International Journal of Engineering & Technology, 7 (4.10) (2018) 758-761

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper



An M/G/1 Retrial Queueing System with Phase Type Services and Working Vacations

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Abstract

A multi phase retrial queue with optional re-service and multiple working vacations is considered. The Probability Generating Function (PGF) of number of customers in the system is obtained by supplementary variable technique. Various system performance measures are discussed.

Keywords: *retrial queues, K - phase service, working vacations.*

1. Introduction

In queueing literature, retrial queueing system has notable amount of works has been done. Retrial queueing system plays a major role in communication systems and networks. Recently, some of the authors like Arivudainambi et al. [1], Artalejo and Gomez-Corral [2], Choudhury and Deka [3], Rajadurai et al. [5, 6], Servi and Finn [7] have developed a queueing models with different aspects.

In this paper, "we have extended the concepts of multi phase of service, optional re-service, multiple working vacations, vacation interruption, breakdowns and repair from the work of Arivudainambi et al. [1]".

2. Model Description

"In this section, the detailed model description as given below:

- ***The arrival process:*** The primary customers arrive from outside of the system according to a Poisson process with rate λ.
- ***The retrial process:*** If an arriving primary customer finds the server free, the customer starts his service immediately. Otherwise, if the server is busy (on working vacation or breakdown), the customers join into an orbit. An arbitrary distribution of inter-retrial times is $R(t)$ and its corresponding Laplace Stieltjes Transform (LST) $R^*(\vartheta)$.
- ***The multiple working vacation process:*** As soon as the orbit becomes empty, the server starts a working vacation. The vacation time describes an exponential distribution with parameter θ. If customers arrive in working vacation period, the service rate of the server is lesser than to regular service. If any customers in orbit at a lower speed service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy period which means vacation interruption. Otherwise, it continues the vacation. At end of a vacation any customers in the orbit, the server switch over to normal working level. Otherwise, the server begins another vacation. The service time in working vacation period follows a general random variable S_v with distribution function $S_v(t)$ and LST $S_v^*(\vartheta)$.
- ***The service process and optional re-service process*:** In this model, we assume that a single server provides K - optional phases of service in succession. The First Phase Service (FPS) is followed by i (i=1,2,…,k) phases of service. After completion of i^{th} phase of service the customer may go to $(i+1)^{th}$ phase with probability p_i or with probability r_i may re-join into i^{th} phase immediately to get a same service again (optional re-service) or may leave the system with probability 1- p_i - r_i = q_i for (i=1,2,…,k), if the service is successful. After completion k^{th} phase service the customers may re-join to k^{th} phase with probability r_k or leaves the system with probability q_k = 1-r_k. The service time follows for all the phases has a general distribution. It is denoted by the random variable S_i with distribution function $S_i(t)$ having LST $S_i^*(\vartheta)$, first and second moments are $\beta_i^{(1)}$ and $\beta_i^{(2)}$ for (i=1,2,…,k). From this model, the modified service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST $S^*(x) = \prod_{i=1}^{k} \Theta_{i-1} S_i^*(x)$ and the expected value is $\beta^{(1)} = \sum_{i=1}^{k} \Theta_{i-1} \beta_i^{(1)}$.

  where $\Theta_i = p_1 p_2 ... p_i$ and $\Theta_0 = 1$.
- ***The breakdown process***: The busy server may breakdown at any stage of service and the serving phase will fail for a short period of time. The breakdowns i.e. server's life times are generated by exponential distribution with rates α_i for i^{th} phase respectively for (i=1,2,…,k).
- ***The repair process:*** As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the

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primary customers till service channel is repaired. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G_i) distributions of the server for i phases are assumed to be arbitrarily distributed with d.f. $G_i(t)$, having LST $G_i^*(\vartheta)$, first and second moments are $g_i^{(1)}$ and $g_i^{(2)}$ for (i=1,2,…,k).

- Various stochastic processes involved in the system are assumed to be independent of each other".

3. Steady state analysis

In this section, we derive the steady state difference-differential equations and the PGFs for the different states.

3.1. The steady state equations

"In steady state, we consider that $R(0) = 0$, $R(\infty) = 1$, $S_i(0) = 0$, $S_i(\infty) = 1$, $S_v(0)=0$, $S_v(\infty) = 1$ are continuous at $x = 0$ and $G_i(0) = 0$, $G_i(\infty) = 1$ are continuous at $y = 0$ (for i =1,2,…,k).

For the conditional completion rates for retrial, regular service on both phases, lower rates service and repair on both phases are $\theta(x)$, $\mu_i(x)$, $\mu_v(x)$ and $\xi_i(y)$ respectively.

i.e., $\theta(x)dx = \dfrac{dR(x)}{1-R(x)}$; $\mu_i(x)dx = \dfrac{dS_i(x)}{1-S_i(x)}$;

$\mu_v(x)dx = \dfrac{dS_v(x)}{1-S_v(x)}$; $\xi_i(y)dy = \dfrac{dG_i(y)}{1-G_i(y)}$.

Let $R^0(t)$, $S_i^0(t)$, $S_v^0(t)$ and $G_i^0(t)$ be the elapsed times (retrial, normal service, working vacation and repair) at time t

To obtain a bivariate Markov process $\{C(t), N(t);\ t \geq 0\}$, where $C(t)$ is the server state (0,1,2,3,4,5) depending on the server is free, busy on i^{th} phase, re-service on i^{th} phase, on working vacation or repair on i^{th} phase. $N(t)$ denotes the number of customers in the orbit at time t. The sequence of random vectors forms a Markov chain which is embedded and ergodic if and only if $\rho < R^*(\lambda)$, for our system to be stable, where

$$\rho = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} + \lambda \sum_{i=1}^{k} r_i \Theta_{i-1} \beta_i^{(1)} \left(1 + \alpha_i g_i^{(1)}\right) - \sum_{i=1}^{k-1} \Theta_i M_{1i}.$$

For the process $\{N(t),\ t \geq 0\}$, the probabilities and probability densities are $P_0(t) = P\{C(t) = 0,\ N(t) = 0\}$ and corresponding probabilities are $P_n(x,t)$, $\Pi_{i,n}(x,t)$, $\Omega_{i,n}(x,t)$, $Q_n(x,t)$, $R_{i,n}(x,y,t)$. depends on the server states for $t \geq 0$, $x \geq 0$ and $n \geq 1$ and $i = 1,2,...,k$.

Then the limiting densities for the above probabilities, we get $P_n(x)$, $\Pi_{i,n}(x)$, $\Omega_{i,n}(x)$, $Q_n(x)$, $R_{i,n}(x,y)$. for $(x,y) > 0$ and $n \geq 0$, $(i = 1,2,...,k)$

By the method of supplementary variable method, the system of equations are as follows (i=1,2,…,k)

$$(\lambda+\theta)P_0 = \sum_{i=1}^{k-1} q_i \int_0^\infty \Pi_{i,0}(x)\mu_i(x)dx + (1-r_k)\int_0^\infty \Pi_{k,0}(x)\mu_k(x)dx + \sum_{i=1}^{k}\int_0^\infty \Omega_{i,0}(x)\mu_i(x)dx + \int_0^\infty Q_0(x)\mu_v(x)dx + \theta P_0 \tag{1}$$

$$\frac{dP_n(x)}{dx} + (\lambda + a(x))P_n(x) = 0,\ n \geq 1. \tag{2}$$

$$\frac{d\Pi_{i,n}(x)}{dx} + (\lambda + \alpha_i + \mu_i(x))\Pi_{i,n}(x) = \lambda\Pi_{i,n-1}(x) + \int_0^\infty R_{i,n}(x,y)\xi_i(y)dy,\ n \geq 1. \tag{3}$$

$$\frac{d\Omega_{i,n}(x)}{dx} + (\lambda + \alpha_i + \mu_i(x))\Omega_{i,n}(x) = \lambda\Omega_{i,n-1}(x) + \int_0^\infty R_{i,n}(x,y)\xi_i(y)dy,\ n \geq 1. \tag{4}$$

$$\frac{dQ_n(x)}{dx} + (\lambda + \mu_v(x))Q_n(x) = \lambda Q_{n-1}(x),\ n \geq 1. \tag{5}$$

$$\frac{\partial R_{i,n}(x,y)}{\partial y} + (\lambda + \xi_i(y))R_{i,n}(x,y) = \lambda R_{i,n-1}(x,y),\ \ n \geq 1. \tag{6}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P_n(0) = \sum_{i=1}^{k-1} q_i \int_0^\infty \Pi_{i,n}(x)\mu_i(x)dx + (1-r_k)\int_0^\infty \Pi_{k,n}(x)\mu_k(x)dx + \sum_{i=1}^{k}\int_0^\infty \Omega_{i,n}(x)\mu_i(x)dx + \int_0^\infty Q_n(x)\mu_v(x)dx,\ n \geq 1. \tag{7}$$

$$\Pi_{1,n}(0) = \int_0^\infty P_{n+1}(x)a(x)dx + \lambda\int_0^\infty P_n(x)dx + \theta\int_0^\infty Q_n(x)dx,\ n \geq 1. \tag{8}$$

$$\Pi_{i,n}(0) = p_{i-1}\left(\int_0^\infty \Pi_{i-1,n}(x)\mu_{i-1}(x)dx\right),\ n \geq 0,\ \text{for}(i = 2,3,...,k). \tag{9}$$

$$\Omega_{i,n}(0) = r_i \int_0^{\infty} \Pi_{i,n}(x)\mu_i(x)dx,\ n \geq 0,\ \text{for}(i = 1,2,3,...,k). \tag{10}$$

$$Q_n(0) = \begin{cases} \lambda P_0, & n = 0 \\ 0, & n \geq 1 \end{cases} \tag{11}$$

$$R_{i,n}(x,0) = \alpha_i \left(\Pi_{i,n}(x) + \Omega_{i,n}(x)\right),\ \ n \geq 0,\ \text{for}(i = 1,2,...,k). \tag{12}$$

The normalizing condition is

$$P_0 + \sum_{n=1}^{\infty}\int_0^{\infty} P_n(x)dx + \sum_{i=1}^{k}\sum_{n=0}^{\infty}\left(\int_0^{\infty}\Pi_{i,n}(x)dx + \int_0^{\infty}\Omega_{i,n}(x)dx + \int_0^{\infty}\int_0^{\infty} R_{i,n}(x,y)dxdy\right) + \sum_{n=0}^{\infty}\int_0^{\infty} Q_n(x)dx = 1. \tag{13}$$

3.2. The steady state solution

The PGF technique is used here to obtain the steady state solution as follows:

$$P(x,z) = \sum_{n=1}^{\infty} P_n(x)z^n;\ \ \Pi_i(x,z) = \sum_{n=0}^{\infty}\Pi_{i,n}(x)z^n;\ \ \Omega_i(x,z) = \sum_{n=0}^{\infty}\Omega_i(x)z^n;$$

$$Q(x,z) = \sum_{n=0}^{\infty} Q_n(x)z^n;\ R_i(x,y,z) = \sum_{n=0}^{\infty} R_{i,n}(x,y)z^n;$$

Multiplying the equations (2) - (12) by z^n and summing over n, (n = 0,1,2,...) for (i = 1,2,…,k). Solving the partial differential equations then the limiting PGFs $P(x,z)$, $\Pi_i(x,z)$, $\Omega_i(x,z)$, $Q(x,z)$ and $R_i(x,y,z)$. "

"**Theorem 3.1** *Under the stability condition* $\rho < R^*(\lambda)$, *the stationary joint distributions of the number of customers in the orbit when server being idle, busy, re-service, on working vacation and under repair is given by* (*for* i=1,2,…,k)

$$P(z) = \frac{zP_0\left(1-R^*(\lambda)\right)\left(\left(S_v^*\left(A_v(z)\right)-1\right)+V(z)\left(\sum_{i=1}^{k}\left(q_i + r_i S_i^*\left(A_i(z)\right)\right)\Theta_{i-1}B_i^*\left(A_i(z)\right)\right)\right)}{\left\{z-\left(R^*(\lambda)+z(1-R^*(\lambda))\right)\left(\sum_{i=1}^{k}\left(q_i + r_i S_i^*\left(A_i(z)\right)\right)\Theta_{i-1}B_i^*\left(A_i(z)\right)\right)\right\}} \tag{14}$$

$$\Pi_i(z) = \left\{\lambda P_0\Theta_{i-1}B_{i-1}^*\left(A_{i-1}(z)\right)\left(\begin{matrix}\left(S_v^*\left(A_v(z)\right)-1\right)\\ \left(R^*(\lambda)+z(1-R^*(\lambda))\right)+zV(z)\end{matrix}\right)\left(1-S_i^*\left(A_i(z)\right)\right)\right\} \Big/ A_i(z)\times Dr(z) \tag{15}$$

$$\Omega_i(z) = \left\{\begin{matrix} r_i\lambda P_0\Theta_{i-1}B_{i-1}^*\left(A_{i-1}(z)\right)\left(S_i^*\left(A_i(z)\right)\right)\left(1-S_i^*\left(A_i(z)\right)\right)\\ \times\left(\left(S_v^*\left(A_v(z)\right)-1\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)+zV(z)\right)\end{matrix}\right\} \Big/ A_i(z)\times Dr(z) \tag{16}$$

$$Q(z) = \left\{\lambda P_0 V(z)/\theta\right\} \tag{17}$$

$$R_i(z) = \left\{\begin{matrix}\alpha_i\lambda P_0\Theta_{i-1}B_{i-1}^*\left(A_{i-1}(z)\right)\left(1+r_iS_i^*\left(A_i(z)\right)\right)\left(1-S_i^*\left(A_i(z)\right)\right)\left(1-G_i^*\left(b(z)\right)\right)\\ \left(\left(S_v^*\left(A_v(z)\right)-1\right)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)+zV(z)\right)\end{matrix}\right\} \Big/ b(z)\times A_i(z)\times Dr(z) \tag{18}$$

where

$$P_0 = \frac{R^*(\lambda)-\rho}{\left\{\left(R^*(\lambda)-\rho\right)+\left(1-S_v^*(\theta)\right)\left(\left(\lambda/\theta\right)(1-\rho)+\rho\left(1-R^*(\lambda)\right)+\sum_{i=1}^{k}\Theta_{i-1}\lambda(1+r_i)\beta_i^{(1)}\left(1+\alpha_i g_i^{(1)}\right)\left(R^*(\lambda)+\left(\lambda/\theta\right)\right)\right)\right\}} \tag{19}$$

$$\rho = \sum_{i=1}^{k}\Theta_{i-1}M_{1i} + \lambda\sum_{i=1}^{k} r_i\Theta_{i-1}\beta_i^{(1)}\left(1+\alpha_i g_i^{(1)}\right) - \sum_{i=1}^{k-1}\Theta_i M_{1i}$$

$$M_{1i} = \lim_{z\to 1} dB_i^*[A_i(z)]\Big/dz = \sum_{j=1}^{i} \lambda\beta_j^{(1)}\left(1+\alpha_j g_j^{(1)}\right)$$

$$M_{2i} = \lim_{z\to 1} d^2B_i^*[A_i(z)]\Big/dz^2 = \sum_{j=1}^{i} \lambda^2\left(\alpha_j\beta_j^{(1)}g_j^{(2)} + \beta_j^{(2)}\left(1+\alpha_j g_j^{(1)}\right)^2\right)$$

Proof Integrating the limiting PGF with respect to *x* and y, define the partial PGF as, $P(z)$, $\Pi_i(z)$, $\Omega_i(z)$, $Q(z)$, $R_i(z)$. The value of P_0 can be determined using the normalizing condition by setting $z = 1$ in (14)-(18) and applying L-Hospitals rule whenever necessary and we get

$$P_0 + P(1) + Q(1) + \sum_{i=1}^{k}\left(\Pi_i(1) + \Omega_i(1) + R_i(1)\right) = 1.$$

Corollary: *Under the stability condition* $\rho < R^*(\lambda)$,

PGF of number of customers in the system

$$K_s(z) = P_0 + P(z) + Q(z) + z\sum_{i=1}^{k}\left(\Pi_i(z) + \Omega_i(z) + R_i(z)\right)$$

PGF of number of customers in the orbit

$$K_o(z) = P_0 + P(z) + Q(z) + \sum_{i=1}^{k}\left(\Pi_i(z) + \Omega_i(z) + R_i(z)\right)$$

Substituting the equations (14)-(19) in the above results, then it can be obtained by direct calculation.

4. Performance measures

The excepted number of customers in the orbit (L_q)

$$L_q = K_o'(1) = \lim_{z\to 1}\frac{d}{dz}K_o(z)$$

The excepted number of customers in the system (L_s)

$$L_s = K_s'(1) = \lim_{z\to 1}\frac{d}{dz}K_s(z)$$

The average time a customer spends in the system (W_s) is $L_s = \lambda W_s$

The average time a customer spends in the queue (W_q) is $L_q = \lambda W_q$.

The server availability probability A_v is given by

$$A_v = 1 - \lim_{z\to 1}\left(\sum_{i=1}^{k} R_i(z)\right) = 1 - \sum_{i=1}^{k} R_i(1).$$

The server failure frequency is obtained as

$$F_f = \alpha_i \times \sum_{i=1}^{k}\left(\Pi_i + \Omega_i\right).$$ ”

5. Conclusion

In this work, we analysed an unreliable retrial queueing system with multi phases of service, multiple working vacation and vacation interruption. The steady state results and important system performance measures are obtained. The practical application of this model is in computer and communication network where messages are processed in multi phases by a single server.

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