International Journal of Engineering & Technology, 7 (4.10) (2018) 1015-1017

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

A Note on Boolean Like Algebras

K.Pushpalatha[1*] V.M.L.Hima Bindu[2]

[1]*Department of mathematics, KLEF,Vaddeswaram*
[2]*Department of mathematics, KLEF,Vaddeswaram*
**Corresponding author E mail: kpushpamphil@gmail.com*

Abstract

In this paper we develop on abstract system: viz Boolean-like algebra and prove that every Boolean algebra is a Boolean-like algebra. A necessary and sufficient condition for a Boolean-like algebra to be a Boolean algebra has been obtained. As in the case of Boolean ring and Boolean algebra, it is established that under suitable binary operations the Boolean-like ring and Boolean-like algebra are equivalent abstract structures.

Keywords: *Boolean algebra; Boolean like algebra; Boolean like ring; Boolean ring;*

1. Preliminaries

Following A.L.Foster's, the concept of Boolean-like ring is as follows:

Definition 1.1: A Boolean-like ring B is a commutative ring with unity which satisfies the following conditions.

(1) $a + a = 0$, and

(2) $a(1+a)\,b(1+b) = 0$ for all $a,b \in B$.

we give some examples of Boolean-like rings.

Example 1.2: Every Boolean ring is a Boolean - like ring.

Proof: If B is a Boolean ring, then for all $a \in B$,

$(a+a)^2 = a + a$, whence $a^2 + a^2 + a^2 + a^2 = a + a$

and so $a + a + a + a = a + a$. Thus $a + a = 0$.

Further $a(1+a) = a^2 + a = a + a = 0$.

Hence $a(1+a)b(1+b) = 0$, for all $a,b \in B$.

By a remark, B is a commutative ring with unity.

Thus B is a Boolean-like ring.

But the converse need not be true (For this refer example 1.4).

Example 1.3 : Let R be a ring with unity and characteristic 2. Let B be the set of all central idempotent of R . Then B is a Boolean subring of R. Further $B \times R$ is a Boolean - like ring with addition and multiplication defined as follows:

$(b_1, r_1) + (b_2, r_2) = (b_1 + b_2, r_1 + r_2)$

$(b_1, r_1).(b_2, r_2) = (b_1b_2, b_1r_2 + b_2r_1)$

for all $b_1, b_2 \in B$ and $r_1, r_2 \in R$

Proof: we first prove that B is a Boolean subring of R .

Let $b_1, b_2 \in B$.

We show that $b_1 - b_2 \in B$ and $b_1 b_2 \in B$

$(b_1 - b_2)^2 = b_1 - b_2$ (Since R has characteristic 2)

For $a \in R$, $(b_1 - b_2)a = b_1a - b_2a$

$= ab_1 - ab_2 = a(b_1 - b_2)$

Hence $b_1 - b_2 \in B$.

Also, $(b_1 b_2)^2 = (b_1 b_2)(b_1 b_2) = b_1(b_2b_1)b_2 = b_1(b_1 b_2)b_2 = b_1^2 b_2^2 = b_1 b_2$

Further $(b_1b_2)a = b_1(b_2 a) = b_1(ab_2) = (b_1 a)b_2 = a(b_1b_2)$.

Hence $b_1, b_2 \in B$.

Trivially $1 \in B$ and $e^2 = e$ for all $e \in B$

Therefore B is a Boolean subring of R.

We now verify that $B \times R$ is a Boolean –like ring.

For $b_1, b_2, b_3 \in B$ and $r_1, r_2, r_3 \in R$,

$[(b_1,r_1) + (b_2,r_2)] + (b_3, r_3)$

$= (b_1, r_1) + [(b_2, r_2) + (b_3, r_3)]$

Hence '+' is associative.

Now $(0, 0) \in B \times R$ and $(b_1, r_1) + (0, 0)$

$= (b_1+0, r_1+0) = (b_1, r_1)$

Therefore (0,0) is additive identity of $B \times R$.

For $(b_1, r_1) \in B \times R$

There exists $(-b_1, -r_1) \in B \times R$ such that

$(b_1, r_1) + (-b_1, -r_1) = (b_1 - b_1, r_1 - r_1) = (0, 0)$

Hence $(-b_1, -r_1)$ is the additive inverse of (b_1, r_1)

$(b_1, r_1) + (b_1, r_2) = (b_2, r_2) + (b_1, r_1)$

Therefore '+' is commutative.

Thus $(B \times R, +)$ is an abelian group.

Now $[(b_1, r_1).(b_2, r_2)].(b_3, r_3)$

$= (b_1b_2, b_1r_2 + b_2 r_1).(b_3, r_3)$

$= (b_1b_2b_3, b_1b_2r_3 + b_3(b_1r_2 + b_2r_1)$

$= (b_1,r_1)[(b_2, r_2)(b_3,r_3)]$

Hence '.' is associative.

Also $(1, 0) \in B \times R$ and $(b_1, r_1).(1,0) = (b_1, r_1)$

Further $(b_1, r_1).(b_2, r_2) = (b_1b_2, b_1r_2+b_2r_1) = (b_2b_1, b_2r_1+b_1r_2)$

$= (b_2, r_2).(b_1, r_1)$

To prove the distributive law ,

Consider $(b_1, r_1)[(b_2, r_2) + (b_3, r_3)]$

$= (b_1, r_1)[b_2 + b_3, r_2 + r_3]$

$= [b_1(b_2+b_3, b_1(r_2+r_3)+(b_2 + b_3)r_1]$

Furthermore, $(b_1, r_1)(b_2, r_2) + (b_1, r_1)(b_3, r_3) =$

$(b_1b_2, b_1r_2 + b_2r_1) + (b_1b_3, b_1r_3 + b_3r_1)$

$= (b_1b_2+b_1b_3, b_1r_2+b_2r_1+b_1r_3+b_3r_1)$

Therefore $(B \times R, +, \cdot)$ is a commutative ring with unity.

Suppose $(b_1, r_1) \in B \times R$.

Since R is a ring of characteristic 2,

$(b_1, r_1) + (b_1, r_1) = (b_1 + b_1, r_1+r_1) = (0, 0)$

Also, $(b_1, r_1)[(1,0) + (b_1, r_1)](b_2, r_2)[(1,0) + (b_2, r_2)]$

Copyright © 2018 Authors. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

$= (0, r_1)(0, r_2) = (0, 0)$
Hence $B \times R$ is a Boolean – like ring.
As a particular case of example 1.3, we have the following

Example 1.4 : Let $Z_2 = \{0,1\}$ be the ring of integers modulo 2. Then Z_2 is a commutative ring with unity and its characteristic is 2. Obviously Z_2 is a Boolean ring. Hence $Z_2 \times Z_2$ is a Boolean-like ring under the operations of addition and multiplication defined as in example 1.3 above. This Boolean-like ring is denoted by H_4.
Write 0 = (0,0), 1 = (1,0), p = (0,1) and q = (1,1).
Thus $H_4 = \{0, 1, p, q\}$ and addition and multiplication tables of H_4 are as follows

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

·	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	0	p
q	0	q	p	1

Obviously H_4 is not a Boolean ring.

Theorem 1.5: Each element 'a' of a Boolean–like ring B satisfies $a^4 = a^2$.
Proof: We have that $a(1+a)b(1+b) = 0$ -------- (i)
By taking $a = b$ in (i), we get that $a(1+a)a(1+a) = 0$
$\Rightarrow a^4 = a^2$, since the characteristic of B is 2.

2. Boolean Like Algebras

We now give the following definition:

Definition 2.1: An algebraic structure
$(A, \wedge, \vee, ^1, 0, 1)$ where $\wedge$ and $\vee$ are binary operations, ^{1} is an unary operation and
0 and 1 are elements of A, is called a Boolean-like algebra if the following conditions are satisfied
(1) $\wedge, \vee$ are associative and commutative
(2) $(a \vee b)^1 = a^1 \wedge b^1$; $(a^1)^1 = a$; $0^1 = 1$
(3) $a \wedge 0 = 0$; $a \wedge 1 = a$
(4) $b \wedge c = 0 \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(5) $a \wedge a^1 \wedge b \wedge b^1 = 0$
(6) $(a \wedge a^1) \vee (a \wedge a^1) = 0$
(7) $(a^1 \vee b) \wedge (a \vee b^1) = (a \wedge b) \vee (a^1 \wedge b^1)$
(8) $[(a \wedge b) \wedge (a \wedge c)^1] \vee [(a \wedge c) \wedge (a \wedge b)^1] = [a \wedge b \wedge c^1] \vee [a \wedge b^1 \wedge c]$, for all $a, b, c \in A$
The following result gives the most important elementary properties of elements in a Boolean-like algebra.

Lemma 2.2: In any Boolean–like algebra A, we have the following
(i) $a \vee 0 = a$ (ii) $0 = 1^1$
(iii) $a \vee 1 = 1$ (iv) $(a \wedge b)^1 = a^1 \vee b^1$
(v) $(a \vee a) \wedge (a^1 \vee a^1) = 0$ (vi) $(a \wedge a) \vee (a^1 \wedge a^1) = 1$
(vii) $(a \vee a^1) \wedge (a \vee a^1) = 1$

Proof: (i) By (2) and (3) of definition 2.1
$(a \vee 0)^1 = a^1 \wedge 0^1 = a^1 \wedge 1 = a^1$. Hence $a \vee 0 = [(a \vee 0)^1]^1 = (a^1)^1 = a$
By (2) of definition 2.1, we have that
(ii) $0 = (0^1)^1 = 1^1$
(iii) $a \vee 1 = [(a \vee 1)^1]^1 = (a^1 \wedge 0)^1 = 0^1 = 1$
(iv) $(a \wedge b)^1 = ((a^1 \vee b^1)^1)^1 = a^1 \vee b^1$
(v) By taking $b = a^1$ in (7), we get that
$(a^1 \vee a^1) \wedge (a \vee a) = (a \wedge a^1) \vee (a^1 \wedge a) = 0$
[vi] By (v) we have that $(a \vee a) \wedge (a^1 \vee a^1) = 0$
Therefore $1 = 0^1 = [(a \vee a) \wedge (a^1 \vee a^1)]^1$
$= (a \vee a)^1 \vee (a^1 \vee a^1)^1 = (a^1 \wedge a^1) \vee (a \wedge a)$
(vii) $(a \vee a^1) \wedge (a \vee a^1) = (a \wedge a) \vee (a^1 \wedge a^1) = 1$, follows from (7) and (vi).

Remark 2.3: Every complemented distributive lattice is a Boolean like algebra.

Proof: Let $(L, \wedge, \vee, ^1, 0, 1)$ be a complemented distributive lattice. By the definition of a complemented distributive lattice the conditions (1)to (6) of a Boolean- like algebra are satisfied.
(7) $(a^1 \vee b) \wedge (a \vee b^1)$
$= [(a^1 \vee b) \wedge a] \vee [(a^1 \vee b) \wedge b^1]$
$= 0 \vee (a \wedge b) \vee (a^1 \wedge b^1) \vee 0$
$= (a \wedge b) \vee (a^1 \wedge b^1)$
(8) $[(a \wedge b) \wedge (a \wedge c)^1] \vee [(a \wedge c) \wedge (a \wedge b)^1]$
$= [(a \wedge b) \wedge (a^1 \vee c^1)] \vee [(a \wedge c) \wedge (a^1 \vee b^1)]$
$= (a \wedge b \wedge a^1) \vee (a \wedge b \wedge c^1) \vee (a \wedge c \wedge a^1) \vee (a \wedge c \wedge b^1)$
$= (a \wedge b \wedge c^1) \vee (a \wedge c \wedge b^1)$.
Therefore L is a Boolean-like algebra.
By a theorem and remark 2.3, we get that every Boolean algebra is a Boolean- like algebra.

Theorem 2.4: A Boolean-like algebra $(A, \wedge, \vee, ^1, 0, 1)$ is a Boolean algebra if and only if $a \wedge a = a$ for all $a \in A$.

Proof: Suppose $a \wedge a = a$ for all $a \in A$. Then $(A, \wedge)$ is a semi-lattice, By (5) of definition 2.1 $x \wedge x^1 = 0$, for all $x \in A$. Also, By (iv) of lemma 2.2,
$1 = 0^1 = (x \wedge x^1)^1 = x^1 \vee x$. If $a \wedge b^1 = 0$, for some $a, b \in A$,
Then $a = a \wedge 1 = a \wedge (b \vee b^1) = (a \wedge b) \vee (a \wedge b^1) = a \wedge b$,
by (4) of def 2.1.
Conversely, if $a \wedge b = a$, then $a \wedge b^1 = a \wedge b \wedge b^1 = a \wedge 0 = 0$. Thus, $(A, \wedge, ^1, 0)$ is a Boolean algebra. Conversely, if $(A, \wedge, ^1, 0)$ is a Boolean algebra, then
$a \wedge a = a$ for all $a \in A$, follows from the fact that $(A, \wedge)$ is a semi-lattice.

Corollary 2.5: A Boolean-like algebra is a complemented distributive lattice $\Leftrightarrow a \wedge a = a$, for all a.

Proof: Let B be a Boolean-like algebra. If B is a complemented distributive lattice, then evidently,
$a \wedge a = a$ for all $a \in B$.
Conversely suppose that $a \wedge a = a$ for all $a \in B$.
By the above theorem B is a Boolean algebra. Then, by the theorem [1], B is a complemented distributive lattice.
We now prove that every Boolean-like algebra is a Boolean-like ring under some binary operations.

Theorem 2.6: Let $(A, \wedge, \vee, ^1, 0, 1)$ be a Boolean-like algebra. Define binary operations $+, \cdot$ by $a + b = (a \wedge b^1) \vee (a^1 \wedge b)$; $a.b = a \wedge b$ for all $a, b \in A$. Then $(A, +, \cdot, 0, 1)$ is a Boolean-like ring.

Proof: In order to prove that $(A, +, \cdot, 0, 1)$is a Boolean-like ring,
We have to prove that
1) $(A, +)$ is an abelian group with identity 0
2) $(A, \cdot)$ is a commutative semi group with identity 1.
3) Distributive law $a(b + c) = ab + ac$
for all $a, b, c \in A$
4) $a + a = 0$ for all $a \in A$, and
5) $a(1+a)b(1+b) = 0$ for all $a, b \in A$

Now, a+b = $(a \wedge b^1) \vee (a^1 \wedge b) = (b \wedge a^1) \vee (b^1 \wedge a) = b + a$
Therefore '+' is commutative.
$(a + b) + c =$
$[[(a \wedge b^1) \vee (a^1 \wedge b)] \wedge c^1] \vee [[(a \wedge b^1) \vee (a^1 \wedge b)]^1 \wedge c$
$= (a \wedge b^1 \wedge c^1) \vee (a^1 \wedge b \wedge c^1) \vee (a \wedge b \wedge c) \vee (a^1 \wedge b^1 \wedge c)$ -- (A)
$a + (b + c) =$
$[a \wedge ((b \wedge c^1) \vee (b^1 \wedge c))^1] \vee [a^1 \wedge ((b \wedge c^1) \vee (b^1 \wedge c))]$
$= (a \wedge b \wedge c) \vee (a \wedge b^1 \wedge c^1) \vee (a^1 \wedge b \wedge c^1) \vee (a^1 \wedge b^1 \wedge c$---- (B)
From (A) and (B), $(a + b) + c = a + (b + c)$. Further
$a + 0 = (a \wedge 0^1) \vee (a^1 \wedge 0) = (a \wedge 1) \vee 0 = a \wedge 1 = a$.
Therefore 0 is the additive identity in A.
Also $a + a = (a \wedge a^1) \vee (a^1 \wedge a) = 0$
Thus inverse of a is itself.
Therefore, (A,+) is an abelian group with identity 0. Further
$a(b .c) = a \wedge (b \wedge c) = (a \wedge b) \wedge c =$ (a.b).c and $a.1 = a \wedge 1 = a$
Also, $a.b = a \wedge b = b \wedge a = b.a$
Therefore, (A,.) is a semigroup with identity 1.
Distributive law:
$a.(b + c) = a \wedge [(b \wedge c^1) \vee (b^1 \wedge c)] = (a \wedge b \wedge c^1) \vee (a \wedge b^1 \wedge c)$,
by (4) of def 2.1
$ab + ac = (a \wedge b) + (a \wedge c)$
$= [(a \wedge b) \wedge a \wedge c)^1] \vee [(a \wedge b)^1 \wedge (a \wedge c)]$
$= (a \wedge b \wedge c^1) \vee (a \wedge b^1 \wedge c)$ by (8) of def 3.1
Hence $a(b + c) = ab + ac$.
Observe that $a + a = 0$ for all a is already proved.
Finally, $1+ a = (1 \wedge a^1) \vee (1^1 \wedge a) = a^1 \vee 0 = a^1$
Therefore $a (1+ a) b (1 + b) = aa^1bb^1 = a \wedge a^1 \wedge b \wedge b^1 = 0$
by (5) of def 2.1.
Hence A is a Boolean-like ring.

We now prove that every Boolean-like ring becomes a Boolean –like algebra.

Theorem 2.7: Let (A, +, · ,0,1) be a Boolean-like ring. Define the binary operations $\wedge$ and $\vee$ and complementation ^{1} by $a \vee b = a+b+ab$; $a \wedge b = a.b$ and $a^1 = 1+a$ for all a, b ∈ A. Then the algebraic system $(A,\wedge, \vee, ^1, 0, 1)$ is a Boolean like algebra.

Proof: In order to prove that A is a Boolean like algebra, we need to verify the following.
(1) $\vee$ and $\wedge$ are associative and commutative.
Now $a \vee b = a + b + ab = b + a + ba = b \vee a$,
and $a \wedge b = a.b = b.a = b \wedge a$.
Also, $a \vee (b \vee c) = a + (b + c + bc) + a(b + c + bc)$
$= a+ b + c + bc + ab + ac + abc$
$= (a + b + ab) + c + (a + b + ab)c = (a \vee b) \vee c$
Further, $(a \wedge b) \wedge c = (ab)c = a(bc) = a \wedge (b \wedge c)$
Therefore $\vee$ and $\wedge$ are associative and commutative
(2) $(a \vee b)^1 = a^1 \wedge b^1$; $(a^1)^1 = a$; $0^1 = 1$
Now $(a \vee b)^1 = 1+ (a + b + ab) = 1+ a + b + ab$
$= (1+ a)(1+ b) = a^1b^1 = a^1 \wedge b^1$
Also $(a^1)^1 = (1+ a)^1 = 1 + 1 + a = a$.
Trivially $0^1 = 1$
(3) $a \wedge 0 = 0$; $a \wedge 1 = a$.
Trivially, $a \wedge 0 = a.0 = 0$ and $a \wedge 1 = a.1 = a$
(4) $b \wedge c = 0 \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
Let $b \wedge c = 0$.
Then $a \wedge (b \vee c) = a (b + c + bc) = ab + ac + abc$
$= (a \wedge b) \vee (a \wedge c)$
(5) $a \wedge a^1 \wedge b \wedge b^1 = a (1+ a) b (1+ b) = 0$
(6) $(a \wedge a^1) \vee (a \wedge a^1)$
$= a (1+ a) + a (1+ a) + aa (1+ a) (1+ a) = 0$
[by (1) & (2) conditions of Boolean-like ring]
(7) $(a^1 \vee b) \wedge (a \vee b^1) = (a^1 + b + a^1b) (a + b^1 + ab^1)$
$= a\,^1a + a\,^1b^1 + a^1a\,b^1 + ba + bb^1 + bab^1 +$
$a^1ba + a^1bb^1 + a^1bab^1$
$= bb^1(1 + a^1 + a) + aa^1(1 + b + b^1) + a^1b^1 + ba$
(since $a^1bab^1 = 0$)
$= bb^1(0) + aa^1(0) + (1+a)(1+b) + ba = 1 + a + b$---(i)
$(a \wedge b) \vee (a^1 \wedge b^1) = ab + a^1 b^1 + aba^1b^1 = ab + a^1b^1$
$= ab + (1+a)(1+b) = 1+ a + b$ ----(ii)
From (i) & (ii) , (7) is satisfied.
$(8)(a \wedge b \wedge c^1) \vee (a \wedge b^1 \wedge c) = abc^1 + ab^1c + abc^1ab^1c$
$= abc^1 + ab^1c = ab(1+ c)+a(1+b)c =$
$ab + abc + ac + abc = ab+ac$ ------ (iii)
$[(a \wedge b) \wedge (a \wedge c)^1] \vee [(a \wedge c) \wedge (a \wedge b)^1]$
$= ab\,(ac)^1 + ac(ab)^1 + (ab)(ac)^1(ac)(ab)^1$
$= ab + ac$ ------ (iv)
From (iii) & (iv) , (8) is satisfied
Thus $(A, \wedge, \vee, ^1, 0, 1)$ is a Boolean-like algebra.
As in the case of Boolean ring and Boolean algebra, we now show that the Boolean –like ring and Boolean-like algebra are equivalent structures.

Theorem 2.8: The following abstract structures are equivalent
(i) Boolean-like ring and (ii) Boolean-like algebra.

Proof: Let (A,+,·,0,1) be a Boolean-like ring. By theorem 2.7,we get a Boolean-like algebra $(A,\wedge,\vee,^1,0,1)$ in which the binary operations $\wedge$, $\vee$ are defined by $a \vee b = a + b + ab$; $a \wedge b = ab$ and the complementation ^{1} is defined by
$a^1 = 1+ a$ for all $a,b \in A$
by theorem 2.6,
we obtain a Boolean-like ring out of this Boolean-like algebra. where new operations $+^1$, $\cdot^1$ in A are defined by
$a +^1 b = (a \wedge b^1) \vee (a^1 \wedge b)$; $a \cdot^1 b = a \wedge b$ and $1=1$; $0=0$.
Then $a +^1 b = (a \wedge b^1) \vee (a^1 \wedge b) = a + ab + b + ab = a + b$
and $a \cdot^1 b = a \wedge b = a \cdot b$. Therefore the newly obtained Boolean-like ring is same as the originally given one.
On the otherhand , let $(A,\wedge,\vee,^1,0,1)$ be a Boolean-like algebra. By theorem 2.6, we obtain a Boolean-like ring (A,+,·,0,1) where + and · are defined by
$a + b = (a \wedge b^1) \vee (a^1 \wedge b)$; $a \cdot b = a \wedge b$ for all a , b in A.
As in theorem 2.7, to construct a Boolean-like algebra out of this Boolean-like ring
we introduce new binary operations $\wedge^1$ and $\vee^1$ as
$a \vee^1 b = a + b + ab$; $a \wedge^1 b = ab$ for all a,b in A
$a^1 = 1+a$ and $0=0$; $1=1$.
Now $a \vee^1 b = a + b + ab = 1+ (1+ a + b + ab)$
$= 1+ (1+ a)(1+ b) = (a^1b^1)^1 = a \vee b$
$a \wedge^1 b = ab = a \cdot b = a \wedge b$.
Therefore, $\wedge^1 = \wedge$ and $\vee^1 = \vee$.
This completes the proof.
Thus, the newly obtained Boolean-like algebra is same as the originally given Boolean-like algebra.

Acknowledgement

The author extends her gratitude to Prof.Y.V.Reddy, Retd. Prof, Department of Mathematics, ANU and also the management of Koneru Lakshmaiah Educational Foundation, Vaddeswaram.

References:

[1] Bhattacharya P.B., (1995) etc Basic Abstract Algebra, 2nd edition, Cambridge University Press
[2] Foster, A.L. (1951), p^k–rings and ring-logics, Ann.sci.Norm Pisa 5,279-300.
[3] George F. Simmons (1963), Introduction to Topology and Modern Analysis, McGraw – Hill Book, Company, Inc
[4] Jachim Lambek (1986) , Lectures on Rings and Modules, 3rd edition Chelsea pub. Company, New York, N.Y.
[5] Swaminathan, V (1982), Boolean-like rings Ph.D thesis, A.U.
[6] Thomas W. Hungerford(1974), Algebra, Holt, Rinehart and Winston, INC., New York