a_6}{a_3}A^5+\frac{7}{64}b_1\frac{a_7}{a_3}A^7}{24\omega^2}\cos(5\omega\tau)\\&-\frac{\frac{1}{64}b_1\frac{a_7}{a_3}A^7}{48\omega^2}\cos(7\omega\tau)\end{aligned} \tag{55}$$

Last, substituting Eqs. (37) and (55) into Eq. (31), we obtain the first-order approximate solution of this oscillator:

$$u(\tau)=A\cos(\omega\tau)-\left[\begin{array}{l}-\frac{5}{16}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{1}{4}b_1\frac{a_2}{a_3}A^3\omega^2+\\ \frac{\frac{1}{4}b_1\frac{a_5}{a_3}A^3+\frac{5}{16}b_1\frac{a_5}{a_3}A^5+\frac{21}{64}b_1\frac{a_7}{a_3}A^5}{8\omega^2}\\ -\frac{1}{16}b_1\frac{a_1}{a_3}A^5\omega^2+\frac{1}{16}b_1\frac{a_6}{a_3}A^5+\\ +\frac{\frac{7}{64}b_1\frac{a_7}{a_3}A^7}{24\omega^2}\\ +\frac{\frac{1}{64}b_1\frac{a_7}{a_3}A^7}{48\omega^2}\end{array}\right]\cos(\omega\tau)-$$

$$-\frac{-\frac{5}{16}b_1\frac{a_1}{a_3}A^5\omega^2-\frac{1}{4}b_1\frac{a_2}{a_3}A^3\omega^2+\frac{1}{4}b_1\frac{a_5}{a_3}A^3+\frac{5}{16}b_1\frac{a_5}{a_3}A^5+\frac{21}{64}b_1\frac{a_7}{a_3}A^5}{8\omega^2}\cos(3\omega\tau)-$$

$$-\frac{-\frac{1}{16}b_1\frac{a_1}{a_3}A^5\omega^2+\frac{1}{16}b_1\frac{a_6}{a_3}A^5+\frac{7}{64}b_1\frac{a_7}{a_3}A^7}{24\omega^2}\cos(5\omega\tau)$$

$$-\frac{\frac{1}{64}b_1\frac{a_7}{a_3}A^7}{48\omega^2}\cos(7\omega\tau) \tag{56}$$

where ω is given in Eq. (46).
The higher-order approximations for ω and $u(\tau)$ can be established in a similar manner.

6. Application of the equivalent linearization method

Now, we will apply the equivalent linearizaton method to find the solution of Eq. (28). The linearized equation of Eq. (28) has the form:

$$\ddot{u}+\omega^2u=0 \tag{57}$$

The equation error between the two Eqs. (28) and (57) is:

$$e(u)=(a_1u^4+a_2u^2+a_3)\ddot{u}+a_4u+a_5u^3+a_6u^5+a_7u^7-\ddot{u}-\omega^2u \tag{58}$$

The unknown coefficient ω^2 is determined from the mean square error criterion:

$$\frac{\partial}{\partial\omega^2}\langle e^2(u)\rangle=0$$

it follows that:

$$\omega^2=\frac{\langle a_1u^5\ddot{u}+a_2u^3\ddot{u}+a_3u\ddot{u}-u\ddot{u}\rangle}{\langle u^2\rangle}+\frac{\langle a_4u^2+a_5u^4+a_6u^6+a_7u^8\rangle}{\langle u^2\rangle} \tag{59}$$

The periodic solution of Eq. (57) is:

$$u(\tau)=A\cos(\omega\tau) \tag{60}$$

Using (60), we calculate averaging operators in Eq. (59):

$$\langle u^2\rangle=\langle A^2\cos^2(\omega\tau)\rangle=\frac{A^2\omega}{2\pi}\int_0^{2\pi/\omega}\cos^2(\omega\tau)d\tau=\frac{A^2}{2} \tag{61}$$

$$\langle u^4\rangle=\langle A^4\cos^4(\omega\tau)\rangle=\frac{A^4\omega}{2\pi}\int_0^{2\pi/\omega}\cos^4(\omega\tau)d\tau=\frac{3A^4}{8} \tag{62}$$

$$\langle u^6\rangle=\langle A^6\cos^6(\omega\tau)\rangle=\frac{A^6\omega}{2\pi}\int_0^{2\pi/\omega}\cos^6(\omega\tau)d\tau=\frac{5A^6}{16} \tag{63}$$

$$\langle u^8\rangle=\langle A^8\cos^8(\omega\tau)\rangle=\frac{A^8\omega}{2\pi}\int_0^{2\pi/\omega}\cos^8(\omega\tau)d\tau=\frac{35A^8}{128} \tag{63}$$

$$\langle u\ddot{u}\rangle=-\langle A^2\omega^2\cos^2(\omega\tau)\rangle=$$

$$-\frac{A^2\omega^3}{2\pi}\int_0^{2\pi/\omega}\cos^2(\omega\tau)d\tau=-\frac{A^2\omega^2}{2} \tag{65}$$

$$\langle u^3\ddot{u}\rangle=-\langle A^4\omega^2\cos^4(\omega\tau)\rangle=-\frac{A^4\omega^3}{2\pi}\int_0^{2\pi/\omega}\cos^4(\omega\tau)d\tau=-\frac{3A^4\omega^2}{8} \tag{66}$$

$$\langle u^5\ddot{u}\rangle=-\langle A^6\omega^2\cos^6(\omega\tau)\rangle=-\frac{A^6\omega^3}{2\pi}\int_0^{2\pi/\omega}\cos^6(\omega\tau)d\tau=-\frac{5A^6\omega^2}{16} \tag{67}$$

Substituting Eqs. (61) - (67) into Eq. (59), we get:

$$\omega^2=\frac{\left(-\frac{5}{16}a_1A^6-\frac{3}{8}a_2A^4-\frac{1}{2}a_3A^2+\frac{1}{2}A^2\right)\omega^2}{\frac{1}{2}A^2}+\frac{\frac{1}{2}a_4A^2+\frac{3}{8}a_5A^4+\frac{5}{16}a_6A^6+\frac{35}{128}a_7A^8}{\frac{1}{2}A^2} \tag{68}$$

From Eq. (68), we obtain the approximate frequency of this oscillator:

$$\omega_{ELM}=\sqrt{\frac{64a_4+48a_5A^2+40a_6A^4+35a_7A^6}{40a_1A^4+48a_2A^2+64a_3}} \tag{69}$$

According to Eqs. (60) and (69), we can obtain the approximate solution of this oscillator:

$$u(\tau)=A\cos\left(\sqrt{\frac{64a_4+48a_5A^2+40a_6A^4+35a_7A^6}{40a_1A^4+48a_2A^2+64a_3}}\ \tau\right) \tag{70}$$

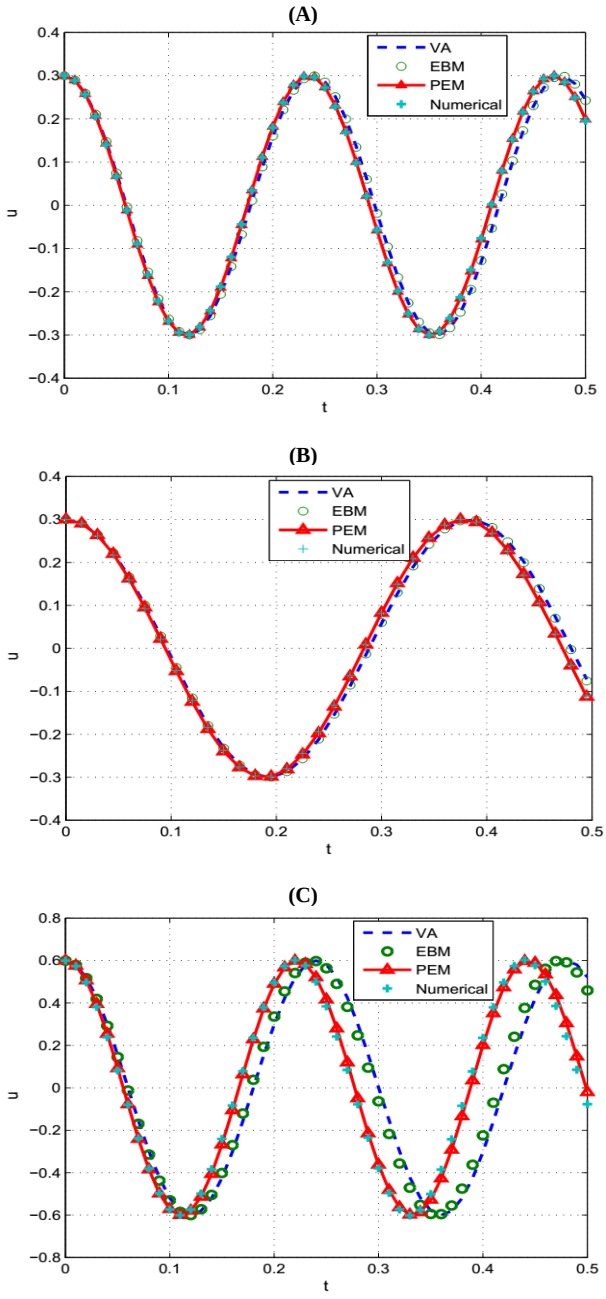
Note that the frequency in Eq. (69) obtained by ELM is the same one in Eq. (46) obtained by PEM.

7. Discussions

In order to validate the effectiveness of the current approaches, Tab. 1 compares the frequencies obtained by the Energy Balance Method ω_{EBM} [5], Variational Approach ω_{VA} [7], Parameter-Expansion Method (in Eq. (46)), Equivalent Linearization Method (in Eq. (60)) and the exact solution ω_{ex} [6] for various parameters N, α, V and initial amplitudes A. The values of dimensionless parameters N, α, V in Tab. 1 correspond to different forces and voltages acting on the microbeam.
Figs. 2(A), 2(B), 2(C) and 2(D) represent comparisons of the analytical solutions of $u(\tau)$ based on time with the Energy Balance Method, Variational Approach and numerical solutions. It could be obtained from the figures that the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions.

Table 1: Comparison of Frequencies Corresponding to Various Parameters of System

case	A	N	α	V	ω_{ex} [6]	ω_{EBM} [5]	ω_{VA} [7]	ω_{PEM} and ω_{ELM}
1	0.3	10	24	0	26.8372	26.3867	26.3644	26.8304
2	0.3	10	24	20	16.6486	16.3829	16.3556	16.6445
3	0.6	10	24	10	28.5382	28.9227	28.5579	28.4030
4	0.6	10	24	20	18.5902	17.5017	17.0940	18.5409



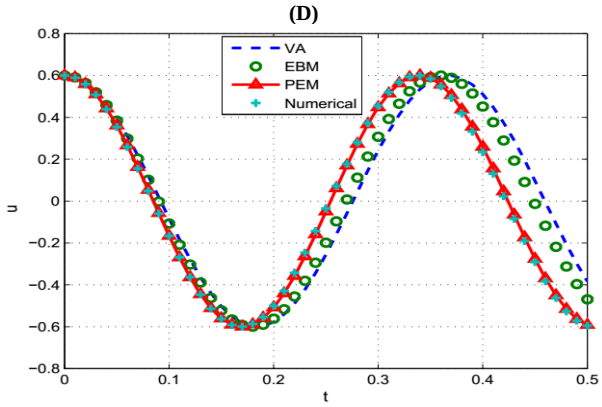
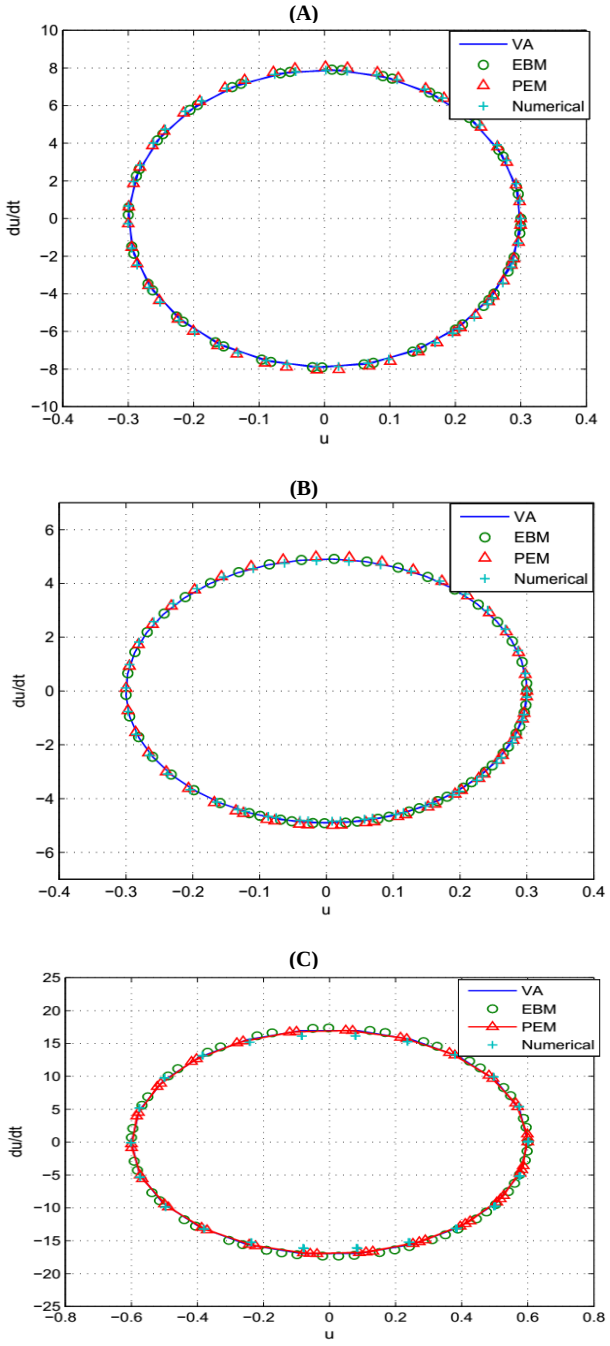


Fig. 2: Comparison Of Solution Corresponding to Various Parameters of System: (A) A=0.3, N=10, A=24, V=0; (B) A=0.3, N=10, A=24, V=20; (C) A=0.6, N=10, A=24, V=10; (D) A=0.6, N=10, A=24, V=20;

Figs. 3(A), 3(B), 3(C) and 3(D) show the comparisons of the analytical solution of $du(\tau)/d\tau$ based on $u(\tau)$ with the energy balance method, variational approach and numerical solutions.



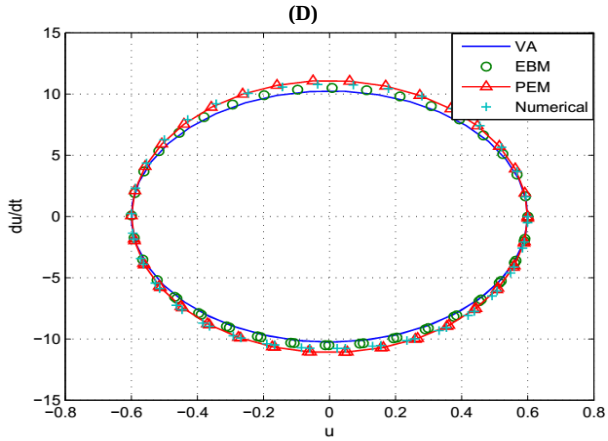


Fig. 3: Comparison Of Phase Solution Corresponding to Various Parameters of System: (A) A=0.3, N=10, A=24, V=0; (B) A=0.3, N=10, A=24, V=20; (C) A=0.6, N=10, A=24, V=10; (D) A=0.6, N=10, A=24, V=20;

Fig. 4 is presented to show the effect of amplitude *A* on the phase plan of the problem. And the effects of *V*, *α* and *N* on the phase plan of the problem are presented in Figs. 5, 6 and 7, respectively. Figs. 8-10 are presented the effects of *V*, *N*, *α* parameters on nonlinear frequency base on amplitude, respectively. From Fig. 8, we see that when the voltage *V* acting on the microbeam increases then the frequency decreases. But from Figs. 9 and 10, the frequency increases when the axial force and the ratio g_0/h increase.

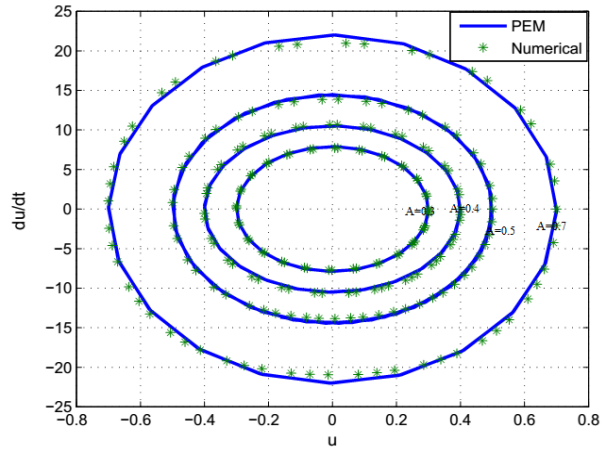


Fig. 4: Effect of Amplitude *A* on the Phase Plan of the Problem for N=10, A=24, V=5.

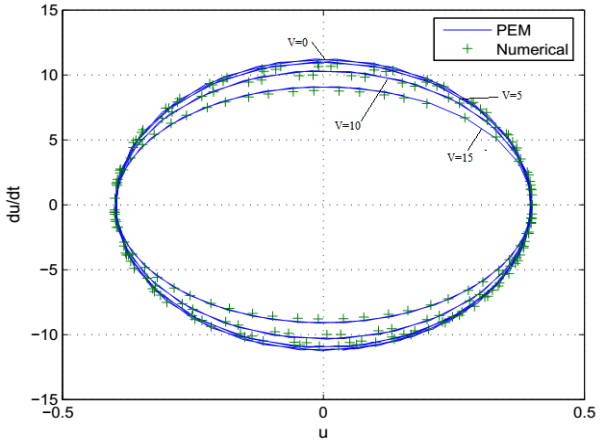


Fig. 5: Effect of V on the Phase Plan of the Problem for N=10, A=24, A=0.4.

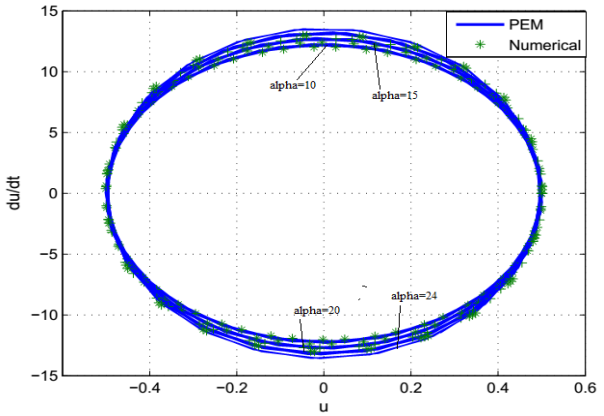


Fig. 6: Effect of α on the Phase Plan of the Problem for N=10, V=10, A=0.5.

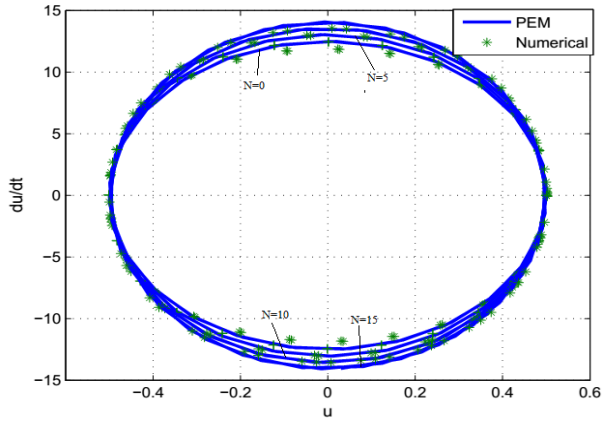


Fig. 7: Effect of N on the Phase Plan of the Problem for A=24, V=10, A=0.5.

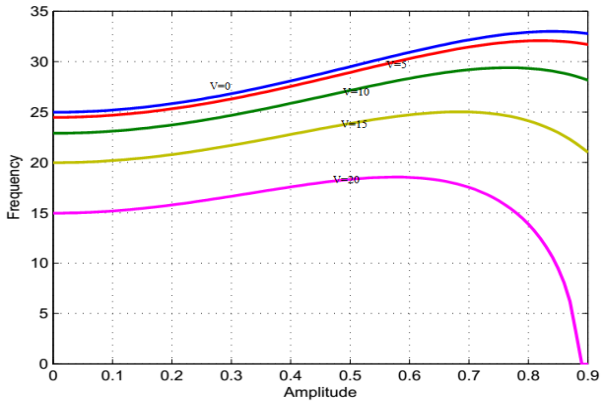


Fig. 8: Effect of V Parameter on Nonlinear Frequency Base on Amplitude for A=24, N=10.

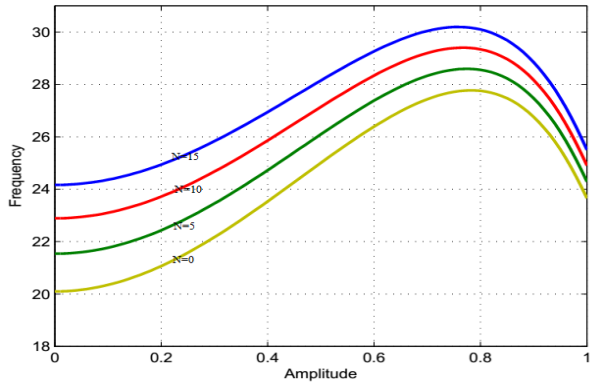


Fig. 9: Effect of N Parameter on Nonlinear Frequency Base on Amplitude for A=24, V=10.

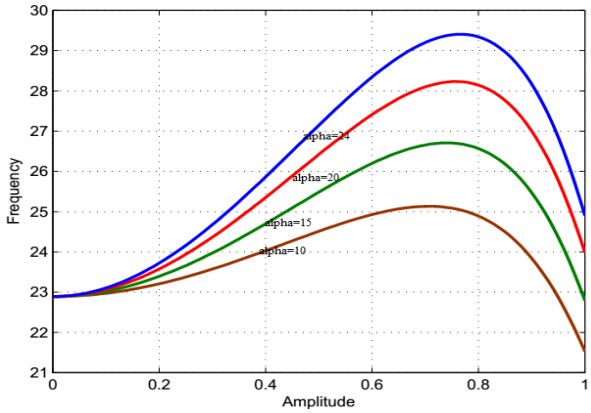


Fig. 10: Effect of α Parameter on Nonlinear Frequency Base on Amplitude for N=10, V=10.

8. Conclusions

In this paper, the Parameter-Expansion Method and the Equivalent Linearization Method were employed to solve the nonlinear governing equation of the microbeam-based micro-electromechanical system where the midplane stretching effect and distributed electrostatic force are both considered. The frequency – amplitude relationships achived by two methods are obtained in the closed forms. Excellent agreement between approximate frequencies can be seen. The high accuracy of two methods is presented by comparing the present solutions with the previous solutions and the exact solutions as well as the numerical solutions.

The effect of the initial amplitude A and the parameters V, α and N on the phase plan of the problem are investigated. And the effects of parameters V, N and α on nonlinear frequency base on amplitude A are also presented.

The results show that the two current methods are very useful in analyzing nonlinear oscillations.

Acknowledgement

The research is supported by Thai Nguyen University of Technology grant for a scientific project (no. “T2018-B28”).

References

[1] E. M. Abdel-Rahman, M.I. Younis, A.H. Nayfeh. “Characterization of themechanical behavior of an electrically actuated microbeam”. *Journal of Micromechanics and Microengineering* , Vol. 12 (2002), pp. 759-766. https://doi.org/10.1088/0960-1317/12/6/306.

[2] J.H. Kuang, C.J. Chen. “Dynamic characteristics of shaped micro-actuators solved using the differential quadrature method”. *Journal of Micromechanics and Microengineering*, Vol.14 (2004), pp. 647-655. https://doi.org/10.1088/0960-1317/14/4/028.

[3] J.H. Kuang, C.J. Chen. “Adomian decomposition method used for solving nonlinear pull-in behavior in electrostatic micro-actuators”. *Mathematical and Computer Modelling*, Vol. 41 (2005), pp. 1479-1491. https://doi.org/10.1016/j.mcm.2005.06.001.

[4] M. I. Younis and A. H. Nayfeh. “A Study of the Nonlinear Response of a Resonant Microbeam to an Electric Actuation”. *Nonlinear Dynamics*, Vol. 31 (2003), pp. 91–117. https://doi.org/10.1023/A:1022103118330.

[5] Fu, Y., Zhang, J., and Wan, L. “Application of the energy balance method to a nonlinear oscillator arising in the microelectromechanical system (MEMS)”. *Current Applied Physics*, Vol.11 (3) (2011), pp.482-85. https://doi.org/10.1016/j.cap.2010.08.037.

[6] Y.H. Qian, D.X. Ren, S.K. Lai, S.M. Chen. “Analytical approximations to nonlinear vibration of an electrostatically actuated microbeam”. *Commun Nonlinear Sci Numer Simulat*. Vol.17 (2012), pp. 1947–1955. https://doi.org/10.1016/j.cnsns.2011.09.018.

[7] Mahdi Bayat, Mahmoud Bayat, Iman Pakar. “Nonlinear vibration of an electrostatically actuated microbeam”. *Latin American Journal of Solids and Structures*. Vol.11 (2014),pp. 534 – 544. https://doi.org/10.1590/S1679-78252014000300009.

[8] J. H. He. “An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering”. *International Journal of Modern Physics B (IJMPB)*, Vol.22(21) (2008), pp. 3487–3578. https://doi.org/10.1142/S0217979208048668.

[9] J. H. He and D. H. Shou. “Application of parameter-expanding method to strongly nonlinear oscillators”. *International Journal of Nonlinear Sciences and Numerical Simulation*, Vol.8 (2007), pp.121–124.

[10] He J H. “Bookkeeping parameter in perturbation methods”. *Int J Non-linear Sci Numer Simul*, (2001); pp.257–264. https://doi.org/10.1515/IJNSNS.2001.2.3.257.

[11] M. Kaya and S. Altay Demirbag. “Application of parameter expansion method to the generalized nonlinear discontinuity equation”. *Chaos, Solitons & Fractals*, Vol. 42(4) (2009), pp.1967–1973. https://doi.org/10.1016/j.chaos.2009.03.143.

[12] H. M. Liu. “Approximate period of nonlinear oscillators with discontinuities by modified lindstedt-poincare method”. *Chaos, Solitons & Fractals*, Vol. 23(2) (2005), pp. 577–579. https://doi.org/10.1016/j.chaos.2004.05.004.

[13] T. Ozis and A. Yildirim. “Determination of periodic solution for a u1/3 force by He’s modified lindstedt-poincaré method”. *Journal of Sound and Vibration*, Vol. 301(1-2) (2007), pp. 415–419. https://doi.org/10.1016/j.jsv.2006.10.001.

[14] S. Q. Wang and J. H. He. “Nonlinear oscillator with discontinuity by parameter-expansion method”. *Chaos, Solitons & Fractals*, Vol. 35(4) (2008), pp. 688–691. https://doi.org/10.1016/j.chaos.2007.07.055.

[15] L. Xu. “Application of he’s parameter-expansion method to an oscillation of a mass attached to a stretched elastic wire”. *Physics Letters A*, Vol. 368(3-4) (2007), pp.259–262. https://doi.org/10.1016/j.physleta.2007.04.004.

[16] L. Xu. “Determination of limit cycle by he’s parameter-expanding method for strongly nonlinear oscillators”. *Journal of Sound and Vibration*, Vol. 302(1-2) (2007), pp.178–184. https://doi.org/10.1016/j.jsv.2006.11.011.

[17] L. Xu. “He’s parameter-expanding methods for strongly nonlinear oscillators”. *Journal of Computational and Applied Mathematics*, Vol. 207(1) (2007), pp.148–154. https://doi.org/10.1016/j.cam.2006.07.020.

[18] J.F. Rhoads, S.W. Shaw, K.L. Turner. “The nonlinear response of resonant microbeam systems with purely-parametric electrostatic actuation”. *Journal of Micromechanics and Microengineering*, Vol.16 (2006), pp. 890-899. https://doi.org/10.1088/0960-1317/16/5/003.

[19] R.C. Batra, M. Porfiri, D. Spinello. “Electromechanical model of electrically actuated narrow microbeams”. *Journal of Microelectromechanical Systems*, Vol.15 (2006), pp. 1175-1189. https://doi.org/10.1109/JMEMS.2006.880204.

[20] Krylov, N., Bogoliubov N., *Introduction to nonlinear mechanics*, New York: Princenton University Press, (1943).

[21] Caughey, T. K., “Equivalent linearization technique”, *The Journal of the Acoustical Society of America*, Vol. 35 (1959), 1706–1711. https://doi.org/10.1121/1.1918794.

[22] Anh, N. D., Schiehlen, W., “New criterion for Gaussian equivalent linearization”, *Euror Journal of Mechanics- A/Solid*, Vol. 16 (1997), 1025– 1039.

[23] Anh, N. D., Di Paola, M., “Some extensions of Gaussian equivalent linearization”, *Proceedings of the International Conference on Nonlinear Stochastic Dynamics*. pp. 5–16. Hanoi, Vietnam (1995).