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Research Paper

Large-amplitude dynamic analysis of heterogeneous thin membrane under different loads

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Abstract

Large amplitude vibration of heterogeneous thin membrane is modeled using the Euler-Bernoulli constitutive law. The obtained nonlinear integro-differential equations are solved using finite difference method. Different results of the problem are illustrated for two different external concentrated loads applied on the middle of the membrane including stepwise and half period sinusoidal forces. The effects of time dependency of the applied loads are investigated. The solution is finally verified and it is concluded that the presented solution can be considered as promising method to obtain large amplitude vibration of thin heterogeneous membrane.

Keywords: *Large Amplitude Vibration; Thin Membrane; Finite Difference Method; Heterogeneity; Sinusoidal Force*.

1. Introduction

During the last decades, different models have been developed in order to study large deflection of solid systems [1, 2]. They include some exact solution for linear and nonlinear systems in special conditions as well as some more complicated problems solved using semi analytical or numerical methods [3]. Different methods such as Finite Element Method (FEM) [4], Finite Difference Method (FDM) [5] and perturbation methods [6] have been employed to solve these nonlinear problems. Their complexity is not just based on nonlinearity of the model because of large deformation of the system. The geometry of the system, its heterogeneity, different boundary conditions, time and spatial dependency of the applied loads are among other factors affect the complexity of the problem. Some packages also have been developed during the last decades to study different solid systems, including large deflection of cantilever beam and thin membrane, such as ABAQUS [7]. The dynamic behavior is more complicated than the static analysis of the systems because of dependency of the variables on both time and space resulting in nonlinear partial differential equations [7].
Several studies have been dedicated to investigate the static and dynamic behavior of thin and thick membrane as one of the most important problems in the area of Elasticity and Mathematical Modeling [7-9]. The small deformation of the problem has exact solution in some of the case such as the homogeneous one-dimensional membrane under concentrated or distributed loads [10]. In the most of the large deformation problems, obtaining the exact solution is not possible and the available models [2], [10] have been solved numerically [10] or using semi analytical approaches [2]. Moreover, the large amplitude dynamic behavior of thin membrane has been investigated as well in specific conditions [7]. Different complexities exist in the natural or artificial systems such as heterogeneity make the analysis of the problem so complicated. There are few articles about analysis of heterogeneous systems especially thin membranes. Therefore, presenting a new dynamic model of heterogeneous membrane to be so general and can be employed to analyze the system with different boundary conditions and under different external loads is so important and useful. This model is presented in this article to analyze dynamic behavior of thin membrane.
The model is developed in order to investigate time and parameter dependency of different variables. In addition, generality of the model let us study the heterogeneous membranes as well. Non-dimensional nonlinear integro-differential equations of the system are developed and non-dimensionalized to be able to obtain more general results.

Discretized form of the equations is obtained using semi implicit finite difference method and solved numerically employing MATLAB 2012. Variation of different variables is obtained with respect to time using the obtained solution and the effects of the dimensionless stiffness are studied as well. As one of the proficiencies of the presented solution, the effects of heterogeneity on the dynamic behavior of the membrane are investigated and existence of critical regions is shown in each case. In order to be sure about the accuracy of the presented solution, it is verified using result obtained by ABAQUS [7].

2. Mathematical modeling

In order to model the transient large deflection behavior of half thin membrane, the infinitesimal element of the membrane is considered in Fig. 1 which shows all of the forces applied on this element. Hence, we can write the equations of motion of this element as follows:

$$\begin{aligned}
&\sum F_x = ma_x \Rightarrow dF_x = \rho ds \frac{\partial^2 x}{\partial t^2} \\
&\sum F_y = ma_y \Rightarrow -dF_y + \omega(s)ds = \rho ds \frac{d^2 y}{dt^2} \\
&\sum M = J\alpha - m\frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right) \\
&\Rightarrow dM - ds\left(F_x \sin(\theta) + F_y \cos(\theta)\right) = J\ddot{\theta} - \rho ds \frac{d}{dt}\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)
\end{aligned} \tag{1}$$

Where ρ is longitudinal mass density and J is the moment of the inertia of the membrane, respectively.

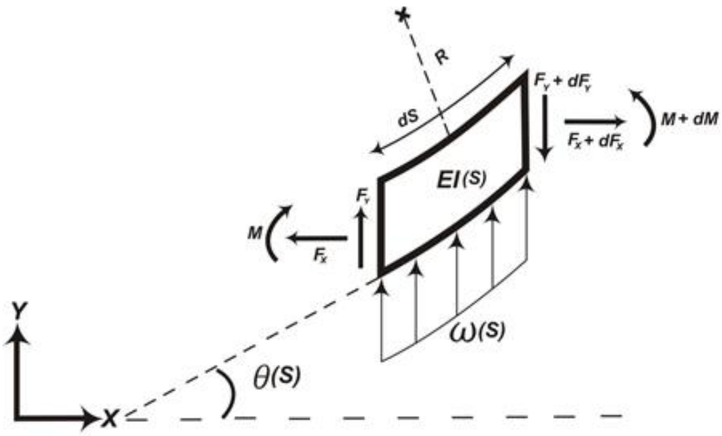

Fig. 1: A schematic model of an element of the membrane.

These equations can be rewritten as follows which are the differential equations of the motion of the membrane.

$$\begin{aligned}
&\frac{\partial F_x}{\partial s} = \rho \frac{\partial^2 x}{\partial t^2} \\
&-\frac{\partial F_y}{\partial s} + \omega(s) = \rho \frac{\partial^2 y}{\partial t^2} \\
&\frac{\partial M}{\partial s} - F_x \sin(\theta) - F_y \cos(\theta) = \frac{J\ddot{\theta}}{\partial s} - \rho \frac{\partial}{\partial t}\left(y\frac{\partial x}{\partial t} - x\frac{\partial y}{\partial t}\right)
\end{aligned} \tag{2}$$

There are six dependent variables including F_x, F_y, M, x, y and θ and just three differential equations. Hence, three other equations should be used to obtain all six variables. There are two geometrical equations define x and y as functions of θ as follows:

$$
\begin{aligned}
x &= \int \cos\theta ds \\
y &= \int \sin\theta ds
\end{aligned}
\tag{3}
$$

In addition, the following equation as the constitutive equation of Euler-Bernoulli beam can be employed to coincide the numbers of equations and dependent variables of the problem:

$$
\frac{M}{EI} = \frac{1}{R} = \frac{\partial\theta}{\partial s} \Rightarrow M = EI\,\frac{\partial\theta}{\partial s}
\tag{4}
$$

The employed geometrical differential equations Eq. (3) and constitutive equation Eq. (4) are employed to obtain the following nonlinear partial integro-differential equations of the system:

$$
\begin{aligned}
&\frac{\partial F_x}{\partial s} = \rho\,\frac{\partial^2}{\partial t^2}\int \cos\theta ds \\
&-\frac{\partial F_y}{\partial s} + \omega(s) = \rho\,\frac{\partial^2}{\partial t^2}\int \sin\theta ds \\
&\frac{\partial}{\partial s}\left(EI\,\frac{\partial\theta}{\partial s}\right) - F_x\,\sin(\theta) - F_y\,\cos(\theta) = \\
&\frac{J\ddot{\theta}}{\partial s} - \rho\frac{\partial}{\partial t}\left(\int \sin\theta ds\,\frac{\partial}{\partial t}\int \cos\theta ds - \int \cos\theta ds\,\frac{\partial}{\partial t}\int \sin\theta ds\right)
\end{aligned}
\tag{5}
$$

After bringing the temporal derivatives into the integrals and taking spatial derivatives from both sides of the two first equations of Eq.**Error! Reference source not found.**, we can obtain the following equations:

$$
\begin{aligned}
&\frac{\partial^2 F_x}{\partial s^2} = -\rho\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \\
&-\frac{\partial^2 F_y}{\partial s^2} + \frac{\partial\omega(s)}{\partial s} = \rho\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) \\
&\frac{\partial}{\partial s}\left(EI\,\frac{\partial\theta}{\partial s}\right) - F_x\,\sin(\theta) - F_y\,\cos(\theta) \\
&= \frac{J\ddot{\theta}}{\partial s} - \rho\left(-\int \sin\theta ds\int\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)ds - \int \cos\theta ds\int\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)ds\right)
\end{aligned}
\tag{6}
$$

The obtained equation should be changed to the non-dimensional form to be able to generalize the solution for different materials and geometries. The non-dimensional forms of the Eq. (6) can be written as:

$$
\begin{aligned}
&\frac{\partial^2 F_x{}^*}{\partial s^{*2}} = -\rho^*\left(\ddot{\theta}^*\sin\theta^* + \dot{\theta}^{*2}\cos\theta^*\right) \\
&-\frac{\partial^2 F_y{}^*}{\partial s^{*2}} + \frac{\partial\omega^*(s^*)}{\partial s^*} = \rho^*\left(\ddot{\theta}^*\cos\theta^* - \dot{\theta}^{*2}\sin\theta^*\right) \\
&\frac{\partial\lambda}{\partial s^*}\left(\frac{\partial\theta^*}{\partial s^*}\right) + \lambda\,\frac{\partial^2\theta^*}{\partial s^{*2}} - F_x{}^*\sin\left(\theta^*\right) - F_y{}^*\cos\left(\theta^*\right) \\
&= \frac{J^*\ddot{\theta}^*}{\partial s^*} - \rho^*\left(-\int \sin\theta^* ds^*\int\left(\ddot{\theta}^*\sin\theta^* + \dot{\theta}^{*2}\cos\theta^*\right)ds^*\right) \\
&-\rho^*\left(-\int \cos\theta^* ds^*\int\left(\ddot{\theta}^*\cos\theta^* - \dot{\theta}^{*2}\sin\theta^*\right)ds^*\right)
\end{aligned}
\tag{7}
$$

Where the dimensionless variables are defined as:

$$s^* = \frac{s}{L};\ \ \omega^*\left(s^*\right) = \frac{\omega\left(s\right)}{\omega_{\max}};\ \ \lambda\left(s^*\right) = \frac{EI\left(s\right)}{\omega_{\max}L^3};\ \ F_x^* = \frac{F_x}{\omega_{\max}L};\ \ F_y^* = \frac{F_y}{\omega_{\max}L}$$
$$\rho^* = \frac{\rho L}{\omega T^2};\ \ J^* = \frac{J}{\omega T^2 L} \tag{8}$$

Where T is the time necessary for system to be reached to the steady state. Henceforth, all of the non-dimensional variables with "*" are shown without this symbol.

Although the presented modeling is true in the case of the systems without dissipation of energy, the damping term are added to the model to modify it. It is added to change the model to a more realistic one and help us verify our result with the available results in the literature. This energy dissipation is taken into considerations as the modifications of the presented model (Eq. (7)) so that a bending dissipation and transverse deflection dissipation terms are considered in the system. The bending dissipation is modeled using a bending damping coefficient C and the transverse deflection dissipation is modeled using the dimensionless parameter D . The considered bending damping affects the behavior of the system as $C\dot{\theta}$. It is considered as a damping moment in the presented model. In addition, the transverse dissipation applies $D\dot{y}$ as damping force. These forces cause the system grows smoothly from its initial state to the final one. This smoothness is different for every system which its value can be set as suitable one to properly model the dynamic behavior of that system.

3. Numerical solution

In order to investigate time-dependent behavior of the membrane, system of nonlinear partial differential equations (Eq. (7)) should be solved using a suitable numerical approach. The time-dependent finite difference scheme is chosen to solve the problem [11]. The approach is based on the implicit time-dependent FDM. In this approach, the equations are discretized as follows to obtain a discrete PDE including the variables at different nodes and time-stages so that the nonlinear PDEs are changed to a linear finite difference equation.

$$\left.\frac{\partial^2 F_x}{\partial s^2}\right|_i^{n+1} = -\rho\left(\ddot{\theta}\Big|_i^n \sin\left(\theta_i^n\right) + \frac{\theta_i^{n+1}-\theta_i^n}{\delta t}\dot{\theta}_i^n \cos\left(\theta_i^n\right)\right)$$
$$-\left.\frac{\partial^2 F_y}{\partial s^2}\right|_i^{n+1} + \left.\frac{\partial\omega\left(s\right)}{\partial s}\right|_i^{n+1} = \rho\left(\ddot{\theta}_i^n \cos\left(\theta_i^n\right) - \frac{\theta_i^{n+1}-\theta_i^n}{\delta t}\dot{\theta}_i^n \sin\left(\theta_i^n\right)\right)$$
$$\frac{\lambda_{i+1}^{n+1}-\lambda_{i-1}^{n+1}}{2\delta s}\left(\frac{\theta_{i+1}^{n+1}-\theta_{i-1}^{n+1}}{2\delta s}\right) + \frac{\theta_{i+1}^{n+1}-2\theta_i^{n+1}+\theta_{i-1}^{n+1}}{\delta s^2} - F_{x,i}^{\ n+1}\sin\left(\theta_i^n\right) - F_y\cos\left(\theta_i^n\right) \tag{9}$$
$$= \frac{J\ddot{\theta}_i^n}{\delta s} + \rho\int\sin\left(\theta_i^n\right)ds\int\left(\ddot{\theta}_i^n\sin\left(\theta_i^n\right) + \frac{\theta_i^{n+1}-\theta_i^n}{\delta t}\dot{\theta}_i^n\cos\left(\theta_i^n\right)\right)ds$$
$$+\rho\int\cos\left(\theta_i^n\right)ds\int\left(\ddot{\theta}_i^n\cos\left(\theta_i^n\right) - \frac{\theta_i^{n+1}-\theta_i^n}{\delta t}\dot{\theta}_i^n\sin\left(\theta_i^n\right)\right)ds$$

Where

$$\ddot{\theta}_i^n = \frac{\theta_i^{n+1}-2\theta_i^n+\theta_i^{n-1}}{\delta t^2},\ \ \left.\frac{\partial\omega\left(s\right)}{\partial s}\right|_i^{n+1} = \frac{\omega_{i+1}^{n+1}-\omega_{i-1}^{n+1}}{2\delta s}$$
$$\left.\frac{\partial^2 F_x}{\partial s^2}\right|_i^{n+1} = \frac{F_x\big|_{i+1}^{n+1} - 2F_x\big|_i^{n+1} + F_x\big|_{i-1}^{n+1}}{\delta s^2},\ \ \left.\frac{\partial^2 F_y}{\partial s^2}\right|_i^{n+1} = \frac{F_y\big|_{i+1}^{n+1} - 2F_y\big|_i^{n+1} + F_y\big|_{i-1}^{n+1}}{\delta s^2}. \tag{10}$$

As it is mentioned, these equations are linearized form of the original nonlinear PDEs (Eq. (7)). The obtained equations for each node and time stage should be gather together to construct a system of equations can be written in the matrix form. The obtained matrix form of the discretized equations for the whole domain at time stage $n+1$ can be solved to obtain the matrix of coefficients and vector of variables including θ at all nodes of the domain. Therefore, instead of solving a static equation iteratively like the one for equilibrium analysis of membrane, the equations are solved at time stages to obtain the time-dependency of the behavior of the membrane.

4. Results and discussion

In this article, force vibration of the thin membrane is investigated under two different applied concentrated loads at the middle of the membrane including stepwise and half period sinusoidal loads. The results of the solution obtained employing the modeling and numerical procedure described in the previous section are presented in this section. The applied loads are considered to be applied in the time period $t:(1-2)$. The maximum values of both of them are considered to be equal to one. Hence, the sinusoidal force is $F=\sin\left(\pi\left(t-1\right)\right)$ which is applied in $t:(1-2)$. The parameter values not mentioned for each diagram are coinciding with the values used to verify the solution described later. The results presented in this section are categorized as time dependency of the variables, parameter dependency of maximum deflection and the effects of heterogeneity on maximum deflection of the membrane.

4.1. Time dependency of the variable

Transient behavior of the membrane is investigated here to help us understanding the dynamic behavior of the system under the considered stepwise and sinusoidal loads. Variations of different variables with respect to time are presented in this section (Figs. 2-5).

Maximum deflection of the membrane is shown in Fig. 2(a) as a function of time. It is clearly shown that the maximum value of this variable is greater in the case of applied stepwise load. It is based on the fact that faster increasing the applied force results in larger oscillation and deflection in the system. It means that when there is a bump, the system will deform more with respect to the case of smooth variation of applied force. Because of the difference of maximum deflection, the rate of increasing also differs so that it is greater for the membrane under stepwise load. Maximum deflection of the system sometimes is very important because of some geometrical criteria in the system. As an example, the space exists for a suspension does not let it to deform more than a specific amount. The behavior of maximum slope is the same as the maximum deflection (Fig. 2(b)). The rate of change and the maximum value is greater for the membrane under sinusoidal load corresponding to stepwise external load. Something is important to be noted in figures, maximum deflection and slope versus time, which is the rate of increasing and decreasing. These rates are different and the system come back to the initial condition in which it is at rest slower.

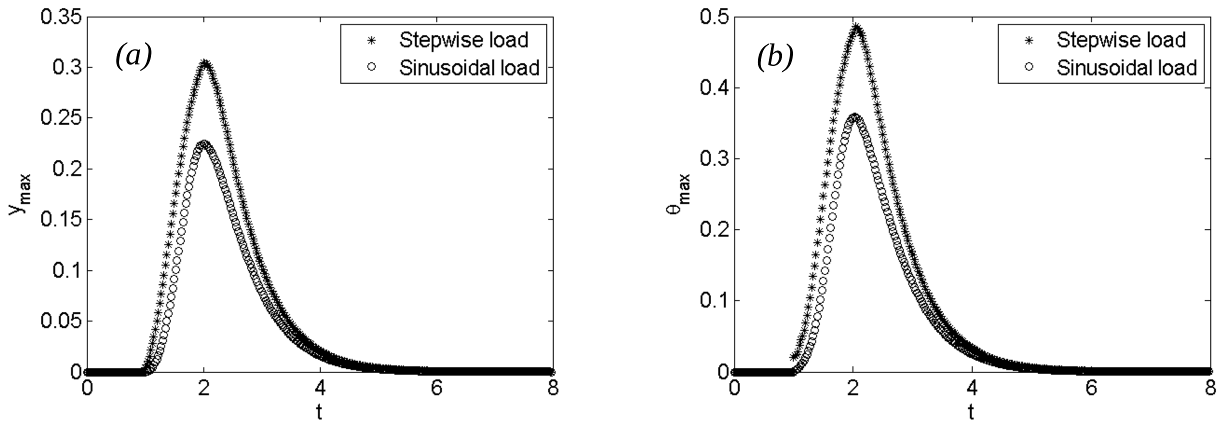

Fig. 2: Time variation of (A) maximum deflection and (B) maximum slope of the membrane under two different loads.

Therefore, we can conclude that the external force applied on the center of the membrane affect the system faster than the boundary condition cause the system to come back to its initial condition.

In addition to the time dependency of maximum deflection and slope of the membrane, maximum horizontal and vertical forces are illustrated in Fig. 3 as functions of time. The internal forces are as important because of definition of stress as a fundamental variable in the system. Stress is defined as force per unit area. Hence, by increasing the forces applied on each intersection, the stress will be increased. If the maximum stress of the domain grows more than critical values as ultimate strength of the system, the material may show different behavior which is not similar to its previous behavior. Therefore, the maximum internal forces should be investigated as critical variables of the system.

The variations of maximum horizontal force with respect to time are presented in Fig. 3(a). It is clearly shown that stepwise external force causes the system to go to more critical region. It is based on the fact that maximum value of the shown variable is greater for the membrane Moreover, there is a secondary stress variation in the system in both cases in which the applied forces are stepwise and sinusoidal loads. These secondary deflections also have peaks which one is related to the applied stepwise load is greater than the other one.
If we want to describe the behavior of the system, there are different important regions in this figure should be explained. The system starts to deform at $t = 1$ and the maximum horizontal force grows smoothly to reach a maximum value between $t = 1$ and $t = 2$. After this peak, the system try to release its stresses and the force becomes zero before finishing the period the external forces applied in between. Although the applied force is just between $t = 1$ and $t = 2$, after this period the system starts to increase its internal stress based on the fact that the stresses are based on the changes in the external forces not on their existence.

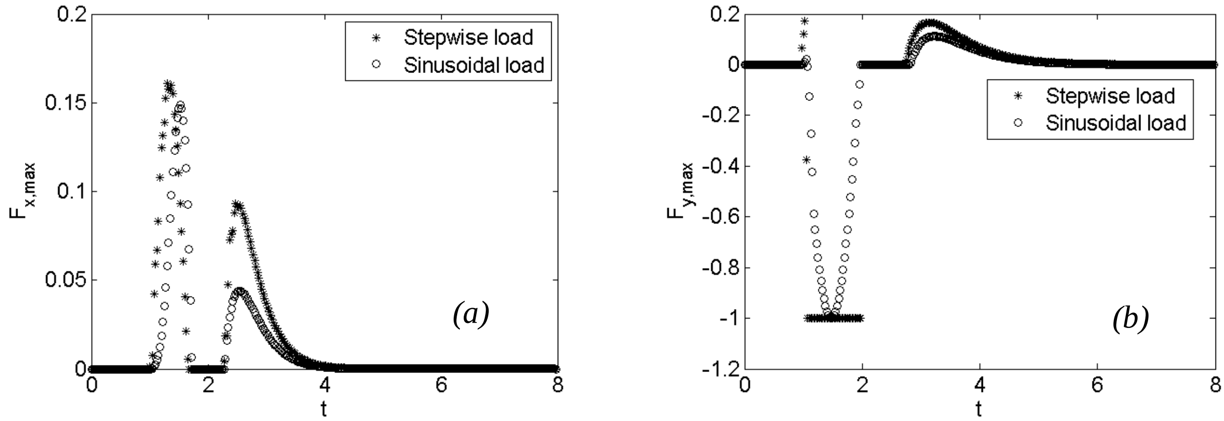

Fig. 3: Time variation of maximum internal (A) horizontal force and (B) vertical force of the membrane under two different loads.

The time dependency of maximum vertical internal force is similar to the horizontal one. There are secondary deformations and the peak values are greater if the system is under stepwise load. There is an important difference between the diagrams of Fig. 3(b). In the case of stepwise load, it is shown that there is a delay in the system so that the maximum internal vertical force reaches to the magnitude of the applied external load while it grows smoothly for the membrane under sinusoidal load. It should be noted that the maximum secondary vertical internal force is much less than the primary one (Fig. 3(b)).

4.2. Parameter dependency of maximum deflection

In addition to the dynamic behavior of the system, effects of parameter values determining the characteristics of the considered material are so important. Hence, the parameter dependency of the maximum deflection of the membrane is investigated and the diagrams are shown in Fig. 4.

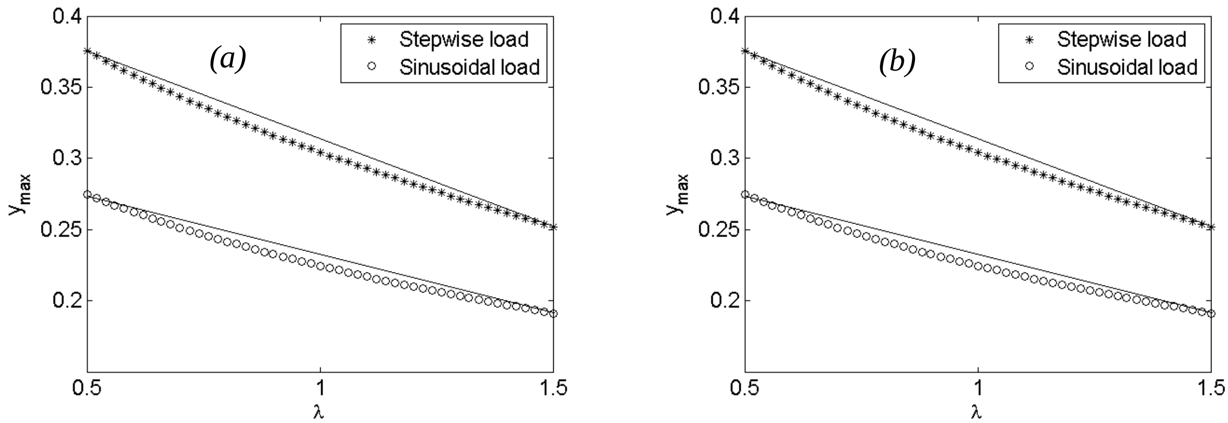

Fig. 4: Maximum deflection of the membrane with respect to (A) λ and (B) C (under two different loads).

Variations of peak values of maximum deflection of the membrane with respect to dimensionless stiffness parameter λ are illustrated in Fig. 4(a) for both applied stepwise and sinusoidal loads. It is shown that the behavior is nonlinear and the effect of this parameter decreases as it increases. It can be seen that for large values of λ, the variable goes toward zero and the dependency decreases slowly. The diagrams presented in Fig. 4(b) are similar to the ones shown in Fig.

4(a) and we conclude that parameters λ and C affects the maximum deflection in the similar way. Therefore, its value and dependency decreases by increasing the bending dissipation considered in the model.
Similar to the previous diagrams, the behavior of the system illustrated in Fig. 4 are different for the two considered applied loads. As it is described before, the peak value of maximum deflection is greater when the stepwise load is applied to the system. In Fig. 4(a), we can see that not only the peak value is larger but also the range of its change is larger in the case of stepwise external force. It is different from the effect of applied load on bending damping dependency of the peak value of the maximum deflection. It can be said that variation of this parameter has same effect on both diagrams of Fig. 4(a).

4.3. Effects of heterogeneity on maximum deflection

There are different natural and artificial systems which can not be modeled using homogeneous models [12], [13]. Hence, it is necessary to have a more general model instead of the available homogeneous ones to be able to investigate these systems. The presented model in this article is employed to investigate some effects of heterogeneity in the membrane. The dimensionless stiffness parameter λ as the heterogeneous parameter, e.g. this parameter only differs in different position of the domain.

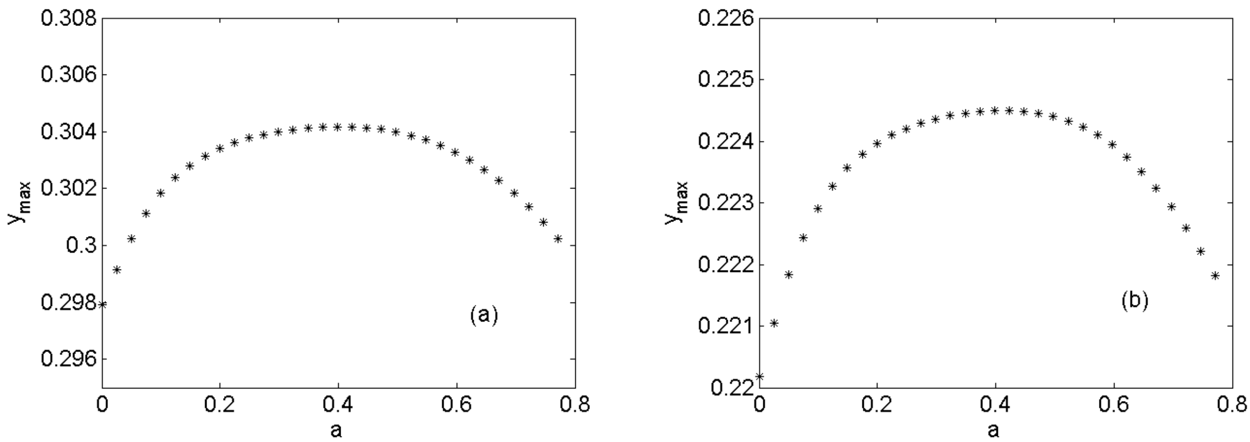

Fig. 5: Maximum deflection versus the beginning position of the heterogeneous region with $\lambda_h = 10/9$; (A) stepwise Load, (B) half period sinusoidal load.

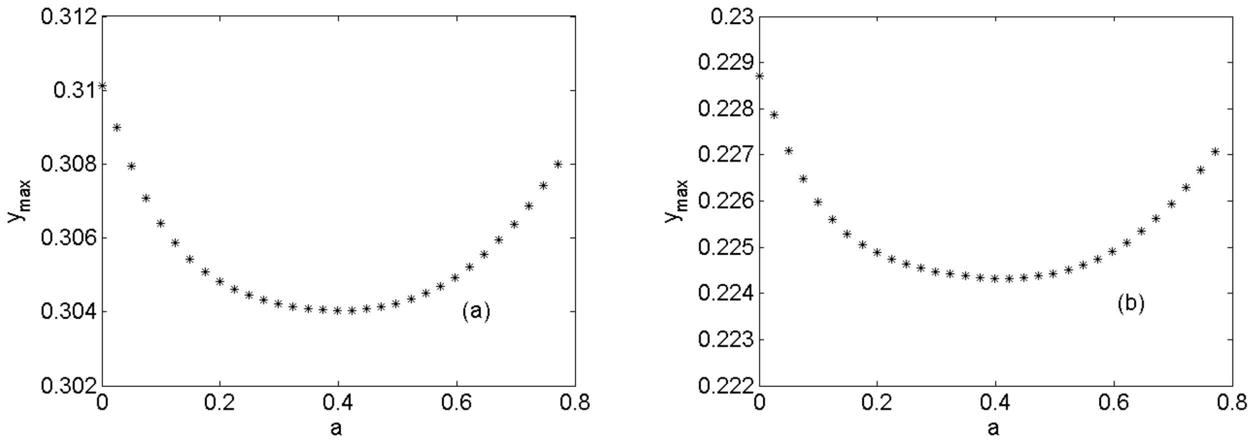

Fig. 6: Maximum deflection versus the beginning position of the heterogeneous region $\lambda_h = 9/10$; (A) stepwise load, (B) half period sinusoidal load.

The heterogeneity is considered as a small region with length $L_h = 0.2$ with $\lambda_h = 9/10$ and $\lambda = 10/9$ which their results are presented in Figs. 5 and 6, respectively. In Fig. 5, variation of maximum deflection with respect to a as the beginning position of the heterogeneous region is illustrated. It is shown that the three is the same position in which the maximum deformation is maximized. It is based on the fact that $\lambda_h = 10/9$ causing the membrane to divide into different parts and its middle deforms easier. We can conclude that there is critical region which is located in the middle of the half membrane. Therefore, changing the properties of this region can affect the behavior of the membrane in the most effective way. The behavior of the membrane in Fig. 6 in which $\lambda_h = 9/10$ is considered as the heterogeneity of the membrane. There are minimum values which also in the middle of the considered range. Moreover, the variations of the variable in both figures are greater when the external stepwise load is applied to the membrane.

4.4. Verification of the solution

In order to show the trustfulness of the presented solution, the presented model is used to obtain the variation of maximum deflection of a cantilever beam subjected to the external force and defined for the parameter values defined in [7]. The results obtained using the presented model is compared with the results obtained employing ABAQUS software [7] (Fig. 7). It is clearly shown that the considered model can be considered as the promising one to investigate the dynamic behavior of the membrane and beam can be investigated by this model.

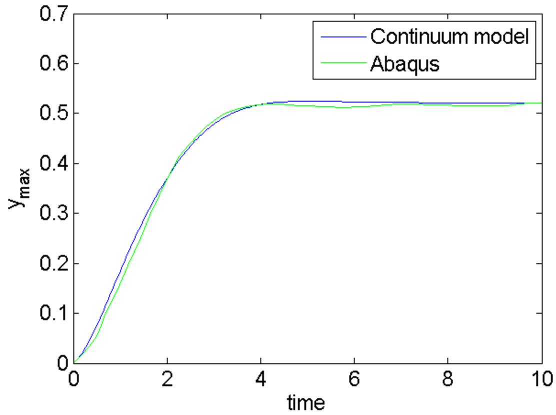

Fig. 7: Comparision of the time variation of the cantilever beam problem with the results available in [7].

5. Conclusion

Dynamic behavior of the thin membrane is modeled in this article. The presented general model can be used to obtain both large and small amplitude vibration of the membrane. In addition, this developed methodology can be employed to study the effects of different concentrated and distributed loads on the system as well as existence of heterogeneity in the domain of solution. The obtained nonlinear integro-differential equations of the problem are changed to the dimensionless form so that the results can be used to analyze the membranes with different sizes and material properties. Finite difference method is employed to discretize the equation and their matrix form is obtained at each time stage. The solution of the problem is used to obtain different results for the two considered applied concentrated loads including stepwise and sinusoidal. The maximum values of different variables of the problem are illustrated as functions of time. The diagrams of maximum deflection and slope of the membrane reveal the existence of one peak in their variations with respect to time. The maximum internal forces have additional peaks after the time period the external loads applied to the system. The variation of peak values is also presented with respect to variation of dimensionless stiffness and bending damping. The nonlinearity of the diagrams is discussed and the effects of the rate of applying the external load are investigated. Moreover, behavior of the membrane by changing the position of the heterogeneous region considered in the membrane is investigated and the critical region of the membrane is shown. Finally, the presented solution is verified with the results obtained by ABAQUS software [7] for cantilever beam problem can be also solved using the presented approach.

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