International Journal of Engineering & Technology, 7 (4.10) (2018) 1001-1006

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

G, $\overline{G}$ Properties – from $\overline{G}$, G

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Abstract

In this paper weprovide a method of determining the chromatic polynomial of $\overline{G}$ without actual construction of $\overline{G}$. A planar graph characterization of graphs whose domatic partition is $\frac{n}{\gamma(G)}$ using $\overline{G}$ properties is established and provide a MATLAB program for identifying just excellent graphs.

Keywords:*Chromatic Polynomial; Complement Graph; Just Excellent; Planar.*

1. Introduction

In [1] M. Yamuna et.al have determined the chromatic polynomial of G^* without actual construction In [2], M. Yamuna et al has introduced just excellent graph . M. Yamuna et.al have proved that just excellent graph is not a domatic subdivision stable graph[3]. In [4], [5],M. Yamuna et al introduced Non domination subdivision stable graphs (NDSS) and characterized planarity of complement of NDSS graphs.. In [6],[7], M. Yamuna et al introduced γ - uniquely colorable graphs and also provided the constructive characterization of γ - uniquely colorable trees and characterized planarity of complement of γ - uniquely colorable graphs.This paper targets to determine properties of $\overline{G}$using properties of G without constructing $\overline{G}$.

2. Terminology

We consider only simple graph with n and m vertices and edges. The graph H is said to be the complement of G if any two vertices arenotadjacent in G, then in H they are adjacent.G**.** uvis obtained by contracting (u, v), where u $\perp$v. Results related to graph theory we refer to [9].

D is a dominating set if every vertex of V – D is adjacent to some vertex of D. Minimum cardinality of D, is said to be a minimum dominating set (MDS). The cardinality of any MDS for G is said to be domination number of G, represented by γ(G). Results related to domination we refer to [10].

We know that in a complete graph, the maximum number of edges is $(n (n-1))/2$. Also E (G) $\cup$ E ($\overline{G}$) = $(n (n-1))/2$. If n = 100, then G $\cup\overline{G}$ has 4950 edges. If G has say thousand edges $\overline{G}$ has 3950 edges. It is obvious that construction $\overline{G}$ is very difficult in such cases. As n increases, this complexity invariably increases. When properties of $\overline{G}$ are required in such cases it is better to determine them from G without drawing $\overline{G}$.

3. Result and discussion

This section gives a recursive technique for finding the chromatic polynomial of $\overline{G}$ from G **.**

This results and discussions is for modifying the existing technique of determining the chromatic polynomial ($P_n(\lambda)$) for any graph, so that $P_n(\lambda)$ of $\overline{G}$ can be determined from G. Throughout this section for any edge e = (u, v) in G the corresponding edge in $\overline{G}$ is denoted by $\bar{e}$ = ($\bar{u}$, $\bar{v}$).

3.1. Modified G – { e }

We know that no edges in G implies an edge in $\overline{G}$. We use this to modify the operation G – { e }. Comparing G and$\overline{G}$, picking two non – adjacent vertices in $\overline{G}$ is equivalent to choosing adjacent vertices in G.

For example consider the graph G in Fig.1 and its complement $\overline{G}$ **.**

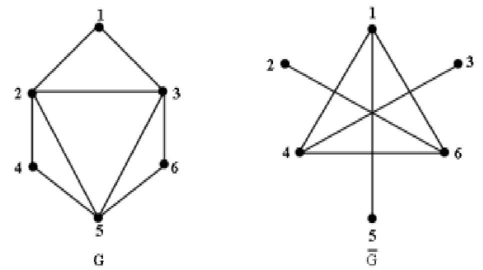

Fig.1

1 not adjacent to 2 in $\overline{G}$. Add an edge between 1 and 2 in $\overline{G}$ to generate graph G_1, that is $G_1 = \overline{G}\cup\{ (1\ 2) \}$.

We aim to generate the graph G_1 from graph G. Since it is just an additional edge available in G_1, removal of this edge in G would satisfy our requirement. Also we observe that $\overline{G - \{ (1\ 2) \}} = G_1$.

So we conclude that adding an edge between non adjacent vertices in $\overline{G}$ is equivalent to removing the edge between same vertex pair in G.

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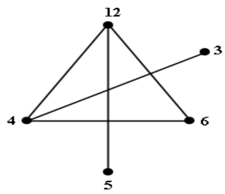
3.2. Type – I Operation

Remove edge e = (u, v) in G. Let $G_1 = G - \{ e \}$. By applying Type – I Operation we create a new graph G_1,so that $\overline{G}_1 = \overline{G} + \{\overline{e}\}$.

3.3. Modified G.uv



$\overline{G}$.12

Fig.2

Merging vertices 1 and 2 of $\overline{G}$, we obtain graph $\overline{G}$.12. We observe that vertices1and2 are together not adjacent to the vertices 3, 5 respectively in $\overline{G}$. The same is true in $\overline{G}$.12 also.

Vertices 4, 6 are adjacent to atleast one of vertex 1 or 2. So in $\overline{G}$.12 vertex 12 is adjacent to the vertex 4 and 6. We have to decide a suitable operation in G which would generate a graph G_2, so that complement of G_2 is G.uv for any two non adjacent vertices u, v in $\overline{G}$. From the above observation, we can decide that for any two adjacent vertices e = (u, v) in G generate a graph G.uv as follows

1.If u and v are collectively ⊥to vertices $v_1, v_2, \ldots v_p$, then in G draw an edge between vertex pairs (uv, v_1), (uv, v_2), … (uv, v_p) respectively.

2.If $u_1, u_2, \ldots u_q$ are vertices adjacent to atleast one of u or v then there is no edge between vertex pairs (uv, u_1), (uv, u_2), … (uv, u_q) respectively.

By applying these two operations, we observe that in the complement of $\overline{G}$.uv there exist no edge between uv and vertices $v_1, v_2, \ldots v_p$. Also there exist edges between uv and vertices $u_1, u_2, \ldots u_q$.

3.4. Type – II Operation

Let e = (u, v). Let $X = \{ v_1, v_2, \ldots v_p \} \subseteq V(G)$ such that u and v are both adjacent to $v_1, v_2, \ldots v_p$. Let $Y = \{ u_1, u_2, \ldots u_q \} \subseteq V(G)$ such that $u_1, u_2, \ldots u_q$ are adjacent to atleast one of u or v.

Generate G.uv by applying the following operations

1.Draw edge between (uv, v_1), (uv, v_2), … (uv, v_p) .

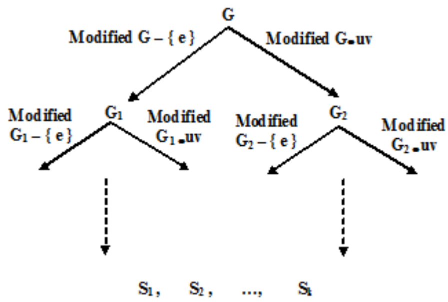
2.There is no edge between (uv, u_1), (uv, u_2), … (uv, u_q).

3.5. Complete Graph Identification

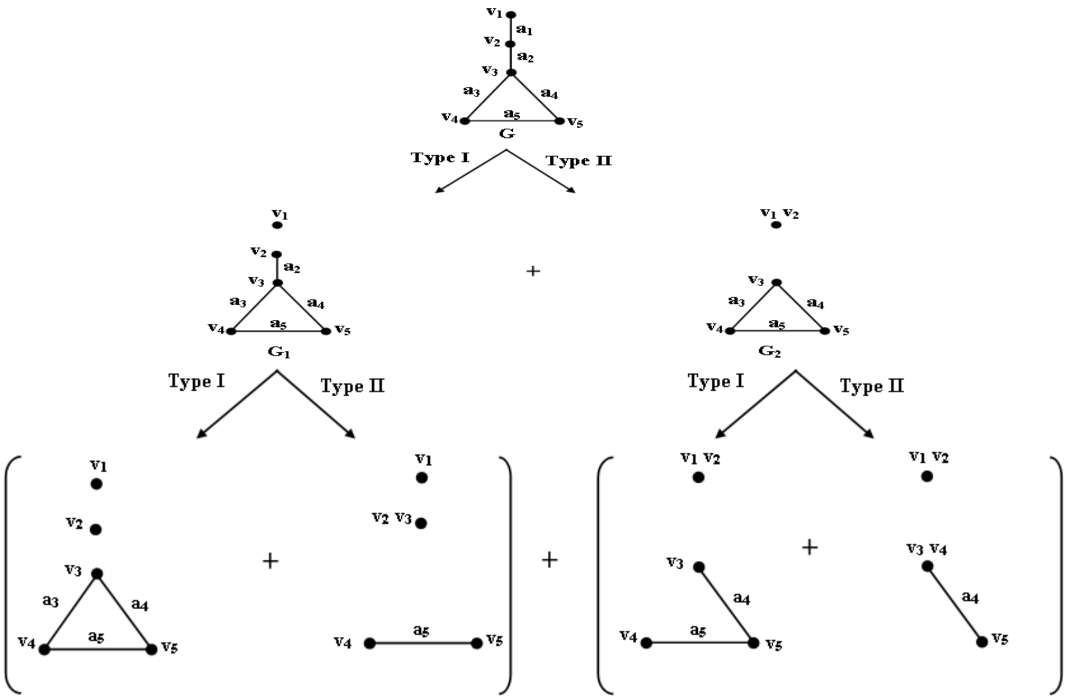
By recursive applying Type I, II operations on G, we reach a graph H such that H is a null graph. This means that $\overline{H}$ is a complete graph. Let G be any graph. Apply modified G – { e } and G. uv on H to generate G_1, G_2 , that is G_1 is the graph obtained by applying Type I operation on G and G_2 is the graph obtained by applying Type II operation on G. Continue this procedure recursively on G_1 and G_2 and on the remaining graphs generated until we cannot continue any further. This meanst hat we now have a sequence of graphs $S_1, S_2, \ldots, S_k$ such that each S_i, i = 1, 2, …, k are null graphs, so that $\overline{S}_i$, i = 1, 2, …, k is a complete graph.

- 1). Chromatic polynomial of $\overline{S}_i = \prod_{k=0}^{n-1}(\lambda - k)$, if S_i is a null graph with n vertices. So $P_n(\lambda)(S_i) = \sum_{i=1}^{k} P_n(\lambda)(\overline{S_i})$. This procedure is summarized in hierarchy.

In Fig. 3 $P_n(\lambda)(\overline{S_i}) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda$



Hierarchy. 1.



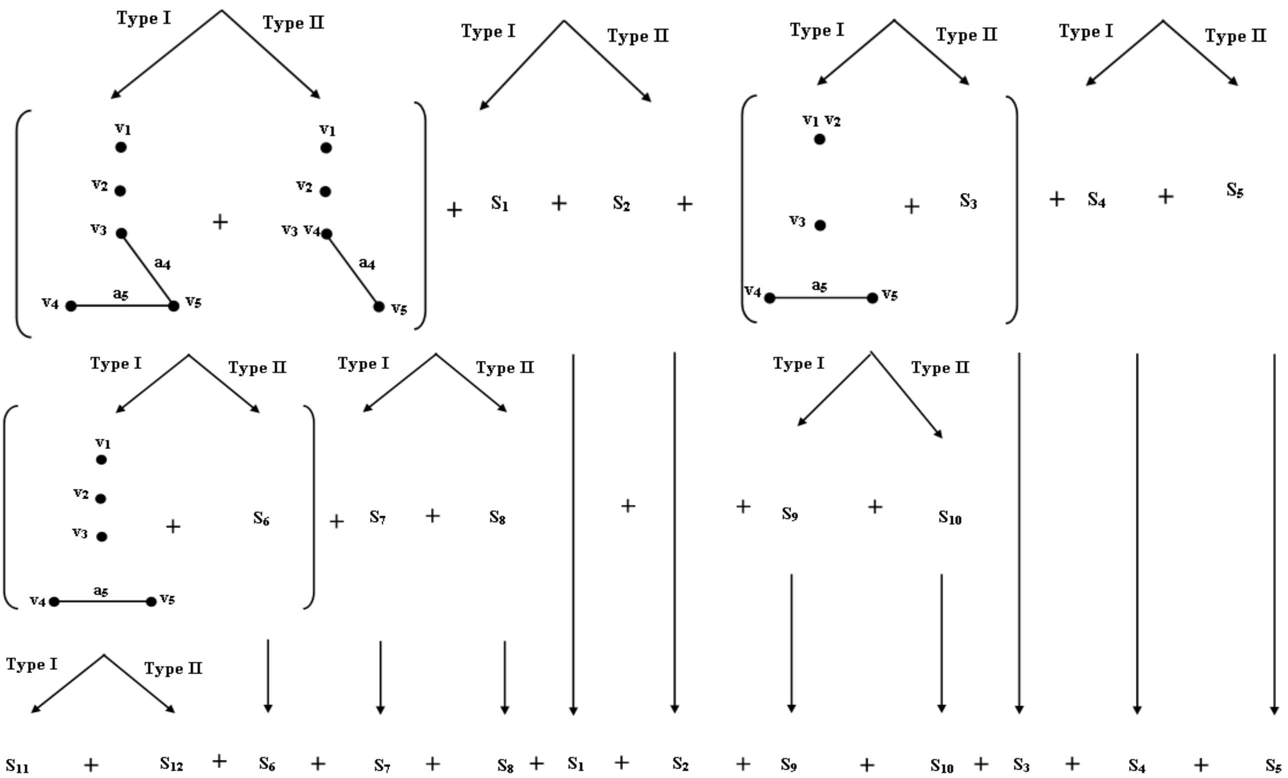


Fig.3

From modified G – e and G.uv , we understand that for every pair of adjacent vertices (non – adjacent in $\overline{G}$)

- $(\overline{G-e}) = \overline{G} + \bar{e}$ and
- $(\overline{G \cdot uv}) = \overline{G}.uv$

that is the recursive process is determined in such a way that at every stage the number of non – adjacent vertices in G =the number of adjacent vertices in $\overline{G}$. The numberof operations involved in determining the chromatic polynomial of $\overline{G}$ fromG is equal to the number of operations involved in determining the same from $\overline{G}$, since the operations are repeated for every adjacent pair in G

4. Planarity of $\overline{G}$ - from G

Planar graphs are often characterized by the following famous theorems that uses Kuratowski's graphs.

R1. G is planar iff G does not contain K_5 or $K_{3,3}$as a subgraph or any graph homeomorphic to K_5 or $K_{3,3}$ as a subgraph.

R2. G is planar iff it does not have a subgraph contractible to K_5 or $K_{3,3}$ [6].

R_3. G is outer planar iff it has no subgraph homeomorphic to K_4 or $K_{2,3}$except $K_4 – x$ [6].

In this section, we shall prove that a just excellent graph is non – planar, when $\gamma(G) \geq 4$ using R_1 and R_2. If $\gamma (G) = 1$, $\overline{G}$ is disconnected, so we consider the cases only when $1 < \gamma (G) \leq k$, for some integer k.

Theorem 1

Let G be a just excellent graph. Let $D = \{u_1, u_2, \ldots u_m\}, m \geq 3$ be a γ - set for G. Let $X_1 = pn (u_1, D) = \{a_1, a_2, \ldots ,a_{k1} \}$, $X_2 = pn (u_2, D) = \{b_1, b_2, \ldots ,b_{k2} \}, \ldots , X_m = pn (u_m, D) = \{m_1, m_2, \ldots ,mk_m \}$. Then

A.$\langle pn (u_i, D)\rangle$ is not complete, for all i = 1, 2, . . . , m.

B.$\langle X_i, X_j\rangle$ is not complete, $\forall$ i $\neq$ j.

C. Assume that condition B is satisfied. Then for some $z_k \in X_k$, it is not possible that (y_j, z_k) is the only non – adjacent vertex pair in (X_j, X_k) such that $y_j \in X_j$, $z_k \in X_k$.

Proof

A.If possible assume that $\langle pn (u_1, D)\rangle$ is complete. Then $D' = D – \{u_1\} \cup \{a_i\}$ is a γ - set for G. D, D′ are γ - sets for G containing $\{u_2, u_3, \ldots u_m\}$, a contradiction, $\Rightarrow\langle pn (u_i, D)\rangle$ is not complete, for all i = 1, 2, . . . , m.

B. If $\exists$ some X_i, X_j, $i \neq j$, $\ni \langle X_i, X_j\rangle$ is complete, then $D' = D – \{u_i\} – \{u_j\} \cup \{x_i\} \cup \{y_j\}$, where $x_i \in pn (u_i, D)$, $y_j \in pn (u_j, D)$ is a γ - set for G. D, D′ are two distinct γ - sets for G containing $\{u_1, u_2, \ldots u_{i-1}, u_{i+1}, \ldots , u_{j-1}, u_{j+1}, \ldots u_m \}$, a contradiction to our assumption that G is just excellent, $\Rightarrow\langle X_i, X_j\rangle$ is not complete, $\forall$ i $\neq$ j.

C.By part B, we know that there exist some $x_i \in pn (u_i, D)$, $y_j \in pn (u_j, D)$ such that x_i not adjacent to y_j. If possible, assume that in (X_j, X_k) , $y_j$not adjacent to $z_k$is the only non – adjacent vertex pair that is, $\forall$vertex in $X_j – \{y_j\}$ is $\perp$ to every vertex in X_k. $D' = D – \{u_j\} – \{u_k\} \cup \{y_a\} \cup \{z_b\}$, $y_a \in X_j$, $z_b \in X_k$, $a \neq j$, $b \neq k$ are γ - sets for G that is, D, D′ are two distinct γ - sets for G containing u_i, a contradiction. Hence statement C is true

Theorem 2

Let G be a just excellent graph, $\ni \gamma (G) = 3$. Then $\overline{G}$ is non – outer planar.

Proof

Let $\gamma (G) = 3 = \{u_1, u_2, u_3\}$. Let $X_1 = pn (u_1, D) = \{a_1, a_2, \ldots ,a_{k1}\}$, $X_2 = pn (u_2, D) = \{b_1, b_2, \ldots ,b_{k2}\}$, $X_3 = pn (u_3, D) = \{c_1, c_2, \ldots ,c_{k3} \}$. We prove that $\overline{G}$ is non – planar by showing that $\overline{G}$contains $K_{2, 3}$ as a subgraph. Let V_1 and V_2 be the vertex set of $K_{2,3}$in$\overline{G}$. Choose $V_2 = \{ u_1, u_2, u_3 \}$.

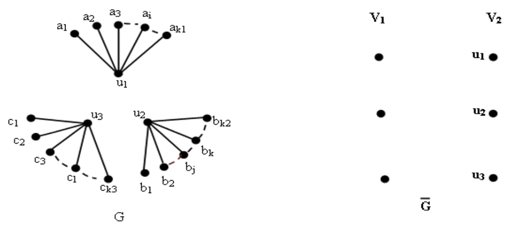


Fig.4

By part B of Theorem 1, we know that there exist
i. a_i, b_j such that a_i not adjacent to b_j,
iib$_k$, c_l such that b_k not adjacent to c_l, $k \neq j$,
This is possible, because pn $(u_i, D) \geq 2$, for all $u_i \in D$. a_i not adjacent to b_j, b_k not adjacent to c_l, in$\overline{G}$. Contract these edges in $\overline{G}$. Let $V_1 = \{ a_ib_j, b_kc_l \}$.$a_i \in X_1$, $b_j \in X_2$, a_i not adjacent to $\{u_2, u_3\}$ and b_j not adjacent to u_1 in G. This means that a_i adjacent to $\{u_2, u_3\}$ and b_j adjacent to $u_1$in $\overline{G}$. That is vertex $a_ib_j \perp$ to $\{ u_1, u_2, u_3 \}$ in $\overline{G}$. Similarly vertex $b_kc_l \perp$ to $\{ u_1, u_2, u_3 \}$ in $\overline{G}$. $\langle u_1, u_2, u_3, a_ib_j, b_kc_l \rangle$ is $K_{2,3}$ in $\overline{G}$, that is $\overline{G}$ is non – outer planar.

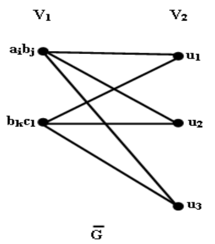


Fig.5

Theorem 3

Let Gbe a just excellentgraph, ∋ γ (G) = 4. Then$\overline{G}$ is non – planar.

Proof

Let γ (G) = 4 = D = $\{u_1, u_2, u_3\}$. Let X_1 = pn $(u_1, D) = \{a_1, a_2, \ldots, a_{k1}\}$, X_2 = pn $(u_2, D) = \{b_1, b_2, \ldots, b_{k2}\}$, X_3 = pn $(u_3, D) = \{c_1, c_2, \ldots, c_{k3}\}$, X_4 = pn $(u_4, D) = \{d_1, d_2, \ldots, d_{k4}\}$. We prove that $\overline{G}$ is non – planar by showing that $\overline{G}$ contains $K_{3,3}$ as a subgraph. Let V_1 and V_2 be the vertex set of $K_{3,3}$in$\overline{G}$.
Choose some $b_i \in X_2$, $a_i \in X_1$. b_i not adjacent to u_1, a_i not adjacent to u_4 in G, implies u_1 adjacent to b_i, u_4 adjacent a_i in $\overline{G}$. Choose $V_1 = \{u_1b_i, u_2, u_3\}$, $V_2 = \{u_4a_i, d_i, d_j\}$ where $d_i, d_j \in X_4$.

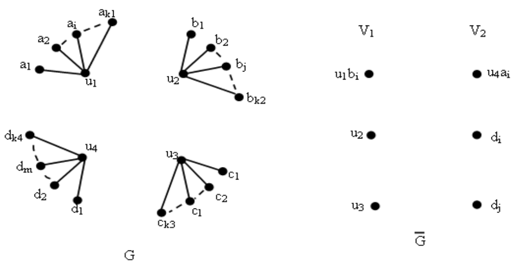


Fig.6

i. u_1 not adjacent to d_i, d_j and u_4 not adjacent to b_i.
ii. u_2 not adjacent to d_i, d_j, a_i.
iii. u_3 not adjacent to d_i, d_j, a_i in G.

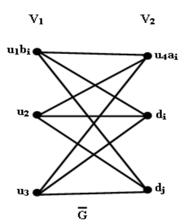


Fig.7

This implies, these vertices are adjacent in $\overline{G}$. Hence $\langle V_1, V_2 \rangle$ is $K_{3,3}$ implies $\overline{G}$ is non – planar.

Note

- Generalizing Theorem 3, we conclude that, if G is just excellent graph ∋ γ (G) $\geq$ 4, then $\overline{G}$ is non – planar.
- When γ (G) = 2, G just excellent, $\overline{G}$may be planar or non – planar.

Example

In Figure 8 G,$\overline{G}$ are planar, G is just excellent.

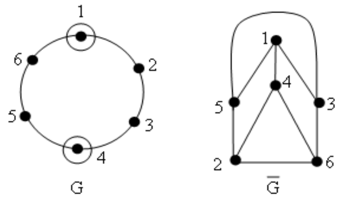


Fig.8

Figure 9 G is just excellent, $\overline{G}$ - {(6, 4)}, {(7, 4)} is a subgraph of$\overline{G}$

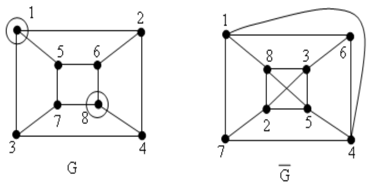


Fig. 9

Contracting (4, 5), (6, 3), (7, 2), we see that $\langle 1, 63, 45, 72, 8 \rangle$ is K_5 and hence $\overline{G}$is non planar.
If G is just excellent graph ∋

i. γ (G) = 2, then $\overline{G}$ may be planar or non – planar.
ii. γ (G) = 3, then $\overline{G}$ is non – outer planar.
iii. γ (G) $\geq$ 4, then $\overline{G}$ is non – planar.

5. Matrix Representation for Just Excellent Graphs

Let G be any graph with n – vertices. Let A represents the adjacency matrix of G. Let N denote a n ×n matrix, where

$$N = [n_{ij}]_{n \times n} = \begin{cases} 1 & \text{if } i = j \\ a_{ij} & \text{the } (i,j)^{th} \text{ entry in the adjacency matrix} \end{cases}$$

Let $x = (x(v_1), x(v_2), \ldots, x(v_n))^T$ be a { 0, 1 } vectors . We know that, if x represents the any dominating set, then Nx≥ 1, that is in a resulting matrix Nx, all the entry values are non zero.

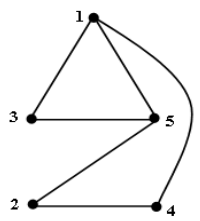


Fig.10

$$\overset{N}{\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} = \overset{Nx}{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

That is $\{ v_1, v_4 \}$ is a dominating set for G.
We know that, a graph G is just excellent if every vertex is contained in a unique γ - set of G.
If k does not divide n, then G is not just excellent. If k divides n, then we proceed as follows.

Consider the non – zero columns of NV. Let Xbe the set of all vectors ∋Nx_i≥ 1, i.e., Ns≥1. Let |s| = q, since p represents all possible permutation, there exist two vectors x_i, x_j, $i \neq j$ such that both x_i and x_j have non – zero entry in corresponding position, this mean that a vertex is contained in more than one γ - set. If | q | = | p |, then we mean to tell that every x_i is a γ - set for G. So atleast one vertex is contained in more than one γ - set, implies G is not just excellent.

If $q/n \neq k$, then R1 is not satisfied implies G is not just excellent.

If $q/n = k$, then add every row of matrix of order n × 1. If every entry of the resulting column matrix is one, then every vertex is contained in exactly one γ - set implies G is just excellent else if atleast one entry of this matrix is not equal to one implies G is not just excellent.

If G is just excellent ∋ γ(G) = 3, then $\overline{G}$is non – outer planar. If G is just excellent ∋ γ(G) ≥ 4, then $\overline{G}$is non – planar.

Example

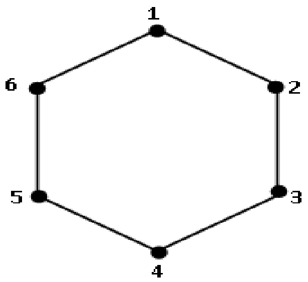


Fig.11

For the graph in Fig. 11, γ(G) = 2. We consider all possible subsets with 2 vertice sand label them as { S_1, S_2, S_3, …, S_{15} } = {{ v_1, v_2}, { v_1, v_3}, { v_1, v_4}, { v_1, v_5}, { v_1, v_6 }, { v_2, v_3 }, { v_2, v_4 }, { v_2, v_5 }, { v_2, v_6 }, { v_3, v_4 }, { v_3, v_5 }, { v_3, v_6 }, { v_4, v_5 }, { v_4, v_6 }, { v_5, v_6 } }.

NV =

$$\begin{pmatrix} 1&1&0&0&0&1\\ 1&1&1&0&0&0\\ 0&1&1&1&0&0\\ 0&0&1&1&1&0\\ 0&0&0&1&1&1\\ 1&0&0&0&1&1 \end{pmatrix} \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&1&1&1&1&1&0&0&0&0&1\\ 0&0&0&1&1&1&0&0&0&1&0&0&0&1&0\\ 0&1&1&0&0&1&0&0&1&0&0&0&1&0&0\\ 1&0&1&0&1&0&0&1&0&0&0&1&0&0&0\\ 1&1&0&1&0&0&1&0&0&0&1&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&1&0&1&0&0&2&1&1&1&2&1&1&1&2\\ 0&0&0&1&1&1&1&1&1&2&1&1&1&2&2\\ 0&1&1&1&1&2&1&1&2&2&0&0&1&1&1\\ 1&1&2&1&2&2&0&1&1&1&0&1&1&1&0\\ 2&2&2&1&1&1&1&1&1&0&1&1&1&0&0\\ 2&1&1&1&1&0&1&1&0&0&2&2&1&1&1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\\ 1&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix} \quad z = \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix}$$

where z is the matrix obtained by adding each rows of S. Since all the entries in the Z matrix is 1, the graph seen in Fig.10 is just excellent.

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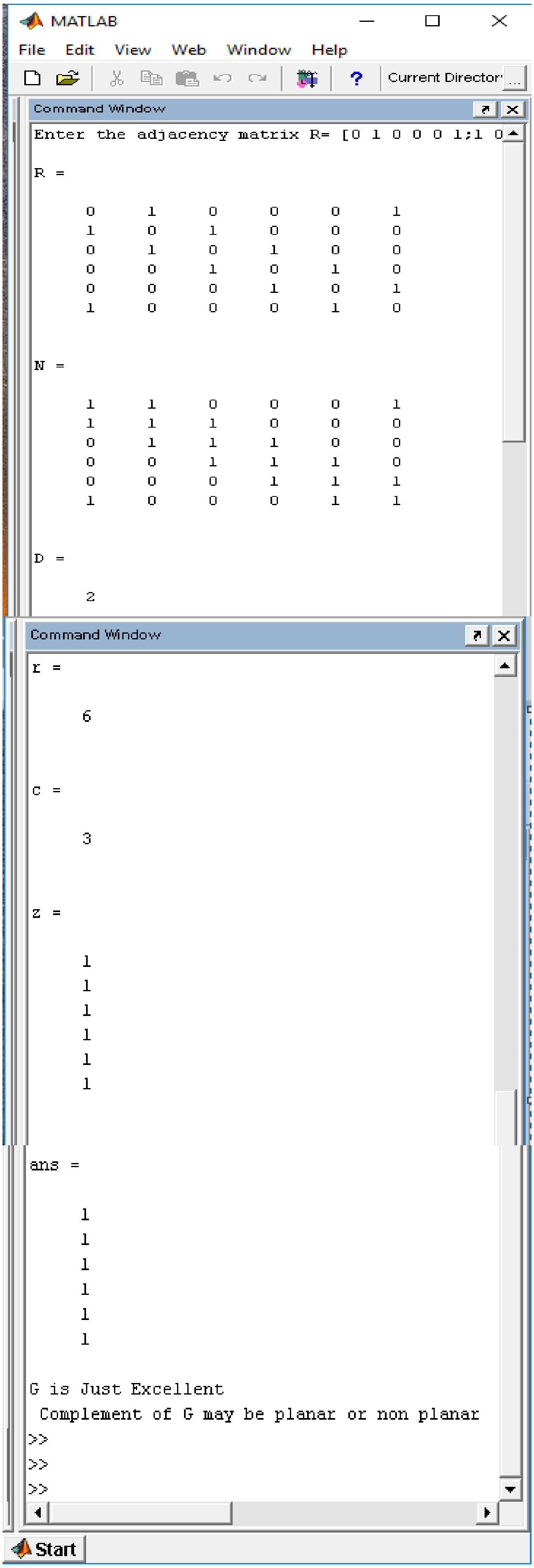
MAT LAB program for just excellent graphs

Snapshot – 1 provides the output for the graph in Fig. 11.



Snapshot - 1

6. Conclusion

The difficulty in constructing $\overline{G}$ particularly, when G has large number of vertices and less number of edges resulted in experimenting if $\overline{G}$ property can be determined without constructing $\overline{G}$. Is it possible to achieve this with the same number of graph operations originally used in $\overline{G}$? In this paper, we have devised a technique of determining the chromatic polynomial of $\overline{G}$ without actual construction of $\overline{G}$, but with the same number of operations used in $\overline{G}$. This result paved way for new method of approaching graph problems. Deciding $\overline{G}$ properties without constructing $\overline{G}$. As an output, non – planarity of $\overline{G}$ is determined from properties of G without knowing the actual structure of $\overline{G}$. We further plan to extend this idea and establish properties of $\overline{G}$ without constructing $\overline{G}$.

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