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Research paper

Methods for Transformation of Rectangular Spatial Coordinates to Geodetic Coordinates

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Abstract

The article gives a brief analysis of methods and algorithms for the transformation of spatial rectangular coordinates to curvilinear coordinates - geodetic latitude, geodetic longitude, geodetic height. Two algorithms for solving the equation for determining longitude are considered. Three formulas used to calculate the height are analyzed, with an estimate of their errors due to the approximate latitude. The shortcomings of mathematical solutions to these problems are revealed. A study of different approaches and methods for solving the transcendental equation for determining the latitude, based on the theory of separation of the root of the equation, is performed. Using this technique, iterative processes were performed to calculate the reduced latitude u, using trigonometric identities, by introducing an auxiliary angle and transforming it to an algebraic quartic equation, which Borkowski solves by the Ferrari's method. The determination of the root isolation interval allowed using the chord method (proportional parts) to determine the latitude. In all cases, estimates of the convergence of the iterative processes that facilitate the comparative analysis of the proposed solutions are obtained. By further decreasing the separation interval of the root, the accuracy of the non-iterative determination of the latitude is improved by the Newton method.

Keywords*: transformation of coordinates, rectangular and geodetic coordinates, latitude, longitude, algorithms, Newton method.*

1. Introduction

When solving geodetic problems in space, it is necessary to move from the rectangular coordinates X,Y,Z of the point Q to its curvilinear coordinates: geodetic latitude B, geodetic longitude L, geodetic altitude H, measured along the normal from the surface of the ellipsoid. A large number of publications are devoted to the solution of this problem [1-5].

Algorithms for such a transition are obtained by transforming a system of equations

$$\begin{cases} X = (N+H)\cdot\cos B\cdot\cos L, \\ Y=(N+H)\cdot\cos B\cdot\sin L, \\ Z = \left(N\cdot\left(1-e^2\right)+H\right)\cdot\sin B, \end{cases} \tag{1}$$

allowing for the given coordinates B,L,H of the point Q to calculate its rectangular Cartesian coordinates X,Y,Z, where $N = a\big/\sqrt{1-e^2\sin^2 B}$ is the radius of curvature of the first vertical; a - the semimajor axis of the ellipsoid; $e=\sqrt{a^2-b^2}\big/a$ - the first eccentricity of the ellipsoid; b - the semiminor axis of the ellipsoid.

The problem of the reverse transition is more difficult to solve. Let us analyze briefly the applied methods of transformations of the system (1) and the results of the research.

2. Transformation methods

In scientific articles and practically in all manuals from the system of equations (1) by dividing the second equation the first expression for the latitude tangent is obtained

$$tgL = Y/X,\ X \neq 0. \tag{2}$$

To calculate the longitude, the solution is proposed [1, 6, 7] $L = arctg(Y/X)$, which is valid only for $0^0 \le L < 90^0$. Since, the geodetic longitude L changes in the range $0^0 \le L \le 360^0$, then the solution of equation (2) is satisfied by the algorithm [8]:

1) if $X=0$; $Y=0$ then L – any, in particular, $L=0$;
2) if $X=0$; $Y\neq 0$, then $L=\begin{cases}\pi/2 & at\quad Y>0; \\ 3\pi/2 & at\quad Y<0;\end{cases}$
3) if $X \neq 0$ then $L_0 = arctg(Y/X)$ and

$$L=\begin{cases} L_0 & \text{at} \quad X>0, Y\ge 0; \\ L_0+2\pi & \text{at} \quad X>0, Y<0; \\ L_0+\pi & \text{at} \quad X<0, Y-\text{any}. \end{cases} \tag{3}$$

The algorithm (3) will be simplified if the equation (2) is transformed to the half-longitude argument [9]:

$$tg\frac{L}{2}=\frac{Y}{X+R}, \quad R=\sqrt{X^2+Y^2}. \tag{4}$$

In this case, unlike (Varao Romao, 1987; Vermeille, 2002), the longitude L is defined as follows [5]:

1) if $Y\ne 0$, then $L_0=2arctg(Y/(X+R))$, and

$$L=\begin{cases} L_0, & \text{if} \quad Y>0; \\ L_0+2\pi, & \text{if} \quad Y<0; \end{cases}$$

2) if $Y=0$, then $L=\begin{cases} 0 & \text{at} \quad X\ge 0; \\ \pi & \text{at} \quad X<0. \end{cases}$

(5)

To determine the altitude H at $Z>0$, the following dependencies [10] are recommended to use

$$H=R/\cos B-N, \text{ if } 0^o\le B\le 45^o, \tag{6}$$

$$H=Z/\sin B-N\left(1-e^2\right), \text{ if } 45^o<B\le 90^o. \tag{7}$$

These formulas, or one of them, are applied for the calculation of H in the works [6, 11-16].

The disadvantage of these solutions is that with a small latitude error ΔB, the height H from them is calculated with a large error ΔH [8, 9].

In the work [17] the altitude is calculated according to the following formula

$$H=R\cos B+T\sin B-N,$$

where $T=Z+e^2N\sin B$.

Consequently, considering the expression T,

$$H=R\cos B+Z\sin B-a\sqrt{1-e^2\sin^2 B}. \tag{8}$$

Using the methods of differential calculus, establish that the latitude B, determined with an error ΔB when calculating the altitude H with formulas (6) - (8), causes, respectively, the following errors [9]

$$\Delta_6 H=(a+H)\,\text{tg}\,B\cdot\Delta B,$$

$$\Delta_7 H=-(a+H)\,\text{ctg}\,B\cdot\Delta B,$$

$$\Delta_8 H=-\frac{1}{2}(a+H)\cdot\Delta B^2. \tag{9}$$

From the conditions for the application of equations (6), (7) it follows that the greatest errors in determining the altitude are formed at $B=45^0$. Then, $|\Delta_6 H|=|\Delta_7 H|=(a+H)\cdot|\Delta B|$, $|\Delta_8 H|=|\Delta_6 H|\cdot|\Delta B|/2$, that is, when calculating the altitude by formula (8), its error decreases by $2/|\Delta B|$ times. For example, if $\Delta B=2{,}5''$ and taking into account the following data (Bowringб 1976): $B=45^0$; $H=2a$, we get $|\Delta_8 H|=0{,}0014m$, whereas $|\Delta_6 H|=|\Delta_7 H|=230m.$

The greatest difficulties arise when determining the geodetic latitude B. Equations (6), (7) produce a transcendental equation for its calculation [11]:

$$\text{tg}\,B=\left(Z+e^2N\sin B\right)/R, \quad R\ne 0. \tag{10}$$

The solution of this equation is performed by the method of successive approximations [10, 18-21], by the methods of expansion in series [13, 22-24], differential corrections [10, 14, 25] and other approximation methods [8].

Exact algorithms for solving equation (10) are obtained by transforming it using trigonometric identities to an algebraic form [7, 26-31], etc. The drawback of these solutions by the Ferrari's method is the bulkiness and complexity of the algorithms obtained and their practical inapplicability for theoretical studies and solving related problems [8]. In addition, since the initial data are approximate, in general, there is no need to apply complex exact algorithms to calculate approximate results.

The solution of equation (10) is most often performed by the method of successive approximations. However, in the derivation of recurrence formulas, many authors do not pay enough attention to, or even do not consider the conditions of convergence of iterative processes, the rate of their approach to the root, the estimation of the accuracy of the results obtained, do not reveal the advantages and disadvantages of the proposed formulas with respect to the known ones. Sometimes analytical methods of functions study are replaced by the method of comparing the results calculated from them with their more accurate values for individual values of the arguments. The analysis at this level is superficial, and the conclusions and proposals made on its basis look unconvincing.

To carry out such studies, it is necessary to separate the root of equation (10), i.e. specify the interval as short as possible, within which it is enclosed. For this purpose, a theory and method [8, 9] were developed, on the basis of which the research was carried out. The root tgB of the equation (10) is separated by geometric and analytical methods. An interval of a very small length $\left[\frac{Z}{R};\frac{Z}{\left(R\left(1-e^2\right)\right)}\right]$ is established, within which it is enclosed at $H\ge 0$ at the latitude belt $\left[-90^0;90^0\right]$.

Since many dependencies in functions with reduced latitude u take a simpler form, then the equation (10) is reduced to the form [12]:

$$\text{tg}\,u=A+C\sin u; \quad A=\frac{Z\sqrt{1-e^2}}{R}; \quad C=\frac{ae^2}{R}. \tag{11}$$

Therefore, the equation (11) is analyzed, and the results obtained are used to construct the solutions of equation (10).

Using the dependence

$$tgu=\sqrt{1-e^2}\cdot tgB, \tag{12}$$

the boundaries of the root $\bar{t}=tg\bar{u}$ of equation (11) are set

$$T_1=\frac{Z\sqrt{1-e^2}}{R}<\text{tg}\,\bar{u}<\frac{Z}{R\sqrt{1-e^2}}=T_2 \tag{13}$$

or, it is equivalent to the function

$$f(t)=t-A-Ct\Big/\sqrt{1+t^2} \tag{14}$$

To carry out the research, approximate equalities obtained from the corresponding expressions at $e=0$ are used:

$$B\approx u;\quad N\approx a;\quad Z\approx(a+H)\sin u;\quad R\approx(a+H)\cos u, \tag{15}$$

Since $T_1=(1-e^2)T_2$, and on any terrestrial ellipsoid $e^2<0{,}01$, then in the numbers T_1 and T_2, after their rounding, the first two significant figures will be the same. This property is also applicable for all numbers enclosed between them, which can be approximately equal T_1 or T_2 at the specified accuracy. This allows us to obtain good estimates for the analysis of the iterative process, taking one of the ends of the segment as an intermediate point. Due to this property, it is possible to compare the errors of both one and different functions with respect to the points of the interval [T_1, T_2].
In accordance with (15) obtain the values

$$f(T_1)\approx-\frac{ae^2}{a+H}t_o,\qquad f(T_2)\approx\frac{He^2}{a+H}t_o, \tag{16}$$

where t_o - is any number in the segment [T_1, T_2].
It follows from (16) that $f(t)$ has different signs at finite points of the segment for $H>0$, and in this case, the root $\bar{t}$ belongs to this interval. By virtue of the constancy of the sign of the first derivative

$$f'(t)=1-C\cos^3 u\approx1-\frac{ae^2}{a+H}\cos^2 u>0$$

the function $f(t)$ is strictly monotonic, and therefore equation (14) has only one root. In this case: if $Z>0$, then $0<T_1<\bar{t}<T_2$, if $Z<0$, then $T_2<\bar{t}<T_1<0$, if $Z=0$, then $T_1=\bar{t}=T_2=0$.

The iterative process $t_k=\varphi(t_{k-1})$ by formula (11), where $\varphi(t)=A+C\sin u=A+Ct\Big/\sqrt{1+t^2}$, converges to the root $\bar{t}$ monotonically, with the speed of a geometric progression with a denominator

$$q=\varphi'(t_0)=C\cos^3 u_0\approx\frac{ae^2}{a+H}\cos^2 u_0<1.$$

The dependence between the errors $\varepsilon_k=t_k-\bar{t}$ of two successive approximations to the root $\bar{t}$ is expressed by the equality $\varepsilon_k=q\cdot\varepsilon_{k-1}$, which is reduced to the form $\varepsilon_k=q^k\cdot\varepsilon_0$, where $\varepsilon_0=t_0-\bar{t}\approx f(t_0)$. But the error of the function t_k and its argument u_k are related by the dependence

$$\Delta u_k=\varepsilon_k\cdot\cos^2 u_0, \tag{17}$$

Therefore,

$$\Delta u_k=\left(\frac{ae^2}{a+H}\right)^k(\cos u_0)^{2k+2}\cdot f(t_0). \tag{18}$$

Taking into account the expressions (16), determine the latitude errors (18) at the k-th iteration $\Delta_1 u_k$ with the initial approximation $t_0=T_1$ and $\Delta_2 u_k$ with the initial approximation $t_0=T_2$:

$$\Delta_1 u_k=-\left(\frac{ae^2}{a+H}\right)^{k+1}\sin u_o\cdot\cos^{2k+1}u_o. \tag{19}$$

$$\Delta_2 u_k=\frac{a^k e^{2k+2}\cdot H}{(a+H)^{k+1}}\sin u_o\cdot\cos^{2k+1}u_o. \tag{20}$$

The obtained characteristics (19), (20), related by the relation $\Delta_2 u_k=-\frac{H}{a}\cdot\Delta_1 u_k$, allowed determining the latitude errors $\Delta u\approx\Delta B$ at each iteration, determining its marginal errors depending on the latitude u and altitude H, performing a comparative analysis by the analytical methods of the proposed solutions of equations (10) and (11).

Equation (11) can be represented in different forms by means of trigonometric identities. Replacing $C\sin u$ with $C\cos u\cdot tgu$, we get $tgu=A+C\cos u\cdot tgu$, then

$$tgu=\frac{A}{1-C\cos u}=\varphi_1(t) \tag{21}$$

Since the derivative $\varphi_1'(t)\approx-\frac{ae^2}{a+H}\sin^2 u$ at $H>-a+ae^2$ satisfies the condition $-1<\varphi_1'(t)<0$, then two successive approximations t_{k-1} and t_k of an iterative process $t_k=\varphi_1(t_{k-1})$ will be located on opposite sides of the root $\bar{t}$ of equation (21). In this case, a simple error estimate is allowed $\varepsilon_k\le|t_k-t_{k-1}|$. The same approach is used to study the iterative process [9]:

$$\varepsilon_k=q_1^k\cdot\varepsilon_0=(-1)^k\left(\frac{ae^2}{a+H}\right)^k\sin^{2k}\cdot u_0\cdot f(t_0).$$

Therefore, considering (16) and (17) we get

$$\Delta_3 u_k=(-1)^{k+1}\left(\frac{ae^2}{a+H}\right)^{k+1}\cdot\sin^{2k+1}u_o\cos u_o,\ \text{ if }\ t_o=T_1, \tag{22}$$

$$\Delta_4 u_k=(-1)^k\frac{a^k\cdot e^{2k+2}\cdot H}{(a+H)^{k+1}}\cdot\sin^{2k}u_o\cos u_o,\ \text{ if }\ t_o=T_2. \tag{23}$$

By the methods of differential calculus, the complete characteristics of the iterative process are defined by expressions (22), (23) [8, 9] and comparative analysis is performed with other solutions.
By introducing an auxiliary angle $\varphi=arctgA$, the equation (11) reduces to the form [8]:

$$\sin(u-\varphi)=p\sin 2u,\ \text{where}\ p=C\Big/\left(2\sqrt{1+A^2}\right). \tag{24}$$

Calculation of latitude is performed by the iterative method
$u_k=\varphi+\arcsin(p\sin 2u_{k-1})$, $u_0=\varphi$,
with the following accuracy estimate

$$\Delta u_k = -\frac{1}{2}\left(\frac{ae^2}{a+H}\right)^{k+1} \cos^k 2u_0 \sin 2u_0. \quad (25)$$

This method of calculating geodetic latitude is recommended in the State Standards of Russia.

The initial approximation $u_0 = \varphi$, whose error is estimated by equations (19) or (25) at $k = 0$, reaches its maximum value $\Delta u_0 = \frac{1}{2}e^2 \cdot \rho'' = 690'' \approx 11'$ at $u = 45^0$ and $H = 0$. The maximum errors Δu_k in the iterations $k = 1,2,3...$, according to (25), are $\Delta u_1 = 2{,}8''; \Delta u_2 = 0{,}012''; \Delta u_3 = 0{,}000067''; \Delta u_4 = 0{,}0000004''$.

Comparing the error expressions (24), (19), (22), see that in calculating the latitude by the formula (24), the iteration process converges with the same rate as in the formulas (11), (21) at $t_0 = T_1$.

Since $\Delta u \approx \Delta B$ then all estimates of the reliability of the results for the given latitude will also be valid for geodetic latitude.

By means of the identity $C \sin u = C \sin u\left(\sin^2 u + \cos^2 u\right)$ the equation (11) is represented in the form $\mathrm{tg}\, u = A + C \sin^3 u + C\, \mathrm{tg}\, u \cos^3 u$, then

$$\mathrm{tg}\, u = \frac{A + C \sin^3 u}{1 - C \cos^3 u}, \quad (26)$$

or when considering the coefficients (11)

$$\mathrm{tg}\, u = \frac{Z\sqrt{1-e^2} + ae^2 \sin^3 u}{R - ae^2 \cos^3 u}. \quad (27)$$

It is proved (Medvedev, 1994) that the definition of the root of equation (26) by the method of successive approximations $t_k = \varphi_2(t_{k-1})$ is equivalent to applying the Newton method for solving equation (11), that is

$\varphi_2(t) = t - f(t)/f'(t)$.

Geometrically this means that t_k is an abscissa of the point of intersection with the tangent axis t drawn to the graph of the function $f(t)$ at $t = t_{k-1}$. Therefore, this method of solution is also called the method of tangents. The iterative process will converge monotonically to the root $\bar{t}$ from the end of the segment $t_0 = T_2$ at a rate

$$q_2 = \varphi_2''(t_0) \approx f''(t_0) = \frac{3}{2} \cdot \frac{ae^2}{a+H} \sin u_0 \cos^3 u_0 < 1,$$

the error ε_k of each subsequent approximation decreases in proportion to the square of the error ε_{k-1} of the previous approximation, ensuring quadratic convergence [9]:

$$\varepsilon_k = q_2 \cdot \varepsilon_{k-1}^2 = \frac{3}{2} \cdot \frac{ae^2}{a+H} \sin u_0 \cos^3 u_0 \cdot \varepsilon_{k-1}^2.$$

This equality expressed through $\varepsilon_0 \approx f(t_0)$, takes the form $\varepsilon_k = q_2^{2^k - 1} \cdot \varepsilon_0^{2^k}$ and in accordance with (17)

$$\Delta u_k = \left[f(t_o)\right]^{2^k} \cdot \left(\frac{3}{2}\frac{ae^2}{(a+H)} \sin u_o \cos^3 u_o\right)^{2^k - 1} \cdot \cos^2 u_o. \quad (28)$$

Assuming $t_o = T_1$ and considering (16), from (28) we have

$$\Delta_1 u_k = \left(\frac{3}{2}\right)^{2^k - 1} \cdot \left(\frac{ae^2}{a+H}\right)^{2^{k+1}-1} \cdot \sin^{2^{k+1}-1} u_o \cos^{2^{k+1}-1} u_o. \quad (29)$$

The function $\mathrm{y}_1(u) = (\sin u \cos u)^{2^{k+1}-1}$ on the interval $\left[0^o, 90^o\right]$ takes on the greatest value when $u = 45^o$ and $\max\, \mathrm{y}_1 = 1/2^{2^{k+1}-1}$, while $\max\left(\frac{ae^2}{a+H}\right) = e^2$ at $H = 0$.

Therefore,

$$\max \Delta_1 u_k = 3^{2^k - 1} \cdot e^{2^{k+2}-2} / 2^{3 \cdot 2^k - 2}.$$

Similarly, if $t_o = T_2$, then

$$\Delta_2 u_k = \left(\frac{3}{2}\right)^{2^k - 1} a^{2^k - 1} e^{2^{k+2}-2} H^{2^k} (\sin u_o \cos u_o)^{2^{k+1}-1} / (a+H)^{2^{k+1}-1}$$

(30)

The function $\mathrm{y}_2(H) = H^{2^k} / (a+H)^{2^{k+1}-1}$ has a maximum at $H = a \cdot 2^k / \left(2^k - 1\right)$. Then

$$\max \Delta_2 u_k = \left(\frac{3}{2}\right)^{2^k - 1} \cdot \frac{\left(2^k\right)^{2^k} \cdot \left(2^k - 1\right)^{2^k - 1} \cdot e^{2^{k+2}-2}}{2^{2^{k+1}-1} \cdot \left(2^{k+1} - 1\right)^{2^{k+1}-1}}.$$

From (29) and (30) obtain the dependence $\Delta_2 u_k = (H/a)^{2^k} \cdot \Delta_1 u_k$. It follows that when constructing an algorithm for determining the root from (26) by the method of repeated approximations, it is necessary to apply both initial approximations T_1 and T_2, since at $H > a$ we have $\Delta_2 u_k > \Delta_5 u_k$, and at $0 < H \le a$ on the contrary – $\Delta_2 u_k \le \Delta_1 u_k$.

The formula (27) by substitution (12) is transformed into the form

$$\mathrm{tg}\, B = \frac{Z + be'^2 \sin^3 u}{R - ae^2 \cos^3 u}, \quad (31)$$

which was derived by Bowring (1976) for a special case when $tgu = T_2$ by the expansion of the function $f(t)$ (14). He indicated the reduced latitude u by the symbol θ. The error of formula (31) considering $\Delta u \approx \Delta B$ is obtained from (30) at $k = 1$

$$\Delta B = \frac{3}{2}\frac{aH^2 e^6}{(a+H)^3}(\sin u \cos u)^3. \quad (32)$$

Its greatest value $\Delta B'' = e^6 \rho''/36 = 0{,}0017''$ is achieved at $u = 45^o$ and $H = 2a$.

If the altitude *H* is estimated with the calculated value of *B* from formulas (6-8), then the expression (32) must be substituted into equalities (9) to determine the errors ΔH and carry out the studies [8].

In [24], to calculate the latitude, it is suggested not the formula (11), but the following formula

$$t=\varphi_3(t)=A+\frac{AC}{\sqrt{1+t^2}}+\frac{tC^2}{1+t^2}, \tag{33}$$

with initial approximation

$$t_0=A\left(1+ae^2\right)\Big/\sqrt{R^2+\left(1-e^2\right)Z^2}. \tag{34}$$

The equality (33) is obtained from (11): $t=\varphi(t)=A+Ct\big/\sqrt{1+t^2}$ if the denominator $\sqrt{1+t^2}$ is left unchanged in it, and the quantity t in the numerator is replaced by the expression $\varphi(t)$ from (11).

To confirm the effectiveness of this technique, derive a derivative, which in the notation adopted here has the form

$$\varphi_3'(t)=C\frac{1-t^2}{\left(1+t^2\right)^2}-\frac{ACt}{\left(\sqrt{1+t^2}\right)^3}\approx\left(\frac{ae^2}{a+H}\right)^2\cos 2u-\frac{ae^2}{a+H}\sin^2 u. \tag{35}$$

And then mistakenly the greatest value $\varphi_3'(t)$ is estimated by the absolute value of the first term $\left(\frac{ae^2}{a+H}\right)^2\le e^4$, and not the second one $\frac{ae^2}{a+H}\le e^2$. As a result, the conclusion about the rate of convergence of the iterative process turns out to be wrong.

Vincenty [25] analyzes the formulas (27) and (33). To this end, he compiles a table of errors of the given latitude in seconds, which arise with initial approximations t_0 to the root at $t_0=T_2$ (13) and (34). He solves this problem by calculating these formulas at latitudes $u=30^0;45^0;60^0$ and altitudes $H=-10;0;10;1000;12800km$, with further comparing of the results with their more accurate values. Then he sets the necessary number of iterations, after which the error $|\Delta H|=1mm$.

Such a research path does not allow establishing either the rate of convergence of the iterative processes or their maximum errors. More complete and accurate information is obtained by analytical methods of research, which lead to the goal much easier and faster. Thus, when calculating by formula (33), the latitude error of the initial approximation (34), which is the result of the first iteration by formula (11), is determined from equation (19) at $k=1$:

$$\Delta u_0=-\left(\frac{ae^2}{a+H}\right)^2\sin u_0\cdot\cos^3 u_0, \tag{36}$$

and the iteration process in accordance with (35) and $\Delta u_k=q^k\cdot\Delta u_0$:

$$\Delta u_k=(-1)^{k+1}\left(\frac{ae^2}{a+H}\right)^{k+2}\left(\sin u_0\right)^{2k+1}\cdot\cos^3 u_0. \tag{37}$$

When calculating the latitude by formula (27), the error of the initial approximation is derived from equality (32). For subsequent iterations, the values Δu_k are determined using the relation (30).

The Vincenty table is obtained from (32) and (36), and the analysis of the iteration processes of the solutions of equations (27) and (33) is performed according to the regularities (30), (37).

Equation (11) is reduced by a substitution $t=tg\left(\frac{\pi}{4}-\frac{u}{2}\right)$ to algebraic form

$$t^4+2(A-C)t^3+2(A+C)t-1=0, \tag{38}$$

which K. Borkowski [32] solves by the Ferrari's method. To solve equation (38) in [33] Newton's method is applied and comparison with algorithms from [15, 32, 34] is carried out. However, general estimates in the form (28-30), which facilitate the conduct of a full analysis, are not given.

The definition of the root $\bar{t}$ of equation (11), i.e. the establishment of the interval $\left[T_1,T_2\right]$ within which it was enclosed at $H\ge 0$ allowed using the chord method (the method of proportional parts) to calculate the latitude.

In the chord method, the end of the segment in which the signs of $f(t)$ and $f''(t)$ are opposite is considered as the initial approach to the root. Since $f''(t)=3Ct\cos^5 u\approx\frac{3ae^2}{a+H}t\cdot\cos^4 u$ then such a point is $t_o=T_1$. Since the successive approximations to the root $\bar{t}$ occur starting from the end of T_1 and the value of T_2 does not change, then the recurrence formula has the following form [8]:

$$t_k=t_{k-1}-\frac{f\left(t_{k-1}\right)\cdot\left(t_{k-1}-T_2\right)}{f\left(t_{k-1}\right)-f\left(T_2\right)},\qquad t_o=T_1. \tag{39}$$

From the geometric point of view, in the chord method, the arc of a curve $f(t)$ on an interval $\left[t_{k-1},T_2\right]$ is replaced by a contracting chord. Set the abscissa t_k of the point of intersection of the chord with the axis t as an approximate value of the root.

The latitude Δu_k error is determined by the equality

$$\Delta u_k=(-1)^{k+1}\cdot\left(\frac{3}{2}\right)^k\cdot\frac{a^{k+1}\cdot e^{4k+2}\cdot H^k}{(a+H)^{2k+1}}\left(\sin u_o\cdot\cos u_o\right)^{2k+1}. \tag{40}$$

The investigations (40) on the extremum with respect to variables u and H derive

$$\max\left|\Delta u_k''\right|=\frac{1}{2}\left(\frac{3}{8}\right)^k\frac{k^k\cdot e^{4k+2}\cdot(k+1)^{k+1}}{(2k+1)^{2k+1}}\cdot\rho''.$$

Therefore,

at $k=1$ $\quad\max\left|\Delta u_1''\right|=\rho''\cdot e^6/36=0{,}0017''$,

at $k=2$ $\quad\max\left|\Delta u_2''\right|=\rho''\cdot(0{,}3)^5\cdot e^{10}\approx 7''\cdot 10^{-9}$.

Thus, the equation (10) derived from the system (1), with the help of trigonometric identities and substitutions, is transformed into other equations (11), (21), (24), (26), (38) equivalent to it with respect to the root $\bar{t}$. Since as a result of the transformations their analytical expression changes, then the iterative processes based on them differ both in the rate of convergence and in the types of approximations to the root. In simple iterative methods, approaches to the root converge linearly with the rate $q \le e^2$. This is realized in algorithms with one-sided convergence according to formulas (10), (11), in iterative processes with two-sided convergence - according to formulas (21), (33), (24). If we set $\varepsilon_{k-1} \le 10^{-m}$ for definiteness in the dependence $\varepsilon_k = q \cdot \varepsilon_{k-1}$, then considering $q \le e^2 \approx 7 \cdot 10^{-3}$ we obtain $\varepsilon_k = 7 \cdot 10^{-(m+3)} \approx 10^{-(m+2)}$.

From here on it follows that in the simple iteration method each new approximation provides an increase in two valid decimal signs. In comparison, note that when calculating the root by the tangent method (26), the error ε_k in the *k-th* approximation decreases in proportion to the square of the error ε_{k-1} of the previous approximation, ensuring quadratic convergence. In this case, for the same ε_{k-1} and $\max q_3 = \frac{9\sqrt{3}e^2}{32} \approx 3 \cdot 10^{-3}$ the error $\varepsilon_k \le 3 \cdot 10^{-(2m+3)}$.

Consequently, each new approximation increases the number of valid decimal signs by $m+3$, i.e. more than twice. In the chord method (39), the rate of convergence is linear, where $q_x = -\frac{3}{8}\frac{ae^4H}{(a+H)^2}\sin^2 2u_0$, therefore at $H = a$ $\max|q_x| = \frac{3e^4}{32} \approx 7 \cdot 10^{-5} < 10^{-4}$, for. In accordance with this $\varepsilon_k = |q_x| \cdot \varepsilon_{k-1} < 10^{-(m+4)}$ and in each iteration, the increment will be four decimal signs. Thus, the convergence of the approximation process in the chord method is twice as fast as in the simple iteration method, but less than in the tangent method approximately by $m-1$ sign.

The general expressions for the latitude calculation errors (19), (20), (22), (23), (25), (29), (30), (37), (40) derived by the iteration method using formulas (11), (21), (24), (26), (33), (39) allow studying the convergence of iterative processes, establishing latitude accuracy by iterations, determining its greatest errors, and highlighting its greatest errors, and also performing a comparative analysis of the proposed solutions and establishing a connection between them.

For this purpose, the latitude error (40) in the chord method will be denoted by $\Delta_x u_k$, and the errors (29), (30) in the tangent method by $\bar{\Delta}_1 u_k$ and $\bar{\Delta}_2 u_k$.

Equations (40) and (30) produce the expression for the connection of latitude errors determined by the chord formula (39) and the tangent formula (26) for $t_0 = T_2$:

$$\frac{\bar{\Delta}_2 u_k}{\Delta_x u_k} = (-1)^{k+1}\left(\frac{3}{2}\right)^m a^{m-1}e^{4m}H^{m+1}(\sin u_0 \cos u_0)^{2m}\big/(a+H)^{2m}, \tag{41}$$

where $m = 2^k - k - 1$.

The results of the research (41) are not given in this paper. By substitution of $\bar{\Delta}_2 u_k = (H/a)^2 \cdot \bar{\Delta}_1 u_k$ in (41) determine the dependence between the errors $\bar{\Delta}_1 u_k$ and $\Delta_x u_k$.

Similarly, by dividing (29) by (19), find

$$\frac{\bar{\Delta}_1 u_k}{\Delta_1 u_k} = -\left(\frac{3}{2}\right)^{m+k}\left(\frac{ae^2}{a+H}\right)^{2m+k}(\sin u_0)^{2(m+k)}(\cos u_0)^{2m};\ m = 2^k - k - 1.$$

Similarly, establish

$$\frac{\Delta_x u_k}{\Delta_1 u_k} = (-1)^k\left(\frac{3}{2}\right)^k \frac{e^{2k} \cdot H^k}{(a+H)^k}(\sin u_0)^{2k}$$

and from the ratio $\Delta_2 u_k = -\frac{H}{a} \cdot \Delta_1 u_k$ obtain the equations of the connection between the errors $\bar{\Delta}_1 u_k$ and $\Delta_2 u_k$; $\Delta_x u_k$ and $\Delta_2 u_k$.

The same way, find the dependencies between the errors of other formulas.

It should be noted that the accuracy with which latitude is determined after the second iteration by formula (26) is achieved only in the sixth to seventh iterations in the calculations by formulas (11), (21) with linear convergence. Therefore, based on equation (26) with quadratic convergence, it is possible to construct a non-iterative high-precision algorithm for computing latitude.

For this purpose, the isolation interval $[T_1, T_2]$ of the root of equation (11) decreases [35] to $[T_3, T_4]$. Assume $Z > 0$ and $t_0 = T_1$ after the first iteration using (11) and (21) obtain $T_3 = A + C\sin u_1$; $T_4 = A/(1 - C\cos u)$ or taking into account the expressions A and C from (11)

$$T_3 = \frac{Z\sqrt{1-e^2}}{R}\left(1 + \frac{ae^2}{b_0}\right); T_4 = \frac{Z\sqrt{1-e^2}}{R(1 - ae^2/b_0)};$$
$$b_0 = \sqrt{R^2 + Z^2(1-e^2)}. \tag{42}$$

The value T_3 in [24] is set as the initial approximation (34).

Using the dependencies (15) by formula (11), determine the values of the function $f(t)$ at the endpoints of the interval $[T_3, T_4]$.

$$f(T_3) \approx -\left(\frac{ae^2}{a+H}\right)^2 \cdot t_o \cos^2 u_o,\quad f(T_4) \approx \left(\frac{ae^2}{a+H}\right)^2 \cdot t_o \sin^2 u_o. \tag{43}$$

It follows from equality (43) that the function $f(t)$ takes on values with opposite signs not only at $H > 0$ but also in the entire region of convergence of the iterative process [9]: $-a + ae^2 < H < +\infty$.

When calculating the latitude from (26), use dependence (28) at $k = 1$ for estimating its error.

$$\Delta u = \frac{3}{2}[f(t_o)]^2 \cdot \frac{ae^2}{a+H}\sin u_o \cos^5 u_o. \tag{44}$$

Taking the values T_3 and T_4 (42) for t_o, the expression (44) considering (43) is transformed accordingly to the forms

$$\Delta_3 u = \frac{3}{2} \cdot \left(\frac{ae^2}{a+H}\right)^5 \sin^3 u_o \cos^7 u_o; \tag{45}$$

$$\Delta_4 u = \frac{3}{2} \cdot \left(\frac{ae^2}{a+H}\right)^5 \sin^7 u_o \cos^3 u_o. \tag{46}$$

The odd function $y_1(u) = \sin^3 u \cos^7 u$ takes on the greatest value $y = 1029\sqrt{21}/10^5 \approx 0{,}047$ at latitude $u = arctg\sqrt{7/3} \approx 57^0$. Since

$y_2(u)=\sin^7 u\cos^3 u = y_1(90^0-u)$ then $y_2(u)$ takes on this highest value at $u\approx 90^0-57^0=33^0$, therefore $\max|\Delta_3 u|=\max|\Delta_4 u|\approx\left(\frac{ae^2}{a+H}\right)^5\cdot 0{,}0705$. The function $y_3(H)=\frac{1}{a+H}$ in interval $\left(-a+ae^2;+\infty\right)$ decreases monotonically and at $H\to+\infty$ $\Delta u\to 0$. At $H=-a+ae^2$, when the point is 43 km from the center of the ellipsoid, $y_3(H)=\frac{1}{ae^2}$ and $\max|\Delta u|\approx 4^0$. But already at $H\geq -4500km$ $|\Delta u|\leq 0{,}00005''$.

For points of the ground surface at $|H|\leq 10km$ $\max|\Delta u''|=\rho''\cdot e^{10}\cdot 0{,}0705\approx 2''\cdot 10^{-7}$. In this case, when calculating the altitude H by formula (8), its error (9) equals $|\Delta H|=\frac{1}{2}(a+H)\Delta u^2=ae^2\cdot 0{,}0025м\approx 0{,}0002mm$. Therefore, high-precision results are obtained.

The root isolation interval $[T_1,T_2]$ can be reduced by the same formulas (11) and (21), but with the help of the initial approximation $t_0=T_2$ [5]. In work [5], based on the expressions (23), (8), non-iterative high-precision algorithms were obtained, expressed only through the initial data X,Y,Z.

3. Conclusions

Based on the results of the above-mentioned analysis of the methods for converting spatial rectangular coordinates X,Y,Z to geodetic B,L,H, it is proposed:

to use the algorithm (5) to calculate the geodetic longitude L;

to determine the geodetic latitude $B=arctg\left(tgu/\sqrt{1-e^2}\right)$ to carry out according to formula (26) with the initial approximation $t_0=T_3$ or $t_0=T_4$ (42);

to find the geodetic altitude H by formula (8).

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