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Research paper

The solution of convolution-typed Volterra integral equation by G-transform

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Abstract

We would like to consider the solution of convolution-typed Volterra integral equation by using G-transform, a generalized Laplace-typed transform. The tool of G-transform is analyzed to be well applied to convolution-typed Volterra integral equation.

Keywords: *initial value problem; G-transform; Volterra integral equation*

1. Introduction

It is a well-known fact that Volterra integral equation is equivalent to the initial value problem of ordinary differential equation(ODE). Since the tool used to find the solutions in this article is G-transform, a generalized Laplace-typed transform, we would like to check it up first. In a word, G-transform is a comprehensive form of Laplace-typed transform, and this transform has a strong point in the choice of an integer α. This means that we can freely choose α for a variety of problems.
The form of Laplace-typed transform is

$$\int_0^\infty k(s,t)f(t)dt,$$

and Laplace transform has the kernel $k(s,t) = e^{-st}$ as we know already[8]. This Laplace transform can be rewritten as

$$\int_0^\infty e^{-\frac{t}{u}}f(t)dt$$

by $s = 1/u$, and so we proposed the general form of Laplace-typed transform by

$$u^\alpha \int_0^\infty e^{-\frac{t}{u}}f(t)dt$$

as a natural extension[4]. In this comprehensive form, the integer value can be suitably selected in various problems. The value $\alpha = 0$, Laplace transform, has a strong point in the transforms of derivatives, and while, $\alpha = -2$ has a strong point in the transforms of integrals[7]. Sumudu transform[3, 11] has a value of $\alpha = -1$ and Elzaki one[3, 5] has a value of $\alpha = 1$ in the above form.
Theories on integral transforms provide a reasonable tool for solving differential equations[1], and these transforms are meaningful not only in the given space by the inverse transform, but also in the transformed space in itself[4]. The utility of integral transforms can be easily seen in computed tomography(CT) scan or magnetic resonance imaging(MRI). Normally, we obtain the projection data by integral transform, and produce the image with the inverse transform[6]. Volterra's population model is studied for population growth of a species within a closed system by Biazar and Hosseini[2], Sohrabi and Ranjbar deal with the integral-algebraic equations that it is coupled system of Volterra integral equations in [10], and Medina concerns the stability for linear Volterra difference equations in Banach spaces[9].
In this article, we would like to consider the solution of convolution-typed Volterra integral equation by using the tool of G-transform. The obtained result is as follows; The solution of convolution-typed Volterra integral equation

$$y(t) - \int_0^t y(\tau)h(t-\tau)\,d\tau = f(t)$$

can be represented as

$$y(t) = G^{-1}(\frac{F}{1-H}),$$

where $Y = G(y)$, $H = G(h)$, and $F = G(f)$.

2. the solution of convolution-typed Volterra integral equation by G-transform

Convolution-typed Volterra integral equation has the form of

$$y(t)-\int_0^t y(\tau)h(t-\tau)\,d\tau=f(t), \tag{1}$$

where h is the kernel. A useful method to solve these equations is the Adomian decomposition method, and the integral transforms method is possible as well. In here, we preferentially would like to use G-transform method, a generalized Laplace-typed transform, and chosen integer value is $\alpha=-2$ in the definition of G-transform.

Theorem 1. *The equation (1) has the solution of the form of*

$$y(t)=G^{-1}(\frac{F}{1-H})$$

by G-transform, where $Y=G(y)$, $H=G(h)$, and $F=G(f)$.

Proof. Let * be the convolution, and let $Y=G(y)$, $H=G(h)$, and $F=G(f)$. Since equation (1) can be rewritten as

$$y(t)-(y*h)(t)=f(t),$$

taking G-transform on both sides of the equation (1), we have $Y-YH=F$. Organizing this equality, we have $(1-H)Y=F$. Thus,

$$Y=\frac{F}{1-H}$$

and gives the solution

$$y(t)=G^{-1}(\frac{F}{1-H}).$$

□

In the above theorem, we have used the relation $\pounds(f)\pounds(g)=\pounds(f*g)$ for $f*g$ is the convolution of f and g. Note that G-transform has a form of

$$u^{\alpha}\int_0^{\infty}e^{-\frac{t}{u}}f(t)dt,$$

and after this sentence, we would like to use G_{-2}-transform as a representative of G-transform. Using

$$G_{-2}(f)=u^{-2}\int_0^{\infty}e^{-\frac{t}{u}}f(t)dt=\frac{1}{u^2}\cdot F(\frac{1}{u})$$

for Laplace transform $\pounds(f)=F(s)$, we can obtain the table of G_{-2}-integral transforms as follow;

Table 1: Table of G_{-2}-integral transforms[4, 7]

	f(t)	$G_{-2}(f)$
1	1	$1/u$
1	t	1
3	t^n	$n!,\ u^{n-1}$
4	e^{at}	$\frac{1}{u(1-au)}$
5	$\sin at$	$\frac{a}{1+u^2a^2}$
6	$\cos at$	$\frac{1}{u(1+u^2a^2)}$
7	$\sinh at$	$\frac{a}{1-u^2a^2}$
8	$\cosh at$	$\frac{1}{u(1-u^2a^2)}$
9	$e^{at}\cos at$	$\frac{1-au}{u^3[(\frac{1}{u}-a)^2+a^2]}$
10	$e^{at}\sin at$	$\frac{a}{u^2[(\frac{1}{u}-a)^2+a^2]}$
11	te^{at}	$\frac{1}{(1-au)^2}$

Lemma 2. *(The convolution of f and g for G_{-2}-transform[4])*

$$G_{-2}(f*g)=u^2G_{-2}(f)G_{-2}(g)$$

where $*$ is the convolution of f and g.

On the other hand, Note that Volterra integral equation is equivalent to the initial value problem of ODE. Consider a linear ODE with variable coefficients

$$y''+A(t)y'+B(t)y=g(t)$$

with the initial conditions $y(t_0)=y_0$, $y'(t_0)=y_1$. Integrating it twice with respect to y, we obtain

$$y(t)=f(t)+\int_{t_0}^t K(t,\tau)y(\tau)\,d\tau \tag{2}$$

where,

$$K(t,\tau) = -A(\tau) + (\tau - t)[B(\tau) - A'(\tau)], \tag{3}$$

$$f(t) = \int_{t_0}^{t} (t-\tau)g(\tau)d\tau + [A(t_0)y_0 + y_1](t - t_0) + y_0. \tag{4}$$

Let us look at the following examples with this in mind.

Example 3. *(The solution of a linear ODE by using G-transform) Consider a homogeneous linear ODE*

$$y'' + \alpha^2 y = 0$$

with the initial conditions $y(0) = 0$ and $y'(0) = \alpha$, where α is a constant.

Solution. From (2), (3), and (4), we have

$$y(t) = \alpha t - \alpha^2 \int_0^t (t-\tau)y(\tau)d\tau,$$

where $K(t,\tau) = \alpha^2(\tau - t)$ and $f(t) = y_1 t + y_0 = \alpha t$. This integral equation can be represented by convolution as

$$y(t) = \alpha t - \alpha^2 (y * t).$$

Note that $G_{-2}(t) = 1$ and $G_{-2}(sin\ at) = a/(1+u^2a^2)$. Taking G_{-2}-transform, we have

$$Y = \alpha - \alpha^2 u^2 Y$$

for $G(y) = Y$. Thus,

$$Y = \frac{\alpha}{1+\alpha^2u^2}$$

and the solution is

$$y(t) = sin\ \alpha t.$$

Example 4. *Consider the integral equation*

$$y(t) + 2e^t \int_0^t e^{-\tau} y(\tau)\, d\tau = te^t.$$

Solution. This equation is equal to

$$y(t) + 2\int_0^t e^{t-\tau} y(\tau)\, d\tau = y(t) + 2(y * e^t) = te^t. \tag{5}$$

In this article, it is precisely G_{-2}-transform, but let us use G_{-2}-transform with G-transform easily. Taking G_{-2}-transform, we have

$$Y + 2u^2 Y \frac{1}{u(1-u)} = \frac{1}{(1-u)^2}$$

because of $G(e^{at}) = \frac{1}{u(1-au)}$ and $G(te^{at}) = \frac{1}{(1-au)^2}$. Organizing this equality, we have

$$Y(1 + \frac{2u^2}{u(1-u)}) = \frac{1}{(1-u)^2},$$

hence

$$Y = \frac{1}{1-u^2}.$$

Thus,

$$y = G^{-1}(Y) = sin\ ht,$$

where h is hyperbolic function.

Of course, this result is equal to the that of Laplace transform; The equation (5) is represented as

$$Y + 2Y\frac{1}{s-1} = \frac{1}{(s-1)^2}$$

by Laplace transform. Organizing the equality, we obtain

$$Y = \frac{1}{s^2-1}$$

and the solution is $y = sin\ ht$, where h is hyperbolic function.

In [7], we have obtained the solutions of the examples by using G_{-2}-transform as below;

Example 5. *Volterra integral equation of the second kind*

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)d\tau = t$$

has the solution of the form of

$$y(t) = t + \frac{1}{6}t^3.$$

Example 6. *Volterra integral equation of the second kind*

$$y(t) - \int_0^t (1+\tau)y(t-\tau)d\tau = 1 - \sinh t$$

has the solution

$$y(t) = \cosh t$$

.

Example 7. *The solution of*

$$y(t) + \int_0^t (t-\tau)y(\tau)d\tau = 1$$

is $y = \cos t$.

(Checking by the direct calculation) Expanding the given equation, we have

$$y(t) + t \cdot \int_0^t y(\tau)d\tau - \int_0^t \tau y(\tau)d\tau = 1.$$

Differentiating twice with respect to t, we have $y''(t) + y(t) = 0$. Thus, from the $y(0) = 1$ and $y'(0) = 0$ obtained by calculating course, we have the solution $y = \cos t$.

Example 8. *The solution of*

$$y(t) - \int_0^t y(\tau)d\tau = 1$$

is $y = e^t$.

Solution. (By the direct calculation) Differentiating both sides, we obtain $y'(t) - y(t) = 0$ and the solution is $y = e^t$.

(By the property of integration in G_{-2}-transform) We have showed that

$$G_{-2}[\int_0^t f(\tau)d\tau] = uF(u)$$

in [4], where $F(u) = G_{-2}[f(t)]$. Of course, this states that the integration of a function $f(t)$ corresponds to multiplication of $G_{-2}(f)$ by u for G_{-2}-transform. Hence, taking G-transform, we have

$$Y - uY = \frac{1}{u}$$

for $Y = G_{-2}(y)$. Organizing this equality, we have

$$Y = \frac{1}{u(1-u)}$$

and the solution is

$$y = e^t$$

because of $G_{-2}(1) = \frac{1}{u}$ and $G_{-2}(\ e^{at}) = \frac{1}{u(1-au)}$.

(By the property of convolution) Since the given equation is the same as $y - y * 1 = 1$, by lemma 2,

$$Y - u^2 \cdot Y \cdot \frac{1}{u} = \frac{1}{u}.$$

By the simple calculation, we have the solution

$$y = e^t.$$

3. Conclusion

The used transform, G-transform, is a generalized Laplace transform and it is well applied to the convolution-typed Volterra integral equations.

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