International Journal of Engineering & Technology, 7 (4.36) (2018) 379-384

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

Trilinear Finite Element Solution of Three Dimensional Heat Conduction Partial Differential Equations

Erwin Sulaeman[1*], S.M. Afzal Hoq[1,4], Abdurahim Okhunov[2], Marwan Badran[3]

[1]Department of Mechanical Engineering, International Islamic University Malaysia
[2]Department of Science in Engineering, International Islamic University Malaysia
[3]Department of Mechatronics Engineering, International Islamic University Malaysia,
[4]Department of Civil Engineering, Southern University Bangladesh, Bangladesh
**Corresponding author E-mail: esulaeman@iium.edu.my*

Abstract

Solution of partial differential equations (PDEs) of three dimensional steady state heat conduction and its error analysis are elaborated in the present paper by using a Trilinear Galerkin Finite Element method (TGFEM). An eight-node hexahedron element model is developed for the TGFEM based on a trilinear basis function where physical domain is meshed by structured grid. The stiffness matrix of the hexahedron element is formulated by using a direct integration scheme without the necessity to use the Jacobian matrix. To check the accuracy of the established scheme, comparisons of the results using error analysis between the present TGFEM and exact solution is conducted for various number of the elements. For this purpose, analytical solution is derived in detailed for a particular heat conduction problem. The comparison shows promising result where its convergence is approximately O(h^2) for matrix norms L^1, L^2 and L^∞.

Keywords: *Trilinear finite element method, hexahedron finite element, Galerkin method, 3D-Laplace equation, error analysis, heat conduction.*

1. Introduction

Finite element method (FEM) is computational technique commonly used to solve partial differential equations (PDE) which occurs in mathematical modeling of boundary value problems. The main feature of the method is its capability to solve complicated geometry, load or boundary conditions of the model by dividing the physical domain into finite sub-domains or elements and transforms the system into a set of linear equations. The accuracy of FEM is usually increase by increasing the number of elements. Therefore numerical simulation by using FEM in general requires digital tool for its processing. It is not surprising that the tremendous progress of application using FEM is in parallel with the revolutionary advancement of computer and digital technology. Currently FEM is used in engineering, applied mathematics, economics, medical among others [1-8].

The Galerkin's finite element method (GFEM) is one of FEM that utilizing a weight residual methods. Most of the GFEM utilize Jacobian matrix for the derivative of the function with respect to the natural coordinate system used to derive the stiffness matrix of the element [9-12]. The Jacobian matrix is very useful for arbitrary element geometry. However, the formulation involving Jacobian matrix will increase the computational time for each element. For a structured mesh, where most of the domain geometry is regular, it is possible to construct the element stiffness matrix directly without the necessity to use the Jacobian matrix. This procedure is conducted in the present work to solve a 3D steady state heat conduction problem which is solved by discretizing the physical domain into a number of 8-nodes brick elements. A simplified stiffness matrix that can be used for a homogenous cubic domain problem that allow for a structured grid mesh generation with uniform distribution of element sizes is presented where its scheme can reduce significantly the CPU time. The exact solution for a particular problem is derived and compared to the present GFEM results.

2. Mathematical Formulation

The three dimensional steady state heat conduction equation in homogenous domain without the heat source can be expressed as the following Laplace's equation [2]:

$$\nabla^2 u = \frac{\partial^2 u}{\partial^2 x}(x,y,z) + \frac{\partial^2 u}{\partial^2 y}(x,y,z) + \frac{\partial^2 u}{\partial^2 z}(x,y,z) = 0$$
$$0 < x < L_x, \quad 0 < y < L_y \quad \text{and} \quad 0 < z < L_z \qquad (1)$$

where u is the temperature at each point in the domain, u_b and f_b are the Dirichlet and Neumann boundary conditions, respectively. By application of the wighted residual approach. The PDE of (1) can be written as

$$I = -\int w\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) d\Omega + \oint_\Gamma w \frac{\partial u}{\partial n} d\Gamma \qquad (2)$$

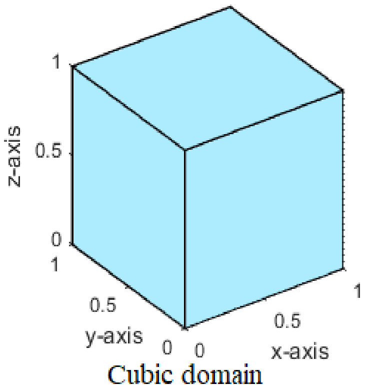
where w is the weighted function formulated using Galerkin approach. By further performing the integration by part procedure, Eq. (2) can be transformed to become the so-called weak formulation of Eq. (2):

$$I = -\int w\left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z}\right) d\,\Omega + \oint_\Gamma w \frac{\partial u}{\partial n} d\,\Gamma \qquad (3)$$

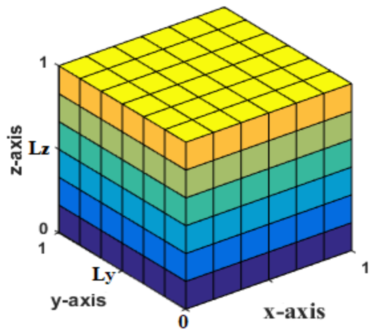
As the key steps in FEM, the solution to Eq. (3) is performed by dividing the domain Ω into sub-domains such that it reduces Eq. (3) into the task of solving a system of linear equations.

3. Basis function and stiffness matrix

Consider a heat conduction of a box shown in Figure 1(a) and its division into sub-domain as shown in Figure 1(b). The domain is assumed to be homogenous and has no heat source such that Eq. (1) can be used.



1(a)



1(b)

Fig. 1: (a) Physical domain of Ω bounded by Γ, (b) physical structural mesh

Assumed that the temperature $u = f(x, y, z)$ is a trilinear function as follows:

$$u = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 zx + a_8 xyz \tag{4}$$

which has 8 unknown coefficients of a. Since the hexahedron element has eight sets of node coordinates (x_i, y_i, z_i) and nodal variables u_i at the vertices of the element, then the coefficients a can be obtained. The value of the variable u at arbitrary location (x, y, z) within the elemental box domain region is approximated by the interpolation/basis function above which can be written in a matrix form as follows

$$u = \begin{bmatrix} 1 & x & y & z & xy & yz & xz & xyz \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \tag{5}$$

Hence, substituting the x, y and z coordinate values at each nodal point gives:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 & x_1y_1 & y_1z_1 & z_1x_1 & x_1y_1z_1 \\ 1 & x_2 & y_2 & z_2 & x_2y_2 & y_2z_2 & z_2x_2 & x_2y_2z_2 \\ 1 & x_3 & y_3 & z_3 & x_3y_3 & y_3z_3 & z_3x_3 & x_3y_3z_3 \\ 1 & x_4 & y_4 & z_4 & x_4y_4 & y_4z_4 & z_4x_4 & x_4y_4z_4 \\ 1 & x_5 & y_5 & z_5 & x_5y_5 & y_5z_5 & z_5x_5 & x_5y_5z_5 \\ 1 & x_6 & y_6 & z_6 & x_6y_6 & y_6z_6 & z_6x_6 & x_6y_6z_6 \\ 1 & x_7 & y_7 & z_7 & x_7y_7 & y_7z_7 & z_7x_7 & x_7y_7z_7 \\ 1 & x_8 & y_8 & z_8 & x_8y_8 & y_8z_8 & z_8x_8 & x_8y_8z_8 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} \tag{6}$$

where u_i in Eq. (8) are the temperature at each corner point of the 8-nodes brick element.

In Equation (6), -we substitute the geometric coordinates of the 8 corner points as follows:

$$\begin{aligned} &x_1 = -b,\ y_1 = c,\ z_1 = d,\ x_2 = -b,\ y_2 = c,\ z_2 = d \\ &x_3 = b,\ y_3 = c,\ z_3 = -d,\ x_4 = -b,\ y_4 = c,\ z_4 = -d \\ &x_5 = b,\ y_5 = -c,\ z_5 = d, x_6 = b,\ y_6 = -c,\ z_6 = d \\ &x_7 = b,\ y_7 = -c,\ z_7 = -d,\ x_8 = b,\ y_8 = -c,\ z_8 = -d \end{aligned} \tag{7}$$

to give

$$\begin{aligned} u = {} & H_1(x,y,z)\ u_1 + H_2(x,y,z)\ u_2 + H_3(x,y,z)\ u_3 + \\ & H_4(x,y,z)u_4 + H_5(x,y,z)u_5 + H_6(x,y,z)u_6 \\ & + H_7(x,y,z)u_7 + H_8(x,y,z)u_8 \end{aligned} \tag{8}$$

where $H_i(x, y, z)$ is the shape function for trilinear hexahedron element which can be express as functions of the three dimensional geometry coordinates as follows:

$$\begin{aligned} H_1 &= \frac{(b-x)(c-y)(d+z)}{8bcd} \\ H_2 &= \frac{(b+x)(c-y)(d+z)}{8bcd} \\ H_3 &= \frac{(b+x)(c-y)(d-z)}{8bcd} \\ H_4 &= \frac{(b-x)(c-y)(d-z)}{8bcd} \\ H_5 &= \frac{(b-x)(c+y)(d+z)}{8bcd} \\ H_6 &= \frac{(b+x)(c+y)(d+z)}{8bcd} \\ H_7 &= \frac{(b+x)(c+y)(d-z)}{8bcd} \\ H_8 &= \frac{(b-x)(c+y)(d-z)}{8bcd} \end{aligned} \tag{9}$$

Where the shape function meets the following conditions

$$\left. \begin{aligned} & H_i(x_i, y_j, z_k) = \delta_{ijk} \\ & \sum_i^1 H_i = 1 \end{aligned} \right\} \tag{10}$$

Therefore, the stiffness matrix can be obtained as

$$\left(K^e\right)=\int_{\Omega^e}\begin{Bmatrix}\frac{\partial H_1}{\partial x}\\ \frac{\partial H_2}{\partial x}\\ \cdot\\ \cdot\\ \\ \frac{\partial H_8}{\partial x}.\end{Bmatrix}\left\{\frac{\partial H_1}{\partial x}\,\frac{\partial H_2}{\partial x}\cdots\cdot\frac{\partial H_8}{\partial x}\right\}+$$

$$\begin{Bmatrix}\frac{\partial H_1}{\partial y}\\ \frac{\partial H_2}{\partial y}\\ \cdot\\ \cdot\\ \cdot\\ \frac{\partial H_8}{\partial y}.\end{Bmatrix}\left\{\frac{\partial H_1}{\partial y}\,\frac{\partial H_2}{\partial y}\cdots\cdot\frac{\partial H_8}{\partial y}\right\}+\begin{Bmatrix}\frac{\partial H_1}{\partial z}\\ \frac{\partial H_2}{\partial z}\\ \cdot\\ \cdot\\ \cdot\\ \frac{\partial H_8}{\partial z}.\end{Bmatrix}\left\{\frac{\partial H_1}{\partial z}\,\frac{\partial H_2}{\partial z}\cdots\cdot\frac{\partial H_8}{\partial z}\right\}\Bigg] \quad (11)$$

which yields an 8-by-8 symmetric matrix

$$\left[K^e\right]=\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18}\\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28}\\ & & K_{23} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38}\\ & & & K_{44} & K_{45} & K_{46} & K_{47} & K_{48}\\ & & & & K_{55} & K_{56} & K_{57} & K_{58}\\ & & sym & & & K_{66} & K_{67} & K_{68}\\ & & & & & & K_{77} & K_{78}\\ & & & & & & & K_{818}\end{bmatrix} \quad (12)$$

where, in the present work, each element matrix is formulated by using a direct integration

$$K^e_{i,j}=\int_{\Omega^e}\left(\frac{\partial H_i}{\partial x}\frac{\partial H_j}{\partial x}+\frac{\partial H_i}{\partial y}\frac{\partial H_j}{\partial y}+\frac{\partial H_i}{\partial z}\frac{\partial H_j}{\partial z}\right)d\Omega \quad (13)$$

4. Analytical solution of the particular heat conduction problem

Consider three dimensional heat conduction equations for steady state problems of a cubic box shown in Fig. 2 which lead to Helmholtz equation (Ref-13).

$$\nabla^2\mathrm{T}=\frac{\partial^2\mathrm{T}}{\partial x^2}+\frac{\partial^2\mathrm{T}}{\partial y^2}+\frac{\partial^2\mathrm{T}}{\partial z^2}=0 \quad (14)$$

Where T is the temperature at each point in the domain similar to the parameter u in (1). In this particular problem, all surface boundaries have the temperature of $T = 0$ except at the top surface ($z = L_z$) that exhibits a sinusoidal function as follow:

$$T=T_0\sin\left(\frac{\pi x}{L_x}\right).\sin\left(\frac{\pi x}{L_y}\right) \quad (15)$$

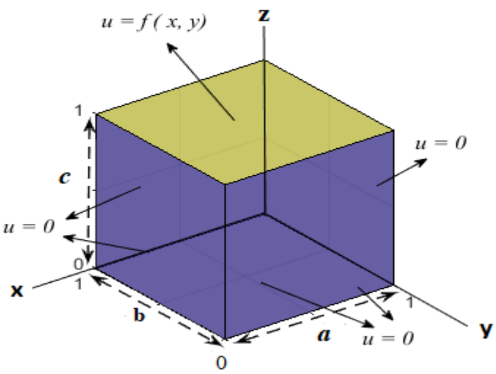


Fig. 2: Physical domain for the analytical solution

Following Ref (5) for solving 2D heat problems, the technique is extended to treat 3D problem where we assumed the solution satisfying the Laplace equation and the boundary conditions can be expressed as follows:

$$T(x,y,z)=P(x).Q(y).R(z) \quad (16)$$

Such that

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=QR\frac{\partial^2 R}{\partial x^2}\\ \frac{\partial^2 u}{\partial y^2}&=PR\frac{\partial^2 R}{\partial y^2}\\ \frac{\partial^2 u}{\partial z^2}&=PQ\frac{\partial^2 R}{\partial z^2}\end{aligned}\right\} \quad (17)$$

Substitution of equation (17) into (14) yields

$$QR\frac{\partial^2 P}{\partial x^2}+PR\frac{\partial^2 Q}{\partial y^2}+PQ\frac{\partial^2 R}{\partial z^2}=0 \quad (18)$$

$$\frac{1}{P}\frac{\partial^2 P}{\partial x^2}+\frac{1}{Q}\frac{\partial^2 Q}{\partial y^2}+\frac{1}{R}\frac{\partial^2 R}{\partial z^2}=0 \quad (19)$$

Therefore, in order to satisfy the conditions in (19), each term in the left hand side of (19) should be equal to a constant such that

$$\left.\begin{aligned}\frac{1}{P}\frac{\partial^2 P}{\partial x^2}&=\alpha^2\\ \frac{1}{Q}\frac{\partial^2 Q}{\partial y^2}&=\beta^2\\ \frac{1}{R}\frac{\partial^2 R}{\partial z^2}&=\gamma^2\end{aligned}\right\} \quad (20)$$

Then, substitution of (20) into (19) yields

$$\alpha^2+\beta^2+\gamma^2=0 \quad (21)$$

Equation (20) can be rewritten as

$$\left.\begin{aligned}\frac{\partial^2 P}{\partial x^2}+\alpha^2 P&=0\\ \frac{\partial^2 Q}{\partial y^2}+\beta^2 Q&=0\\ \frac{\partial^2 R}{\partial z^2}+\gamma^2 R&=0\end{aligned}\right\} \quad (22)$$

The solution of equation (22) for P is

$$P=A_x\sin\alpha x+B_x\cos\alpha x \quad (23)$$

By considering the boundary conditions at x =0 or $x = L_x$

$T=0$ or

$$P_{(x)}\, Q_{(y)}\, R_{(z)} = 0 \quad (24)$$

Or $P = 0$ at $x = 0$ or $x = L_x$.

For $P=0$ when x=0 yields

$$B_x = 0$$

For $P=0$ when $x = L_x$

$$A_x \sin\alpha L_x = 0 \quad (25)$$

Since for non-trivial solution $A_x \neq 0$

$$\sin\alpha\ L_x = 0$$

Therefore

$$\alpha = \frac{n\pi x}{L_x}\ ;\ n = 1, 2, 3....$$

Therefore equation (23) becomes

$$P = \sum A_{x_n} \sin\frac{n\pi x}{L_x}\ ;\ n = 1, 2, 3,....$$

Similarly, for the solution of equation (27) for Q is

$$Q = \sum A_{y_n} \sin\frac{m\pi y}{L_y}\ ;\ m = 1, 2, 3,....$$

Therefore we have

$$\left.\begin{aligned} \alpha_n &= \frac{n\pi}{L_x} \\ \beta_m &= \frac{m\pi}{L_y} \end{aligned}\right\}$$

Since

$$\alpha^2 + \beta^2 + \gamma^2 = 0$$

$$\gamma^2 = -(\alpha^2 + \beta^2)$$

$$= -\left[\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_x}\right)^2\right]$$

$$\gamma^2 = \eta^2$$

Where:

$$\eta^2 = \left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_x}\right)^2$$

Therefore, the equation (7) for R becomes

$$\frac{\partial^2 R}{\partial z^2} - \eta^2 R^2 = 0$$

The solution is

$$R = A_z \sinh\eta z + A_z \cosh\eta z \quad (24)$$

Refer to the boundary conditions

T=0 when $P\,Q\,R_{(z=0)} = 0$

when z=0 or $R_{(z=0)} = 0$

Or

$$A_z \sinh(0) + B_z \cosh(0) = 0$$

$$or\ \ B_z = 0$$

Therefore

$$R = A_z \sinh\eta z \quad (26)$$

The 2nd boundary conditions at $z = L_z$

$$T = T_0 .\sin\frac{\pi x}{L_x}.\sin\frac{\pi y}{L_y} \quad (27)$$

$$PQR = T_0 .\sin\frac{\pi x}{L_x}.\sin\frac{\pi y}{L_y} \quad (28)$$

$$\left[\sum A_{x_n} \sin\frac{n\pi x}{L_x}\right].\left[\sum A_{y_m} \sin\frac{n\pi y}{L_y}\right] \times \left[A_z.\sinh\eta z\right]$$

$$= T_0 .\sin\frac{\pi x}{L_x}.\sin\frac{\pi y}{L_y} \quad (29)$$

By similarity of the left and right hand sides of (29), the following should holds: n=1 , m=1 so that

$$\left(A_x.A_y.A_z\right).\sin\frac{\pi x}{L_x}.\sin\frac{\pi y}{L_y}.\sinh(\eta.L_z) =$$

$$T_0 .\sin\frac{\pi x}{L_x}.\sin\frac{\pi y}{L_y} \quad (30)$$

$$\left(A_x.A_y.A_z\right) = \frac{T_0}{\sinh(\eta.L_z)} \quad (31)$$

Therefore, the analytical solution for this particular problem is

$$T = \frac{T_0 .\sin\frac{\pi x}{L_x}.\ \sin\frac{\pi y}{L_y}\ .\sinh(\eta z)}{\sinh(\eta.L_z)} \quad (32)$$

5. Validation and convergence of the results

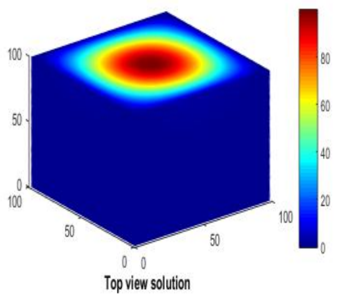
In order to evaluate the accuracy of the present GFEM procedure, a numerical simulation of the steady state heat conduction problem of cubic domain shown in Figure 2 is conducted and the results are compared with the analytical results described in previous section. To check the convergence, the length L_x, width L_y and height L_z of the box are set equal to 1 respectively so that the number of elements in x, y and z are set equal also $n_x = n_y = n_z$. The boundary conditions at the side surfaces are set similar to the analytical problem, i.e all surface boundaries have the temperature of $T = 0$ except the top surface ($z = L_z$) that exhibits a bi-sinusoidal function as shown in (15).

The error is evaluated by comparing the difference between the numerical and analytical results in three different ways. In this case, three matrix norms L^1, L^2 and L^∞ are used which are defined as follow:

$$L^1 = \frac{\sum_i^N \left|\tilde{u}_i - \bar{u}_i\right|}{N}$$

$$L^2 = \sqrt{\frac{\sum_i^N \left(\tilde{u}_i - \bar{u}_i\right)^2}{N}}$$

$$L^\infty = \max\sum_i^N \left|\tilde{u}_i - \bar{u}_i\right|$$



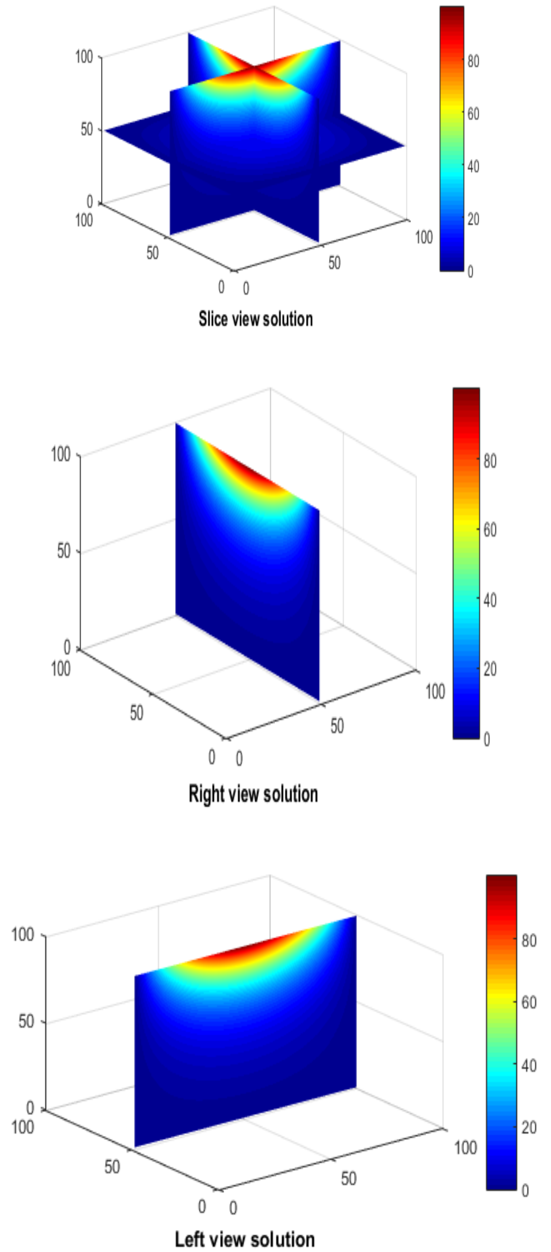


Fig. 3: Temperature distribution based on the analytical solution of Eq. 32. (a) Boundary conditions, (b) Slice view at $x = Lx/2$ and $y = Ly/2$, (c) Slice view at $y = Ly/2$, (d) Slice view at $x = Lx/2$.

The GFEM temperature distribution results is shown in Figure 3. The error of the present GFEM by comparisons with the analytical solution is presented in Table 1. Figure 4 shows the error as function of the number elements. As expected, it is found that the error is consistently reducing by increasing the number of elements or by decreasing the size of elements.

Table.1 L^1, L^2 and L^∞ Error Analysis of the solution

Total element $n_x * n_y * n_z$	Element Size (L /n)	$\|\tilde{u}_i - \bar{u}_i\|_0$ (10^{-3})	$\|\tilde{u}_i - \bar{u}_i\|_2$ (10^{-3})	$\|\tilde{u}_i - \bar{u}_i\|_\infty$ (10^{-3})
8*8*8	0.125	1.0427	1.8958	7.1883
16*16*16	0.062	0.31775	0.5084	1.7609
24*24*24	0.041	0.1506	0.23203	0.78329
32*32*32	0.031	0.087457	0.13235	0.44093
40*40*40	0.025	0.057048	0.08543	0.28207
48*48*48	0.020	0.040121	0.05967	0.19575

To assess the rate of convergence of the error, a general regression procedure [10, 11] is performed. The regression result shows that matrix norm L^1, L^2 and L^∞ can be approximated as

$$L^1 = 0.0477\, h^{1.821}$$
$$L^2 = 0.1062\, h^{1.9308}$$
$$L^\infty = 0.4665\, h^{2.0095}$$

where h is the size of the element length. This indicate that the GFEM result converges to the exact solution with the convergence rates of almost O(h^2) for matrix norm L^1, L^2 and L^∞. This remarkable rate demonstrates the fast convergence of the present GFEM formulation.

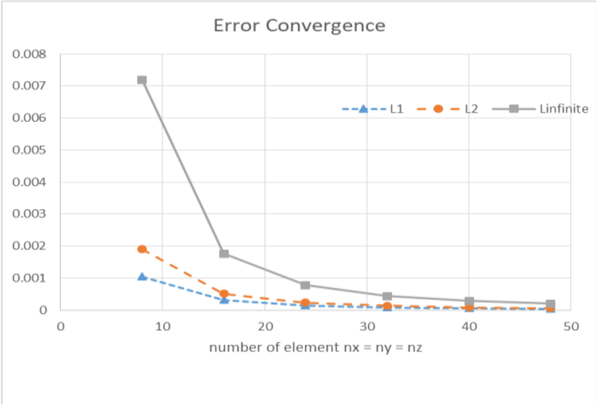


Fig. 4: Convergence of the error analysis results

6. Conclusion

The present work describes a finite element method for three dimensional heat conduction using eight-nodes hexahedron element models where the formulation is conducted without the necessity to use the Jacobian matrix. In order to assess the accuracy of the present work, comparison with analytical results is conducted. The analytical results in the form of temperature distribution on rectangular box with Dirichlet boundary conditions is derived in detailed by using the separation of variable method. The results shows that the convergence of the present TGFEM method is approximately O(h^2) for matrix norm L^1, L^2 and L^∞.

Acknowledgement

The support of International Islamic University Malaysia under the research grant RIGS17-033-0608 is gratefully acknowledged.

References

[1] Richter T. & Vexler B (2013).Efficient numerical realization of discontinuous Galerkin methods for temporal discretization of parabolic problems. *Numerische Mathematik*. 124(1), 151–182.

[2] Yong WK & Hyochoong B (2000). The Finite Element method using Matlab. *CRC Press, Boca Raton London*.

[3] Antonopoulou DC & Plexousaki M (2010), Discontinuous Galerkin methods for the linear Schrödinger equation in non-cylindrical domains. *Numerische Mathematik* 115(4), 585-608.

[4] Dow JO (1999) A Unified Approach to The Finite Element Method and Error Analysis Procedures, Academic Pers, London.

[5] Schotzau D & Schwab C (2000), Time Discretization of Parabolic Problems by the HP-Version of the Discontinuous Galerkin Finite Element Method. *SIAM Journal on Numerical Analysis* 38(3), 837-75.

[6] Dongfang Li & Chengjian Z. (2014), Error estimates of discontinuous Galerkin methods for delay differential equations. *Applied Numerical Mathematics* 82, 1–10.

[7] Manas, KD., Babuska, IM., & Oden, JT. (2001) Solution of stochastic partial differential equations using Galerkin finite element method. *Compt. Methods Appl. Mech. Eng*. 190, 6359–72.

[8] Gil S, Saleta ME & Tobia. (2002), Experimental Study of the Neumann and Dirichlet Boundary Conditions in Two-Dimensional Electrostatic Problems. *American Journal of Physics*, 70 (12), 1208-1213.

[9] Burden RL & Faires JD (2001). Numerical Analysis. 9th ed., *Brooks/Cole,* Boston.

[10] Hoq SMA, Sulaeman E & Okhunov A (2016), Error Analysis of Heat Conduction Partial Differential Equations using Galerkin's Finite Element Method. *Indian Journal of Science and Technology* 9 (36).

[11] Ainsworth M & Oden JT(1997), A posterior error estimation in finite element method, *Computer Methods in Applied Mechanics and Engineering* 142 (1–2), 1-88.

[12] Lijun Yi. (2015) An L^2-error Estimate for the h-p Version Continuous Petrov-Galerkin Method for Nonlinear Initial Value Problems. *East Asian Journal on Applied Mathematics East Asian Journal on Applied Mathematics* 5(4), 301-311.

[13] Kreyszig, E. (2011) Advanced Engineering Mathematics. 10th edn. *John Wiley & Sons*: New York.