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Research paper

On gaussian diophantine quadruples

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Abstract

This paper concerns with the problem of constructing gaussian diophantine quadruples with the property that the product of any two distinct gaussian integers added with 1 and 4 in turn is a perfect square. The construction of gaussian diophantine quadruple (A,B,C,D) is illustrated through employing the non-zero distinct integer solutions of the system of double diophantine equations. The repetition of the above process leads to the generation of sequences of gaussian diophantine quadruples with the given property.

Keywords: *Diophantine Quadruple; Double Diophantine Equations; Gaussian Diophantine Quadruples; Integer Solutions; Pell Equations.*

1. Introduction

The construction of the sets with property that the product of any two of its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m distinct non-zero integers $\{a_1, a_2, ..., a_m\}$ is called a Diophantine m-tuple with property $D(n)$ if $a_i a_j + n$ is a perfect square for all $1 \le i \le j \le m$ [1]. Various mathematicians discussed the construction of different formulations of Diophantine triple and diophantine quadruples with property $D(n)$ for any arbitrary integer n and also for polynomials in n [2-15].

A set $\{a_1, a_2, ..., a_m\} \subset Z(i) - \{0\}$ is said to have this property $D(z)$ if the product of its any two distinct elements increased by z is a square of a Gaussian integer. If the set $\{a_1, a_2, ..., a_m\}$ is a complex diophantine quadruple then the same is true for the set $\{-a_1, -a_2, ..., -a_m\}$. Particularly in [16], the authors have analyzed the problem of the existence of the complex diophantine quadruples. In this context, one may refer [17-25].

In this communication, we construct sequences of gaussian diophantine quadruples with properties $D(1)$ and $D(4)$.

2. Method of analysis

2.1. Problem 1:

Let A, B be two gaussian integers represented by

$$A = kp \pm k + ikq \ , \ B = kp \pm (2n + k) + ikq$$

where k, p, q and n are non-zero integers.

Note that

$$AB + n^2 = (kp \pm (n + k) + ikq)^2 = r^2 \text{ (say)}$$

Therefore, the pair (A, B) is a gaussian diophantine 2-tuple with property $D(n^2)$

Consider C to be a gaussian integer such that

$$AC + n^2 = \alpha^2 \qquad (1)$$

$$BC + n^2 = \beta^2 \qquad (2)$$

Assume

$$\alpha = A + r \ , \ \beta = B + r \qquad (3)$$

Substituting (3) in (1) and (2) and subtracting one from the other, observe that

$$C = A + B + 2r = 4kp \pm 4(n + k) + i\,4kq$$

It is observed that the triple (A, B, C) is a gaussian diophantine 3-tuple with property $D(n^2)$. When $n = 1, n = 2$, the above triple (A, B, C) can be extended to diophantine quadruple with their corresponding properties.

2.1.1. Diophantine quadruple with property $D(1)$:

Let $n = 1$. Then the triple (A, B, C) is given by

$$A = kp \pm k + ikq$$

$$B = kp \pm (2 + k) + ikq$$

$$C = 4kp \pm 4(k + 1) + i\,4kq$$

which is a gaussian diophantine triple with property $D(1)$.

If (A, B, C) is a diophantine triple with property $D(1)$ then the fourth tuple D is given by

$$D = (A + B + C) + 2(ABC + \alpha\beta r) \quad (4)$$

where

$$AC + 1 = \alpha^2 \;,\; BC + 1 = \beta^2 \;,\; AB + 1 = r^2$$

Employing the above formula (4), the fourth tuple D is given by

$$D = 4(kp \pm (k+1) + ikq)\left(4\left(kp \pm (k+1) + ikq\right)^2 - 1\right)$$

Note that (A, B, C, D) is a gaussian diophantine quadruple with property $D(1)$.
Now, consider

$$B = kp \pm (2 + k) + ikq$$

$$C = 4kp \pm 4(k+1) + i\,4kq$$

$$D = 4(kp \pm (k+1) + ikq)\left(4\left(kp \pm (k+1) + ikq\right)^2 - 1\right)$$

which is a gaussian diophantine triple with property $D(1)$.
Using the above formula (4), the fourth tuple E is given by

$$\begin{aligned} E = {} & 256(kp + ikq)^5 \pm (1280k + 1536)(kp + ikq)^4 + (2560k^2 + 6144k + 3552)(kp + ikq) \\ & \pm (2560k^3 + 9216k^2 + 10656k + 3936)(kp + ikq)^2 \\ & + (1280k^4 + 6144k^3 + 10656k^2 + 7872k + 2081)(kp + ikq) \\ & \pm (256k^5 + 1536k^4 + 3552k^3 + 3936k^2 + 2081k + 420) \end{aligned}$$

Observe that (B, C, D, E) is a gaussian diophantine quadruple with property $D(1)$. Proceeding in this way, one may generate a sequence of gaussian diophantine quadruples (A, B, C, D), (B, C, D, E),......with property $D(1)$.
Some numerical examples of gaussian diophantine quadruples for the choices of $(k, p, q) = (1,2,1), (2,1,2), (2,2,3), (3,3,2)$ are respectively given below in Table: 1.

Table1: Numerical Examples

(A, B, C, D)	(B, C, D, E)
$(3+i, 5+i, 16+4i, 816+748i)$,	$(5+i, 16+4i, 816+748i, 142023+346401i)$,
$(1+i, -1+i, 4i, -20i)$	$(-1+i, 4i, -20i, -323+289i)$
$(4+4i, 6+4i, 20+16i, -1860+3760i)$,	$(6+4i, 20+16i, -1860+3760i, -3067352-459644i)$,
$(-4i, -2+4i, -4+16i, 756-848i)$	$(-2+4i, -4+16i, 756-848i, -330660+43140i)$
$(6+6i, 8+6i, 28+24i, -6636+10632i)$,	$(8+6i, 28+24i, -6636+10632i, -17446806-6132282i)$,
$(2+6i, 6i, 4+24i, -1716-3192i)$	$(6i, 4+24i, -1716-3192i, 1291422+1667526i)$
$(12+6i, 14+6i, 52+24i, 12636+45192i)$,	$(14+6i, 52+24i, 12636+45192i, -87594576+138411462i)$,
$(6+6i, 4+6i, 20+24i, -6660+3720i)$	$(4+6i, 20+24i, -6660+3720i, -1522398-6699066i)$

2.1.2. Diophantine quadruple with property $D(4)$:

Let $n = 2$. Then the triple (A, B, C) is given by

$$A = kp \pm k + ikq$$

$$B = kp \pm (4 + k) + ikq$$

$$C = 4kp \pm 4(2 + k) + i\,4kq$$

which is a gaussian diophantine triple with property $D(4)$.
If (A, B, C) is a diophantine triple with property $D(4)$ then the fourth tuple D is given by

$$D = (A + B + C) + \frac{1}{2}(ABC + \alpha\beta r) \quad (5)$$

where

$$AC + 4 = \alpha^2 \;,\; BC + 4 = \beta^2 \;,\; AB + 4 = r^2$$

Employing the above formula (5), the fourth tuple D is given by

$$D = 4(kp \pm (k+2) + ikq)\left(\left(kp \pm (k+2) + ikq\right)^2 - 1\right)$$

Note that (A, B, C, D) is a gaussian diophantine quadruple with property $D(4)$.
Now, consider

$$B = kp \pm (4 + k) + ikq$$

$$C = 4kp \pm 4(2 + k) + i\,4kq$$

$$D = 4(kp \pm (k+2) + ikq)\left(\left(kp \pm (k+2) + ikq\right)^2 - 1\right)$$

which is a gaussian diophantine triple with property $D(4)$.
Using the above formula (5), the fourth tuple E is given by

$$\begin{aligned} E = {} & 16(kp + ikq)^5 \pm (80k + 192)(kp + ikq)^4 + (160k^2 + 768k + 888)(kp + ikq)^3 \\ & \pm (160k^3 + 1152k^2 + 2664k + 1968)(kp + ikq)^2 \\ & + (80k^4 + 768k^3 + 2664k^2 + 3936k + 2081)(kp + ikq) \\ & \pm (16k^5 + 192k^4 + 888k^3 + 1968k^2 + 2081k + 840) \end{aligned}$$

Observe that (B, C, D, E) is a gaussian diophantine quadruple with property $D(4)$. Proceeding in this way, one may generate a sequence of gaussian diophantine quadruples (A, B, C, D), (B, C, D, E), ...with property $D(4)$.
Some numerical examples of gaussian diophantine quadruples for the choices of $(k, p, q) = (1,2,1), (2,1,2), (2,2,3), (3,3,2)$ are respectively given below in Table: 2.

Table 2: Numerical Examples

(A, B, C, D)	(B, C, D, E)

$(3+i\ ,7+i\ ,20+4i\ ,420+292i)$,	$(7+i\ ,20+4i\ ,420+292i\ ,43995+60465i)$,
$(1+i\ ,-3+i\ ,-4+4i\ ,12+4i)$	$(-3+i\ ,-4+4i\ ,12+4i\ ,169-143i)$
$(4+4i\ ,8+4i\ ,24+16i\ ,-312+1456i)$,	$(8+4i\ ,24+16i\ ,-312+1456i\ ,-366644+119428i)$,
$(4i\ ,-4+4i\ ,-8+16i\ ,360-80i)$	$(-4+4i\ ,-8+16i\ ,360-80i\ ,-18504-32124i)$
$(6+6i\ ,10+6i\ ,32+24i\ ,-1440+3720i)$,	$(10+6i\ ,32+24i\ ,-1440+3720i\ ,-1998976+40134i)$,
$(2+6i\ ,-2+6i\ ,24i\ ,-888i)$	$(-2+6i\ ,24i\ ,-888i\ ,-42630+126150i)$
$(12+6i\ ,16+6i\ ,56+24i\ ,4872+13224i)$,	$(16+6i\ ,56+24i\ ,4872+13224i\ ,-5847660+13478790i)$,
$(6+6i\ ,2+6i\ ,16+24i\ ,-1488+264i)$	$(2+6i\ ,16+24i\ ,-1488+264i\ ,125694-243258i)$

2.2. Problem 2:

Let A, B be two gaussian integers represented by

$$A=p+q+i\ ,\ B=p-q+i$$

where p and q are non-zero integers.
Note that

$$AB+q^2=(p+i)^2=r^2 \text{ (say)}$$

Therefore, the pair (A,B) is a gaussian diophantine 2-tuple with property $D(q^2)$
Consider C to be a gaussian integer such that

$$AC+q^2=\alpha^2 \quad (6)$$

$$BC+q^2=\beta^2 \quad (7)$$

Assume

$$\alpha=A+r\ ,\ \beta=B+r \quad (8)$$

Substituting (8) in (6) and (7) and subtracting one from the other, observe that

$$C=A+B+2r=4(p+i)$$

It is observed that the triple (A,B,C) is a gaussian diophantine 3-tuple with property $D(q^2)$. When $q=1,q=2$, the above triple (A,B,C) can be extended to diophantine quadruple with their corresponding properties.

2.2.1. Diophantine quadruple with property $D(1)$:

Let $q=1$. Then the triple (A,B,C) is given by

$$A=p+1+i$$

$$B=p-1+i$$

$$C=4(p+i)$$

which is a gaussian diophantine triple with property $D(1)$.
If (A,B,C) is a diophantine triple with property $D(1)$ then the fourth tuple D is given by

$$D=(A+B+C)+2(ABC+\alpha\beta r) \quad (9)$$

where

$$AC+1=\alpha^2\ ,\ BC+1=\beta^2\ ,\ AB+1=r^2$$

Employing the above formula (9), the fourth tuple D is given by

$$D=4(p+i)\left(4(p+i)^2-1\right)$$

Note that (A,B,C,D) is a gaussian diophantine quadruple with property $D(1)$.
Now, consider

$$B=p-1+i$$

$$C=4(p+i)$$

$$D=4(p+i)\left(4(p+i)^2-1\right)$$

which is a gaussian diophantine triple with property $D(1)$.
Using the above formula (9), the fourth tuple E is given by

$$E=256(p+i)^5-256(p+i)^4-32(p+i)^3+64(p+i)^2+(p+i)-3$$

Observe that (B,C,D,E) is a gaussian diophantine quadruple with property $D(1)$. Proceeding in this way, one may generate a sequence of gaussian diophantine quadruples (A,B,C,D), (B,C,D,E), ...with property $D(1)$.
Some numerical examples are given below:

Table 3: Numerical Examples

p	(A,B,C,D)	(B,C,D,E)
2	$(3+i\ ,1+i\ ,8+4i\ ,24+172i)$	$(1+i\ ,8+4i\ ,24+172i\ ,-7809+4257i)$
3	$(4+i\ ,2+i\ ,12+4i\ ,276+412i)$	$(2+i\ ,12+4i\ ,276+412i\ ,-10304+55873i)$
4	$(5+i\ ,3+i\ ,16+4i\ ,816+748i)$	$(3+i\ ,16+4i\ ,816+748i\ ,61505+224545i)$
5	$(6+i\ ,4+i\ ,20+4i\ ,1740+1180i)$	$(4+i\ ,20+4i\ ,1740+1180i\ ,372802+587073i)$

2.2.2. Diophantine quadruple with property $D(4)$:

Let $q=2$. Then the triple (A,B,C) is given by

$$A=p+2+i$$

$$B=p-2+i$$

$$C=4(p+i)$$

which is a gaussian diophantine triple with property $D(4)$.

If (A,B,C) is a diophantine triple with property $D(4)$ then the fourth tuple D is given by

$$D=(A+B+C)+\frac{1}{2}(ABC+\alpha\beta r) \quad (10)$$

where

$AC+4=\alpha^2$, $BC+4=\beta^2$, $AB+4=r^2$

Employing the above formula (10), the fourth tuple D is given by

$$D=4(p+i)\left((p+i)^2-1\right)$$

Note that (A,B,C,D) is a gaussian diophantine quadruple with property $D(4)$.
Now, consider

$$B=p-2+i$$

$$C=4(p+i)$$

$$D=4(p+i)\left((p+i)^2-1\right)$$

which is a gaussian diophantine triple with property $D(4)$.
Using the above formula (10), the fourth tuple E is given by

$$E=16(p+i)^5-32(p+i)^4-8(p+i)^3+32(p+i)^2+(p+i)-6$$

Observe that (B,C,D,E) is a gaussian diophantine quadruple with property $D(4)$. Proceeding in this way, one may generate a sequence of gaussian diophantine quadruples (A,B,C,D), (B,C,D,E),.....with property $D(4)$.
Some numerical examples are given below:

Table4: Numerical Examples

p	(A,B,C,D)	(B,C,D,E)
2	$(4+i\ ,\ i\ ,\ 8+4i\ ,\ 40i)$	$(i\ ,\ 8+4i\ ,\ 40i\ ,\ -308-71i)$
3	$(5+i\ ,\ 1+i\ ,\ 12+4i\ ,\ 60+100i)$	$(1+i\ ,\ 12+4i\ ,\ 60+100i\ ,\ -979+1969i)$
4	$(6+i\ ,\ 2+i\ ,\ 16+4i\ ,\ 192+184i)$	$(2+i\ ,\ 16+4i\ ,\ 192+184i\ ,\ 1374+10137i)$
5	$(7+i\ ,\ 3+i\ ,\ 20+4i\ ,\ 420+292i)$	$(3+i\ ,\ 20+4i\ ,\ 420+292i\ ,\ 15055+30385i)$

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