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Research paper

Kamenev-Type Oscillation Criteria for Generalized Second Order Sublinear α-Difference Equations

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Abstract

By means of Riccati transformation techniques, authors establish some new oscillation criteria for generalized second order nonlinear α -difference equation $\Delta_{\alpha(\ell)}(p(k\ell+j)(\Delta_{\alpha(\ell)}(u(k\ell+j)))^{\gamma})+q(k\ell+j)u^{\beta}((k-\sigma)\ell+j)=0$, when $0<\beta<1$ and γ are quotient of odd positive integers.

Keywords: *Delay; Sublinear; Superlinear.*

1. Introduction

Difference equations represent a fascinating mathematical area on its own as well as a rich field of the applications in such diverse disciplines. For general background as difference equations with many examples from diverse fields, one can refer to [1].
The theory of difference equations is based on the operator Δ defined as $\Delta u(k)=u(k+1)-u(k)$, $k\in \mathrm{N}=\{0,1,2,\cdots\}$. Even though some authors [1] have suggested the definition of Δ as

$$\Delta u(k)=u(k+\ell)-u(k),\quad \ell\in(0,\infty), \tag{E}$$

no significant progress took place on this line. Jerzy Popenda, et.al., [16] defined Δ_{α} as $\Delta_{\alpha}u(k)=u(k+1)-\alpha u(k)$. Authors in [14], considering the operator Δ defined by (E) as Δ_{ℓ} many interesting results in number theory were obtained [14]. In [15], they generalized the definition of Δ_{α} by $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)}u(k)=u(k+\ell)-\alpha u(k)$ for the real valued function $u(k)$ and $\ell\in(0,\infty)$ and also obtained the solutions of certain types of generalized α -difference equations.
In recent years, the asymptotic behaviour of second order difference equations has been the subject of investigations by many authors [2-12,17-19,21-22].
In this paper, we will be concerned with a class of generalized second order sublinear delay difference equations of the form

$$\begin{aligned}&\Delta_{\alpha(\ell)}(p(k\ell+j)(\Delta_{\alpha(\ell)}(u(k\ell+j)))^{\gamma})\\&+q(k\ell+j)u^{\beta}((k-\sigma)\ell+j)=0,\end{aligned} \tag{1}$$

where $\Delta_{\alpha(\ell)}$ denotes the forward difference operator for any real valued function $u(k\ell+j)$, $k\in(0,\infty)$, $\ell\in(0,\infty)$, $j=k-\left[\frac{k}{\ell}\right]\ell$, γ is quotient of odd positive integers, $0<\beta<1$ is quotient of odd positive integers, σ is a fixed nonnegative integer, $p(k\ell+j)>0$, and $q(k\ell+j)\geq 0$ are real valued functions, and for some $k_0>0$,

$$\sum_{k=k_0}^{\infty}\left(\frac{1}{p(k\ell+j)}\right)^{1/\gamma}=\infty, \tag{2}$$

By a solution of (1) we mean a nontrivial real valued function $u(k)$ defined for $k\geq-\sigma$, and satisfies equation (1) for $k\in(0,\infty)$. Clearly, if

$$u(k\ell+j)=A(k\ell+j)\text{ for }k\in[-\sigma,0] \tag{3}$$

are given, then equation (1) has a unique solution satisfying the initial condition (3).

2. Main Results

Theorem 1 *Assume that (2) holds. Furthermore, assume that there exist a positive real valued functions $\rho(k\ell+j)$ such that for every $\eta\geq 1$ and positive number M.*

$$\begin{aligned}&\limsup_{k\to\infty}\sum_{r=0}^{k}[\rho(r\ell+j)q(r\ell+j)-\alpha(p((r-\sigma)\ell+j))^{1/\gamma}\\&\times\frac{\eta^{1-\beta}((r-\sigma+1)\ell+j)^{1-\beta}(\Delta_{\ell}\rho(r\ell+j))^{2}}{4\beta(M)^{(\gamma-1)/\gamma}\rho(r\ell+j)}]\end{aligned} \tag{4}$$

Then every solution of equation (1) oscillates.
Proof. Suppose to the contrary that $u(k\ell+j)$ is an eventually nonoscillatory solution of (1) such that $u((k-\sigma)\ell+j)>0$ for

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all $k \geq k_0 > 0$. We shall consider only this case, since the substitution $v(k\ell + j) = -u(k\ell + j)$ transforms equation (1) into an equation of the same form. From equation (1) we have for $k \geq k_0$

$$\begin{aligned}&\Delta_{\alpha(\ell)}(p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma}) \\ &= -q(k\ell + j)\, u^{\beta}((k - \sigma)\ell + j) \leq 0,\end{aligned} \tag{5}$$

and so $p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma}$ is an eventually nonincreasing sequence. We first show that $p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma} \geq 0$ for $k \geq k_0$. In fact, if there exists an real $k_1 \geq k_0$ such that $p(k_1\ell + j)(\Delta_{\alpha(\ell)} u(k_1\ell + j))^{\gamma} = c < 0$, then (5) implies that $p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma} \leq c$ for $k \geq k_1$ that is $\Delta_{\alpha(\ell)} u(k\ell + j) \leq (c / p(k\ell + j))^{1/\gamma}$ hence as $k \to \infty$ $u(k\ell + j) \leq \alpha u(k_1\ell + j)$

$$+\sum_{r=k_1}^{k-1}\left[(\alpha - 1)\, u(r\ell + j) + \left(\frac{1}{p(r\ell + j)}\right)^{1/\gamma}\right] \to -\infty \tag{6}$$

which contradicts the fact that $u(k\ell + j) > 0$ for $k \geq k_0$, then $p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma} \geq 0$. Also we claim that $\Delta^2_{\alpha(\ell)} u(k\ell + j) \leq 0$. If not there exists $k_1 \geq k_0$ such that $\Delta^2_{\alpha(\ell)} u(k\ell + j) > 0$ for $k \geq k_1$ and this implies that $\Delta_{\alpha(\ell)} u((k+1)\ell + j) > \Delta_{\alpha(\ell)} u(k\ell + j)$, so that since $\Delta_{\alpha(\ell)} p(k\ell + j) \geq 0$, $p((k+1)\ell + j)(\Delta_{\alpha(\ell)} u((k+1)\ell + j))^{\gamma}$ $> p((k+1)\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma}$ $\geq p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma}$ and this contradicts the fact that $p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma}$ is nonincreasing sequence, then $\Delta^2_{\alpha(\ell)} u(k\ell + j) \leq 0$, and therefore we have for $k \geq k_0$

$$u(k\ell + j) > 0, \Delta_{\alpha(\ell)} u(k\ell + j) \text{ and } \Delta^2_{\alpha(\ell)} u(k\ell + j) \leq 0. \tag{7}$$

Define the sequence $z(k\ell + j)$ by

$$z(k\ell + j) = \alpha^{\left[\frac{k}{\ell}\right]-1} \frac{\rho(k\ell + j)\, p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma}}{u^{\beta}((k - \sigma)\ell + j)} \tag{8}$$

then $z(k\ell + j) > 0$, and

$$\Delta_{\alpha(\ell)} z(k\ell + j) = \frac{\alpha^{\left[\frac{k}{\ell}\right]} \rho(k\ell + j)\, \Delta_{\ell}(p(k\ell + j)(\Delta_{\alpha(\ell)} u(k\ell + j))^{\gamma})}{u^{\beta}((k - \sigma)\ell + j)}$$

$$+ p((k+1)\ell + j)(\Delta_{\alpha(\ell)} u((k+1)\ell + j))\Delta_{\alpha(\ell)}\left(\frac{\alpha^{\left[\frac{k}{\ell}\right]-1} \rho(k\ell + j)}{u^{\beta}((k - \sigma)\ell + j)}\right) \tag{9}$$

From (1) and (9), we have

$$\begin{aligned}\Delta_{\ell} z(k\ell + j) &= -\rho(k\ell + j) q(k\ell + j) + \frac{\Delta_{\ell} \rho(k\ell + j) z((k+1)\ell + j)}{\rho((k+1)\ell + j)} \\ &- \frac{\alpha^{\left[\frac{k}{\ell}\right]} \rho(k\ell + j)\, p((k+1)\ell + j)(\Delta_{\alpha(\ell)} u((k+1)\ell + j))^{\gamma}}{u^{\beta}((k - \sigma + 1)\ell + j)} \\ &\times \frac{\Delta_{\ell} u^{\beta}((k - \sigma)\ell + j)}{u^{\beta}((k - \sigma)\ell + j)}\end{aligned} \tag{10}$$

From (5) and (7), we get

$$\begin{aligned}&p((k - \sigma)\ell + j)(\Delta_{\ell} u((k - \sigma)\ell + j))^{\gamma} \\ &\geq p((k+1)\ell + j)(\Delta_{\ell} u((k+1)\ell + j))^{\gamma} \\ &\text{and } u((k + 1 - \sigma)\ell + j) \geq u((k - \sigma)\ell + j)\end{aligned} \tag{11}$$

and then from (10) and (11), we have

$$\begin{aligned}&\Delta_{\alpha(\ell)} z(k\ell + j) \leq -\rho(k\ell + j) q(k\ell + j) \\ &+ \frac{\Delta_{\ell} \rho(k\ell + j) z((k+1)\ell + j)}{\rho((k+1)\ell + j)} \\ &- \frac{\rho(k\ell + j)\, p((k+1)\ell + j)(\Delta_{\alpha(\ell)} u((k+1)\ell + j))^{\gamma}}{\left(u^{\beta}((k - \sigma + 1)\ell + j)\right)^2} \\ &\times \Delta_{\alpha(\ell)} \alpha^{\left[\frac{k}{\ell}\right]-1} u^{\beta}((k - \sigma)\ell + j)\end{aligned} \tag{12}$$

Now, by using the inequality in [2], for all $u \neq v > 0$ and for $0 < \beta \leq 1$, $u^{\beta} - v^{\beta} \geq \beta u^{\beta - 1}(u - v)$. Then, we have

$$\begin{aligned}&\Delta_{\alpha(\ell)} \alpha^{\left[\frac{k}{\ell}\right]-1} u^{\beta}((k - \sigma)\ell + j) \\ &= \beta(u((k - \sigma + 1)\ell + j))^{\beta - 1} \Delta_{\alpha(\ell)} \alpha^{\left[\frac{k}{\ell}\right]-1} u((k - \sigma)\ell + j).\end{aligned} \tag{13}$$

Substitute from (13) in (12), we have

$$\begin{aligned}&\Delta_{\alpha(\ell)} z(k\ell + j) \leq -\rho(k\ell + j) q(k\ell + j) \\ &+ \frac{\Delta_{\ell} \rho(k\ell + j) z((k+1)\ell + j)}{\rho((k+1)\ell + j)} \\ &- \left(\frac{\rho(k\ell + j)\, p((k+1)\ell + j)\, \beta(u((k + 1 - \sigma)\ell + j))^{\beta - 1}}{\left(u^{\beta}((k - \sigma + 1)\ell + j)\right)^2}\right) \\ &\times \Delta_{\alpha(\ell)} u((k - \sigma)\ell + j)(\Delta_{\alpha(\ell)} u((k+1)\ell + j))^{\gamma}\end{aligned} \tag{14}$$

From (11) and (14), we have

$$\begin{aligned}&\Delta_{\alpha(\ell)} z(k\ell + j) \leq -\rho(k\ell + j) q(k\ell + j) \\ &+ \frac{\Delta_{\ell} \rho(k\ell + j) z((k+1)\ell + j)}{\rho((k+1)\ell + j)} \\ &- \frac{\beta\rho(k\ell + j)(p((k+1)\ell + j))^{1/\gamma} p((k+1)\ell + j)}{(p((k - \sigma)\ell + j))^{1/\gamma}\left(u((k - \sigma + 1)\ell + j)\right)^{1-\beta}} \\ &\times \frac{(\Delta_{\alpha(\ell)} u((k+1)\ell + j))^{\gamma+1}}{\left(u^{\beta}((k - \sigma + 1)\ell + j)\right)} \\ &\Delta_{\alpha(\ell)} z(k\ell + j) \leq -\rho(k\ell + j) q(k\ell + j) \\ &+ \frac{\Delta_{\ell} \rho(k\ell + j) z((k+1)\ell + j)}{\rho((k+1)\ell + j)} \\ &- \left(\frac{\beta\rho(k\ell + j)(p((k+1)\ell + j))^{(1/\gamma)-1}(p((k+1)\ell + j))^2}{(p((k+1)\ell + j))^2 (p((k - \sigma)\ell + j))^{1/\gamma}\left(u((k - \sigma + 1)\ell + j)\right)^{1-\beta}}\right.\end{aligned}$$

$$\times\frac{(\rho((k+1)\ell+j))^2(\Delta_{\alpha(\ell)}u((k+1)\ell+j))^{2\gamma}}{\left(u^{\beta}((k-\sigma+1)\ell+j)\right)^2(\Delta_{\alpha(\ell)}u((k+1)\ell+j))^{\gamma-1}}\Bigg) \quad (15)$$

From (7), we conclude that
$u(k\ell+j)\le\alpha u(k_0\ell+j)+\Delta_{\alpha(\ell)}u(k_0\ell+j)((k-k_0)\ell+j), k\ge k_0$
and consequently there exists a $k_1\ge k_0$ and appropriate constant $\eta\ge 1$ such that $u(k\ell+j)\le\eta(k\ell+j)$ for $k\ge k_1$ and this implies that $u((k-\sigma v+1v)\ell v+j)\le\eta((k-\sigma+1)\ell+j)$ for $k\ge k_2=k_1+\sigma-1$ and, hence

$$\frac{1}{(u((k-\sigma+1)\ell+j))^{1-\beta}}\ge\frac{1}{(\eta((k-\sigma+1)\ell+j))^{1-\beta}}. \quad (16)$$

Since $p(k\ell+j)(\Delta_{\alpha(\ell)}u(k\ell+j))^{\gamma}$ is a positive and increasing function, there exists a $k_2\ge k_1$ sufficiently large such that $p(k\ell+j)(\Delta_{\alpha(\ell)}u(k\ell+j))^{\gamma}\le\frac{1}{M}$ for some positive constant M and $k\ge k_2$, and hence by (5) we have $p((k+1)\ell+j)(\Delta_{\alpha(\ell)}u((k+1)\ell+j))^{\gamma}\le\frac{1}{M}$, so that

$$\frac{1}{(\Delta_{\alpha(\ell)}u((k+1)\ell+j))^{\gamma-1}}\ge(Mp((k+1)\ell+j))^{(\gamma-1)/\gamma} \quad (17)$$

Then from (8), (15), (16) and (17) we have

$$\Delta_{\alpha(\ell)}z(k\ell+j)\le-\rho(k\ell+j)q(k\ell+j)+\frac{\Delta_{\ell}\rho(k\ell+j)z((k+1)\ell+j)}{\rho((k+1)\ell+j)}-\frac{\beta\rho(k\ell+j)M^{(\gamma-1)/\gamma}(z((k+1)\ell+j))^2}{(p((k+1)\ell+j))^2(p((k-\sigma)\ell+j))^{1/\gamma}\eta^{1-\beta}\left((k-\sigma+1)\ell+j\right)^{1-\beta}} \quad (18)$$

$$\Delta_{\alpha(\ell)}z(k\ell+j)\le-\rho(k\ell+j)q(k\ell+j)+\frac{(p((k-\sigma)\ell+j))^{1/\gamma}\eta^{1-\beta}((k-\sigma+1)\ell+j)^{1-\beta}(\Delta_{\ell}\rho(k\ell+j))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell+j)}-\left[\frac{\sqrt{\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell+j)}z((k+1)\ell+j)}{\rho((k+1)\ell+j)\sqrt{\eta((k-\sigma+1)\ell+j)^{1-\beta}p((k-\sigma)\ell+j)}}-\frac{\sqrt{\eta^{1-\beta}((k-\sigma+1)\ell+j)^{1-\beta}(p((k-\sigma)\ell+j))^{1/\gamma}}\Delta_{\ell}\rho(k\ell+j)}{2\sqrt{\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell+j)}}\right]^2$$

$$<-\Big[\rho(k\ell+j)q(k\ell+j)-\frac{(p((k-\sigma)\ell+j))^{1/\gamma}\eta^{1-\beta}((k-\sigma+1)\ell+j)^{1-\beta}(\Delta_{\ell}\rho(k\ell+j))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell+j)}\Big]$$

$$\sum_{r=0}^{k}\Big[\rho(r\ell+j)q(r\ell+j)-\frac{(p((r-\sigma)\ell+j))^{1/\gamma}\eta^{1-\beta}((r-\sigma+1)\ell+j)^{1-\beta}(\Delta_{\ell}\rho(r\ell+j))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(r\ell+j)}\Big]<c_1$$

for all large k, and this is contrary to (4). The proof is complete.

Theorem 2 *Assume that (2) holds. Let* $\rho(k\ell+j)$ *be a real valued function. Furthermore, we assume that there exists a double function* $H(m,k\ell+j); m\ge k\ge 0$ *such that* (*i*) $H(m,m)=0$ *for* $m\ge 0$, (*ii*) $H(m,k\ell+j)>0$ *for* $m>k\ell+j\ge 0$,

(*iii*) $\Delta_{2\alpha(\ell)}H(m,k\ell+j)=H(m,(k+1)\ell+j)-\alpha H(m,k\ell+j)\le 0$ *for* $m\ge k\ell+j\ge 0$. *If*

$$\limsup_{m\to\infty}\frac{1}{H(m,0)}\sum_{k=k_0}^{m-1}\Big[H(m,k\ell+j)\rho(k\ell+j)q(k\ell+j)-\frac{(\rho((k+1)\ell+j))^2}{\rho(k\ell+j)}\times\left(h(m,k\ell+j)-\frac{\Delta_{\ell}\rho(k\ell+j)}{\rho((k+1)\ell+j)}\sqrt{H(m,k\ell+j)}\right)^2\Big]=\infty,$$

where $h(m,k\ell+j)=-\frac{\Delta_{2(\ell)}H(m,k\ell+j)}{\sqrt{H(m,k\ell+j)}}$,

$$\overline{\rho(k\ell+j)}=\frac{\beta\rho(k\ell+j)M^{(\gamma-1)/\gamma}}{(p((k-\sigma)\ell+j))^{1/\gamma}\eta^{1-\beta}\left((k-\sigma+1)\ell+j\right)^{1-\beta}}$$

Then every solution of equation (1) oscillates.

Proof. We proceed as in Theorem 1, we assume that equation (1) has a nonoscillatory solution, say $u((k-\sigma)\ell+j)>0$ for all $k\ge k_0$. From (18) we have for $k\ge k_2$,

$$\rho(k\ell+j)q(k\ell+j)\le-\Delta_{\alpha(\ell)}z(k\ell+j)+\frac{\Delta_{\ell}\rho(k\ell+j)z((k+1)\ell+j)}{\rho((k+1)\ell+j)}-\frac{\overline{\rho(k\ell+j)}}{(\rho((k+1)\ell+j))^2}w^2(k+\ell).$$

$$\sum_{k=n}^{m-1}H(m,k\ell+j)\rho(k\ell+j)q(k\ell+j)\le H(m,k\ell+j)z(k\ell+j)+\sum_{k=n}^{m-1}z((k+1)\ell+j)\Delta_{2(\ell)}H(m,k\ell+j)+\sum_{k=n}^{m-1}H(m,k\ell+j)\frac{\Delta_{\ell}\rho(k\ell+j)z((k+1)\ell+j)}{\rho((k+1)\ell+j)}-\sum_{k=n}^{m-1}\frac{H(m,k\ell+j)\overline{\rho(k\ell+j)}z^2((k+1)\ell+j)}{(\rho((k+1)\ell+j))^2}$$

$$=H(m,k\ell+j)z(k\ell+j)-\sum_{k=n}^{m-1}\Big[z((k+1)\ell+j)+\frac{\rho((k+1)\ell+j)}{2\sqrt{H(m,k\ell+j)\overline{\rho(k\ell+j)}}}\times\left(h(m,k\ell+j)\sqrt{H(m,k\ell+j)}-\frac{\Delta_{\ell}\rho(k\ell+j)H(m,k\ell+j)}{\rho((k+1)\ell+j)}\right)\Big]^2+\frac{1}{4}\sum_{k=n}^{m-1}\frac{(\rho((k+1)\ell+j))^2}{\overline{\rho(k\ell+j)}}\times\left(h(m,k\ell+j)-\frac{\Delta_{\ell}\rho(k\ell+j)\sqrt{H(m,k\ell+j)}}{\rho((k+1)\ell+j)}\right)^2.$$

$$\limsup_{m\to\infty}\frac{1}{H(m,0)}\sum_{k=0}^{m-1}\Big[H(m,k\ell+j)\rho(k\ell+j)q(k\ell+j)-\frac{(\rho((k+1)\ell+j))^2}{\overline{\rho(k\ell+j)}}\times\left(h(m,k\ell+j)-\frac{\Delta_{\ell}\rho(k\ell+j)\sqrt{H(m,k\ell+j)}}{\rho((k+1)\ell+j)}\right)^2\Big]<0,$$

which contradicts to our assumption. Hence the proof.

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