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Research paper

Numerical Solution of Time Fractional Parabolic Differential Equations

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Abstract

In this paper, we study the coupling of an approximate analytical technique called reduced differential transform (RDT) with fractional complex transform. The present method reduces the time fractional differential equations in to integer order differential equations. The fractional derivatives are defined in Jumaries modified Riemann-Liouville sense. Result shows that the present technique is effective and powerful for handling the fractional order differential equations.

Keywords: *Reduced differential transform, fractional derivatives, Riemann-Liouvilles fractional derivatives.*

1. Introduction

In recent years, the fractional calculus [1, 2, 3, 4,12] is an interesting topic in current literature. It has attracted many researchers because of its wide application in diverse fields of engineering and physical sciences. The fractional calculus involves different definitions of the fractional differential operators namely Riemann-Liouvilles derivatives, Caputo derivatives, Riez and Grunwald-Letnikov derivatives. Most of the authors follow the caputo or Riemann – Liouvilles form of fractional derivatives to solve the fractional differential equations because their solutions are practically useful. For example Guy Jumarie [15] proposed a new method obtained by using modified Riemmann-Liouville derivatives to solve the fractional nonlinear partial differential equations. Later Zheng-Biao Li et al [9, 13, 16] proposed fractional complex transform method based on modified Riemmann-Liouville derivatives, a novel and universal approach for finding exact solutions of fractional differential equations. Later, many researchers [10, 11] have explored this topic and applied this technique to solve mathematical models based on fractional calculus.

2. Jumaries Riemann-Liouville Fractional Derivatives

$$D_t^{\beta} f(t) = \begin{cases} \frac{1}{\Gamma(-\beta)} \int_0^t (t-\tau)^{-\beta-1}[f(\tau)-f(0)]d\tau & \beta < 0 \\ \frac{1}{\Gamma(-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta-1}[f(\tau)-f(0)]d\tau, & 0<\beta<1 \\ \left[f^{(\beta-m)}(t)\right]^{(m)}, & m \le \beta \le m+1, \quad m \ge 1 \end{cases}$$

(1)

where $f: R \to R$ is a continuous function.
Here, we mention some important properties of the modified Riemann-Liouvilles derivatives [14]:

$(i) D_t^{\beta}(c) = 0, \alpha > 0, c$ is a constant
$(ii) D_t^{\beta}[cf(t)] = cD_t^{\beta} f(t), \alpha > 0$
$(iii) D_t^{\beta} t^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-\beta)} t^{\alpha-\beta}, \ \alpha > \beta > 0$
$(iv) D_t^{\beta}[f(t)g(t)] = \left[D_t^{\beta} f(t)\right] g(t) + f(t)\left[D_t^{\beta} g(t)\right]$
$(v) D_t^{\beta}[f(h(t))] = f_h'(h(t)) D_t^{\beta} h(t)$

3. Fractional complex transform

$$T = \frac{kt^{\alpha_1}}{\Gamma 1+\alpha_1} \tag{2}$$

$$X = \frac{lx^{\alpha_2}}{\Gamma 1+\alpha_2} \tag{3}$$

$$Y = \frac{my^{\alpha_3}}{\Gamma 1+\alpha_3} \tag{4}$$

$$Z = \frac{nz^{\alpha_4}}{\Gamma 1+\alpha_4} \tag{5}$$

where k, l, m, n are unknown constants and $0 < \alpha_1 \le 1$,

$0 < \alpha_2 \le 1, 0 < \alpha_3 \le 1, 0 < \alpha_4 \le 1$.

4. Outline of the method

The basic definition and properties of reduced differential transform (RDT) are given in [8].
The reduced differential transform of v(x, t) at t = 0 is defined as

$$V_k(x) = \frac{1}{k!}\left[\frac{\partial^k v(x,t)}{\partial x^k}\right]_{t=0} \tag{7}$$

The differential inverse transform of $V_k(x)$ is defined as

$$v(x,t) = \sum_{k=0}^{\infty} V_k(x) t^k \tag{8}$$

and from (7) and (8), we have

$$v(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k v(x,t)}{\partial x^k}\right]_{t=0} t^k \qquad (9)$$

Theorem 1. If $w(x,t)=u(x,t)\pm v(x,t)$ then
$W_k(x)=U_k(x)\pm V_k(x)$
Theorem 2. If $w(x,t)=mv(x,t)$ then $W_k(x)=mV_k(x)$
Theorem 3. If $\mathrm{w(x,t)}=\mathrm{m}\frac{\partial^p v(x,t)}{\partial t^p}$ then
$$\mathrm{W_k(x)}=m\frac{(k+\mathrm{p})!}{k!}\mathrm{V_{k+p}(x)}$$
Theorem 4. If $\mathrm{w(x,t)}=x^p t^q$ then
$$\mathrm{W_k(x)}=x^p\delta(k-\mathrm{q})=\begin{cases}x^p if k=\mathrm{q}\\ 0 \quad if k\neq \mathrm{q}\end{cases}$$
Theorem 5. If $\mathrm{w(x,t)}=\mathrm{m}\frac{\partial^p u(x,t)}{\partial x^p}$ then $\mathrm{W_k(x)}=\mathrm{m}\frac{\partial^p U_k(x)}{\partial x^p}$
Theorem 6. If $\mathrm{w(x,t)}=\mathrm{u}(x,t)v(x,t)$ then
$\mathrm{W_k(x)}=\sum_{p=0}^{k}U_p(x)V_{k-p}(x)$
Theorem7. If $\mathrm{w(x,t)}=x^p t^q\mathrm{u}(x,t)$ then
$\mathrm{W_k(x)}=x^p U_{k-q}(x)$
Theorem 8. If $\mathrm{w(x,t)}=t^q u(x,t)$ then
$\mathrm{W_k(x)}=V_{k-q}(x)$
Theorem 9.If $v(x,t)=e^{au(x,t)}$ then
$$V_k(x)=\begin{cases}e^{aU_0(x)} & if\ k=0\\ a\sum_{\mathrm{k_1=0}}^{\mathrm{k}}\left(1-\frac{\mathrm{k_1}}{\mathrm{k}}\right)\mathrm{V_{k_1}(x)U_{k-k_1}(x)} & \text{if } \mathrm{k}\geq 1\end{cases}$$
The main objective of this work is to illustrate the ability and reliability of fractional complex transform based on Jumaries modified Riemann-Liouville derivatives to achieve the explicit solution of time fractional parabolic equations via the method of RDT

5. Numerical Applications

Example 5.1 Consider the time fractional parabolic equation:

$$D_t^{\beta}v=v_{xx}+e^{-v}+e^{-2v},\quad (x,t)\in[0,1]\times[0,1] \qquad (10)$$

with initial condition

$$v(x,0)=log(x+2) \qquad (11)$$

Using the complex transformation [15], we obtain the following integer order PDE:

$$\frac{\partial v}{\partial T}=v_{xx}+e^{-v}+e^{-2v} \qquad (12)$$

Operating differential transform on (12) and using the theorem (9), we obtained the following recurrence formula

$$(k+1)V_{k+1}(x)=\frac{\partial^2}{\partial x^2}V_k(x)+W_k(x)+G_k(x) \qquad (13)$$

From the initial condition (11), we have

$$V_0(x)=log(x+2) \qquad (14)$$

Where
$$W_k(x)=RDT(e^{-v})=\begin{cases}e^{-v_0} & if\ k=0\\ -\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right)W_{k_1}(x)V_{k-k_1}(x) & if\ k>0\end{cases} \qquad (15)$$

$$G_k(x)=RDT(e^{-2\mathrm{v}})=\begin{cases}e^{-2v_0} & if\ k=0\\ -2\sum_{k_1=0}^{k-1}\left(1-\frac{k_1}{k}\right)G_{k_1}(x)V_{k-k_1}(x) & if\ k>0\end{cases} \qquad (16)$$

Substituting the eqn. (14) in to the eqn. (13) and then using the transformed equations (15) and (16), for k = 0, 1, 2, 3, 4, the first few components of $U_k(x)$ are obtained as follows:

$$V_1(x)=\frac{\partial^2}{\partial x^2}V_0(x)+W_0(x)+G_0(x)$$
$$V_1(x)=\frac{1}{x+2} \qquad (17)$$
$$2\mathrm{V_2(x)}=\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{V_1(x)}-\mathrm{W_0U_1}-2\mathrm{G_0V_1}$$
$$\mathrm{V_2(x)}=-\frac{1}{4(\mathrm{x}+2)^2} \qquad (18)$$
$$3\mathrm{V_3(x)}=\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{V_2}-\mathrm{W_0V_2}-\frac{1}{2}\mathrm{W_1V_1}-2\mathrm{G_0V_2}-\mathrm{G_1V_1}$$
$$\mathrm{V_3(x)}=\frac{1}{9(\mathrm{x}+2)^3} \qquad (19)$$
$$4\mathrm{V_4(x)}=\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{V_3}-\mathrm{W_0V_3}-\frac{2}{3}\mathrm{W_1V_2}-\frac{1}{3}\mathrm{W_2V_1}-2\mathrm{G_0V_3}-\frac{4}{3}\mathrm{G_1V_2}-\frac{2}{3}\mathrm{G_2V_1}$$
$$\mathrm{V_4(x)}=-\frac{1}{16(\mathrm{x}+2)^4} \qquad (20)$$
$$5\mathrm{V_5(x)}=\frac{\partial^2}{\partial \mathrm{x}^2}\mathrm{V_4}-\mathrm{W_0V_4}-\frac{3}{4}\mathrm{W_1V_3}-\frac{1}{2}\mathrm{W_2V_2}-\frac{1}{4}\mathrm{W_3V_1}-2\mathrm{G_0V_4}-\frac{3}{2}\mathrm{G_1V_3}-\mathrm{G_2V_2}-\frac{1}{2}\mathrm{G_3}V_1-\mathrm{G_2V_2}-\frac{1}{2}\mathrm{G_3V_1}$$
$$\mathrm{V_5(x)}=\frac{1}{25(\mathrm{x}+2)^5} \qquad (21)$$

Now, substituting the Eqns. (17) - (21) in to the Eqn. (9), we have
$$v(x,t)=log\left(x+\frac{t^{\beta}}{\Gamma(1+\beta)}+2\right) \qquad (22)$$
which is exactly the same as the results obtained by Galerkin collocation method [5,6] and ADM [7] when $\beta=1$.
Example 5.2 The time fractional parabolic equation
$$xD_t^{\beta}v=v_{xx}+v^2,\quad (x,t)\in[0,1]\times[0,1] \qquad (23)$$

with initial condition

$$v(x,0)=x \qquad (24)$$

Using the complex transformation [15], we obtain the following integer order PDE:

$$x\frac{\partial v}{\partial T}=v_{xx}+v^2 \qquad (25)$$

Operating differential transform on (25) and using the theorem (6), we obtained the following recurrence formula:

$$(k+1)xV_{k+1}(x)=\frac{\partial^2}{\partial x^2}V_k(x)+\sum_{k_1=0}^{k}V_{k_1}(x)V_{k-k_1}(x) \qquad (26)$$

From the initial condition of Eqn. (25), we have
$$V_0(x)=x \qquad (27)$$
According to RDT, using the recursive formula (26) and the initial condition (27), we obtain

$$V_k(x)=x \text{ for } k=1,2,3,4\ldots \qquad (28)$$

This implies
$$v(x,t)=x+xT+xT^2+xT^3+\cdots$$
Since $0<T<1$
$$=\frac{x}{1-T}$$
$$=\frac{x}{1-\left(\frac{t^{\beta}}{\Gamma(1+\beta)}\right)} \qquad (29)$$

which is exactly the same as the results obtained by ADM [17] when $\beta=1$.

6. Conclusion

In this work, we successfully presented the coupling technique to achieve the analytical solution of fractional parabolic differential equations. The fractional complex transform requires no special knowledge of fractional calculus and it is extremely accessible to solve other similar type of nonlinear fractional differential equations.

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