International Journal of Engineering & Technology, 7 (4.10) (2018) 998-1000

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

Planar and Non Planar Construction of γ- Uniquely Colorable Graph

A. Elakkiya[1], M. Yamuna[2]*

[1]Vellore Institute of Technology, Vellore
[2] Vellore Institute of Technology, Vellore
**Corresponding author E-mail: myamuna@vit.ac.in*

Abstract

A uniquely colorable graph G whose chromatic partition contains atleast one γ - set is termed as a γ - uniquely colorable graph. In this paper, we provide necessary and sufficient condition for$\overline{G}$and G* to be γ - uniquely colorable whenever G γ- uniquely colorable and also provide constructive characterization to show that whenever G is γ- uniquely colorable such that |P | ≥ 2, G can be both planarand non planar.

Keywords: *Complement; Dual;Non Planar;Planar;Uniquely colorablegraphs.*

1. Introduction

In [1] Bing Zhou investigated the dominating -χ-color number, $d_\chi(G)$, of a graph G. In [2],[3], M. Yamuna et al introduced γ - uniquely colorable graphs and also provided the constructive characterization of γ -uniquely colorable trees and characterized planarity of complement of γ - uniquely colorable graphs. In [4],[5],M. Yamuna et al introduced Non domination subdivision stable graphs (NDSS) andcharacterized planarity of complement of NDSS graphs

2. Terminology

We consider simple graphs G with n vertices and m edges. K_n is a complete graph with n vertices. K_5 and $K_{3,3}$are called Kuratowski's graph. Results related to graph theory we refer to [6].

Chromatic partition of a graph G is partition the vertices into smallest possible umber of disjoint ,independent sets. A graph G = (V, E) is said to be uniquely colorable ifhas a unique chromatic partition.

D is adominating set if every vertex of V – D is adjacent to some vertex of D. Minimum cardinality of D, is said to be a minimum dominating set (MDS). The cardinality of any MDS for G is said to be domination number of G, represented by γ(G). Results related to domination we refer to [7].

3. Result and Discussion

A uniquely colorable graph G whose chromatic partition contains atleast one γ– set is termed as a γ– uniquely colorable graph.

In Fig. 1 G_1 and G_2 are γ– uniquely colorable graphs. $\overline{\mathbf{G_1}}$ isγ– uniquely colorable while $\overline{\mathbf{G_2}}$ is not γ– uniquely colorable graph. So when G is γ– uniquely colorable, $\overline{\mathbf{G}}$need not be γ– uniquely colorable. In Fig. 2 G_1 and G_2 are γ– uniquely colorable graphs. $G_1^*$is γ– uniquely colorable while $G_2^*$is not γ– uniquely colorable graph. So when G is γ - uniquely colorable ,G^*need not be γ– uniquely colorable. In this paper, we determine the condition for$\overline{\mathbf{G}}$ and G* to be γ– uniquely colorable whenever G is γ– uniquely colorable. We also provide the constructive characterization to show that whenever G is γ uniquely colorable such that | P | ≥ 2, G can be both planar and non planar.

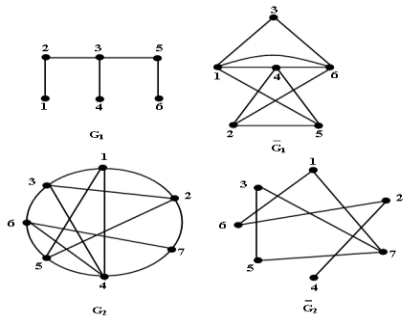


Fig.1

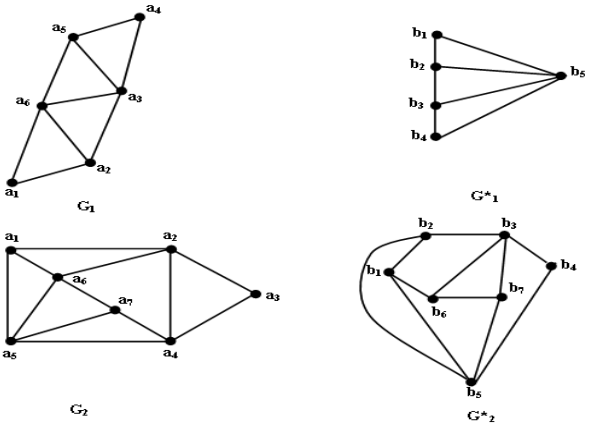


Fig.2

Copyright © 2018Authors. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Theorem 1. Let G be a isγuniquelycolorable graph. $\overline{G}$is alsoγ–uniquely colourable if and only if ∃ a unique smallestpossible partition P = { V_1, V_2, … V_k } of V(G) ə
1. every V_i, i = 1 to k is a clique
2. there exist one V_iə every vertex in V – { V_i } is notadjacent to atleast one vertex in V_i
3. | V_j|≥| V_i |, for every i ≠ j
4. V_i is the smallest set in G satisfying 2

Proof. Assume that $\overline{G}$is a γ– uniquely colourable graph, implies-there exist a partition P_1 = { V_1, V_2, … V_k } for $\overline{G}$such that

- P_1 is unique and smallest possible set.
- every V_i, i = 1 to k, is independent in $\overline{G}$implies every V_i is a clique in G.
- there exist one V_i such that $V_i$is a γ– set for $\overline{G}$. Also | V_i | ≥ | V_j|, for every i ≠ j implies there exist one V_i in G such that every vertex in V – { V_i } not adjacent to atleast onevertex in every V_i.

implies P_1 = { V_1, V_2, … V_k } is a γ– chromatic partition for V (G). P_1 is not unique implies ∃one P_2 = { W_1, W_2, … W_k } in G such that { W_1, W_2, … W_k } is a clique, implies P_2 is also a γ chromatic partition in $\overline{G}$ such that every W_i is independent and | P_1 | = | P_2 |, a contradiction to our assumption that P_1 is unique. P_1 is not smallest, implies one P_3 = { V_1, V_2, … V_k },q < k such that P_3 isa γchromatic partition in G ə U_i, i = 1to q is clique implies P_3 isa γchromatic partition in$\overline{G}$ ə every U_i is independent and | P_3| < | P_1| , a contradiction. P_1 is a γuniquely colorable partition for $\overline{G}$implies there exist one V_i such that V_i is a γ–set for $\overline{G}$, implies every vertex in V – { V_i } is adjacent to atleast one vertex in V_i, implies P_1 is a γ– chromatic partition in G əevery vertex in V – { V_i } is not adjacent to atleast one vertex in V_i. Also, we know that |V_i | ≤ | V_j | for every i ≠ j in $\overline{G}$, implies it is true in G also. If V_i is not the smallest set ə every vertex in V – { V_i } is ⊥ to atleast one vertex in V_i in $\overline{G}$, implies there exist one W contained in V ($\overline{G}$) such that | W |< | V_i | and every vertex in W - V ($\overline{G}$) is⊥ to atleast one vertexin W, a contradiction ⇒ V_i is the smallest set satisfying the property. Hence P_1 is a γ– chromatic partition in G ə the conditions of the theorem are satisfied.

Conversely assume that the conditions of the theorem are satisfied. P is a partition such that it is unique and smallest such that every $V_i$is a clique, implies P_1 is a partition in $\overline{G}$such that every V_i is independent. If P is not a smallest possible partition in $\overline{G}$then there exist one partition P_4 = { R_1, R_2, … R_q }, q < k in $\overline{G}$such that each R_i is independent , implies P_4 is a partition in G such that every $R_i$is a clique such that | P_4 |< | P| , a contradiction. P is not unique in $\overline{G}$, implies there exist a partition P_5 = { S_1, S_2, … S_k} such that each S_i is independent in$\overline{G}$, implies P, P_5 are two possible partition with the same cardinality in G, a contradiction.P is a partition ə there exist one V_i, every vertex in V –{ V_i } is not ⊥ to atleast one V_i , | V_j |≥ |V_i | for any i ≠ j implies P is a partitionin $\overline{G}$əevery vertex in V – {V_i } isadjacent toatleast one V_i, | V_i | ≤ | V_j | , i ≠ j, implies $V_i$is a dominating set for $\overline{G}$. Since V_i is the smallest set satisfying this property, implies V_i is the γ - set for $\overline{G}$

Let P = { R_1, R_2, … R_q }, betheset ofregions of G. Let T ={ r_1, r_2, … r_q }, betheset of vertices in the regions R_1, R_2, ..., $R_q$respectively, that is r_1 is the vertex in the region R_1, r_2 is the vertexin the region R_2, ..., r_q is the vertex in the region $R_q$respectively.

We observe that

- There is a 1-1 mapping between S and T, i.e ∀ R_i ∈ S ∃$r_i$in T, i = 1,..., q.
- ∀ X ⊆ S ∃ a corresponding set in T (say X^*) *, i.e if X ⊆ S ={ R_i, R_p, R_j }, then X ⊆ T = { r_i, r_p, r_j }.
- If a is any edge in G there is a corresponding edge in G^* (say a*).
- Let D ⊆ S ə every region in S – D is⊥ to atleast one region in D ⇒ ∃ D^*⊆ T ə any vertex in T - D^* is ⊥to atleast vertex inD^*.
- D is a smallest cardinality satisfying this property ⇒ D^* is a γ - set for G^*.

Theorem 2. Let G be a γ - uniquely colourable graph. G* isalso γ - uniquely colourable graph if and only if there exist a unique smallest partition P = { R_1, R_2, …, R_k } of R (G) such that
1. everyR_i, i = 1, 2, ...,k is independent.
2. there exist one R_i such that every region in R – { R_i} is adjacent to atleast one region in R_i.
3. | R_j | ≥ |R_i|.

Proof. Assume that G* is γuniquelycolourable graph. If G* is γ - uniquely colourable graph, then there exist a partition P = { V_1, V_2, …, V_k } such that P is a γ - chromatic partition, ⇒
1. every V_i is independent.
2. V_1 is a γ - set for G^*.

1 implies, there exist a set of regions R_1, R_2, …,R_k in G such that every R_i is independent.

2 implies, there exist R_1ə every region in R – { R_1} is adjacent to atleast one region in R_1 and R_1 is the smallest set satisfying this property implies the conditions of the theorem are satisfied.

Conversely, assume that the conditions of the theorem are satisfied. P = { V_1, V_2, …, V_k } is a partition of R (G), implies there exist a partition P_1 = { V_1, V_2, …, V_k } of V(G^*).

1 implies, every V_i, i = 1, 2, ...,k is independent.

2 implies, there exist one V_iə every vertex in V – { V_i} is ⊥to atleast one region in V_i.

3 implies | V_j | ≥|V_i | for all i ≠ j

Since P is a unique partition there exist no other partition of V (G^*) that satisfies all these conditions implies, P_1 is a γ - chromaticpartition for G*.

Planar and Non planar Construction

In this section, we provide constructive characterization to showthat whenever G is γuniquelycolorable such that | P | ≥ 2, G canbe both planarand nonplanar.

Planar Construction when | P | = 2.

Let γ(G) = k_1. Let P = { V_1, V_2}, where V_1 = { a_1,a_2, …,a_{k1}} V_2 ={ b_1, b_2, …,b_{k2}},k_2≥ k_1, k_1≥ 3, k_2≥4.

Constructagraph G_1 as follows

1. V (G_1) = V (G)
2.Considerk$_1$ vertices in V_1 and V_2 say { a_1, a_2, …,a_{k1}} and { b_1, b_2, …,b_{k2}}.

Construct a comb graph with 2k_1 vertices. Label the vertices of this comb as seen in Fig. 3

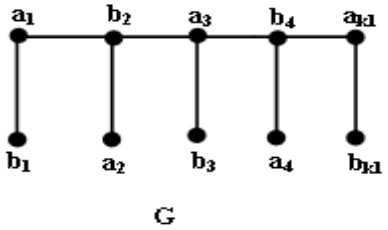


Fig.3

Include the remaining k_2 - k_1 vertices of V_2 as pendant vertices with a_{k1} as the support vertex. The general structure of graph G_1 is as seen in the Fig.4.

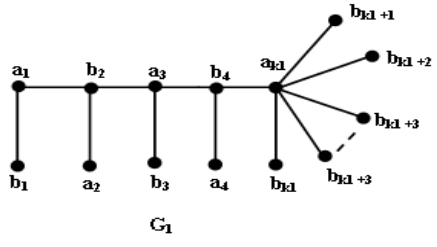


Fig.4

Since we have atleast k_1 pendant vertices, $\gamma(G_1) \geq k_1$, $\{a_1, a_2, \ldots, a_{k1}\}$ is a dominating set for G_1, implies $\gamma(G_1) = k_1$. Since $\langle a_1, a_2, \ldots, a_{k1}, b_1, b_2, \ldots, b_{k2}\rangle$ isacomb, the only possible maximal independent sets are $\{a_1, a_2, \ldots, a_{k1}\}and\{b_1, b_2, \ldots, b_{k2}\}$. $P = \{V_1, V_2\}$ is a partition for G_1 suchthat

1. V_1 is aγ- set for G_1
2. P is the only possible partition for G_1, $\Rightarrow$ G_1 is a γ– uniquely colorable graph.

Non Planar Construction when $|P| = 2$.

Let $\gamma(G) \geq k_1$. $k_1 \geq 6$, $P = \{V_1, V_2\}$ $V_1 = \{a_1, a_2, \ldots, a_{k1}\}$; $V_2 = \{b_1, b_2, \ldots, b_{k2}\}$, $k_2 \geq 6$

Construct a graph G_1 as follows

1.$V(G_1) = V(G)$

2.Considerk_1 vertices in V_1 and k_1 vertices in V_2 say $\{a_1, a_2, a_3, a_4, a_5, a_6\}$, $\{b_1, b_2, b_3, \ldots, b_{k2}\}$. Let $\langle a_1, a_2, a_3, b_1 b_2, b_3\rangle$ is $K_{3,3}$. Include the remaining a_i, b_i, i = 1, 2, 3. Include the remaining b_i, i = 6,7,..,k_2 as arbitrary pendant vertices adjacent to any a_i, i = 1, 2, 3. Graph G_1 is as seen in Fig.5.

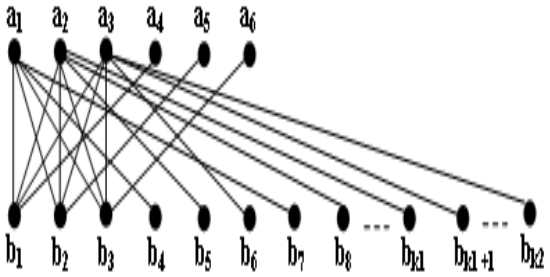


Fig.5.

Since G_1 has atleast k_1 pendant vertices $\{a_4, a_5, \ldots, a_{k1}, b_4, b_5, \ldots, b_{k1}\}$,$\gamma(G_1) \geq k_1$, $\{V_1\}$ dominates G_1. Also $|V_1| = k_1$, implies that V_1isaγ- setfor G_1, since G_1 isabipartite graph $P = \{V_1, V_2\}$ is the onlychromatic partition for G_1 such that V_1 is a γ - set for G_1, implies G_1 is γ - uniquely colorable and non planar.

$\gamma(G) = 3$, $P = \{V_1, V_2\}$, $V_1 = \{a_1, a_2, a_3\}$, $V_2 = \{b_1, b_2, \ldots, b_{k1}\}$,$k_1 \geq 6$,

$\gamma(G) = 4$, $P = |P| = 3V_1 = V_1 = \{a_1, a_2, a_3, a_4\}$, $V_2 = \{b_1, b_2, \ldots, b_{k1}\}$,$k_1 \geq 6$,

$\gamma(G) = 5$, $P = \{V_1, V_2\}$, $V_1 = V_1 = \{a_1, a_2, a_3, a_4, a_5\}$, $V_2 = \{b_1, b_2, \ldots, b_{k1}\}$,$k_1 \geq 6$,are analogus to the above discussion.

$|P| = 3 = P = \{V_1, V_2, V_3\}$, $= K_1$. $|V_2| = k_2$, $|V_3| = k_3$, $k_2, k_3 \geq k_1$.

Planar Construction when $|P| = 3$.

$|P| = 3 = P = \{V_1, V_2, V_3\}$, $= K_1$. $|V_2| = k_2$, $|V_3| = k_3$, $k_2, k_3 \geq k_1$.

Consider a wheel graph with k vertices where $k = k_1 + 2k_i$, where$k_i = \min(k_2, k_3)$. Label the vertices of the wheel in the followingfashion as seen in Fig.6.

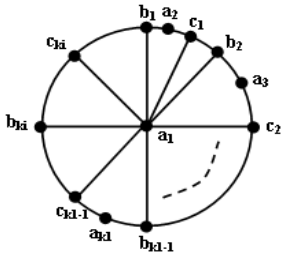


Fig.6

If $k_2 \neq k_3$, then we include the remaining vertices as follows.

Let $k_2 > k_3$. Let $k_2 = k_3 + m$. Label the additional vertices as $\{b_{k3+1}, b_{k3+2}, \ldots, b_{k2}\}$. Include these vertices as seen in Fig.7.

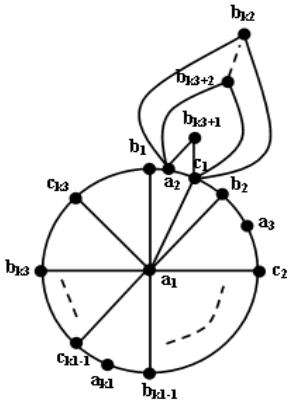


Fig.7

Since $\langle b_1, a_j, c_i\rangle$, i = 1 to k_{1-1}, j = 2 to a_{k1} is P_3eithea$_j$ or b_i or c_i should be included in every possible γ - set for G. $\{a_1, a_2, \ldots, a_{k1}\}$ is a γ- set for G. Also $\{V_1, V_2, V_3\}$ is the only possible chromatic partition for G implies γ - uniquely colorable graph G is planar.

4. Conclusion

In this paper, we provide necessary and sufficient condition for $\overline{G}$and G^* to be γ uniquely colorable and also provide constructive characterization to show that whenever G is γ - uniquely colorable such that $|P| \geq 2$, G can be both planar and non planar.

References

[1] Bing Zhou, "On the maximum number of dominating classes in graph coloring", *Open Journal of Discrete Mathematics*,Vol 6,(2016).pp.70 – 73.

[2] M. Yamuna, A. Elakkiya, "γ - Uniquely colorable graphs", *IOPConf. Series: Materials Science and Engineering* , Vol.263 ,(2017).

[3] M. Yamuna, A. Elakkiya," Planar graph characterization of γ-Uniquely colorable graphs", *IOP Conf. Series: Materials Sci-ence and Engineering* ,Vol263, (2017).

[4] Yamuna, M., Elakkiya, A., "Non domination subdivision stable graphs", *IOP Conf. Series: Materials Science and Engineering*. Vol 263, (2017).

[5] Yamuna, M., Elakkiya, A, "Planar graph characterization of NDSS graphs", *IOP Conf. Series: Materials Science and Engineering* ,Vol 263 ,(2017).

[6] Harary, F,*Graph Theory*, Addison Wesley, Narosa Publishing House, (2001).

[7] Haynes, T.W., Hedetniemi, S. T & Slater, P. J. *Fundamentals of domination in graphs*, New York, Marcel Dekker, (1998).