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Research paper

Numerical Solution of the Non-Linear Singular Systems from Fluid Dynamics Using Leapfrog Method

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Abstract

In this Research article, the new proposed technique for examination of the singular non-linear problem from Fluid dynamics utilizing Leapfrog Method is exhibited. To represent the adequacy of the Leapfrog Method, the different cases in singular non-linear system from Fluid dynamics have been considered and contrasted and the Single Term Haar Wavelet Series and results are observed to be extremely precise. The arrangements of the non-singular particular problems from Fluid dynamics are introduced in the tables. The proposed method can be effortlessly utilized in an advanced Computer.

Keywords: *Fluid Dynamics, Single-Term Haar Wavelet Series, Singular Non-Linear Systems.*

1. Introduction

The singular non-linear structure doesn't concede any explanatory arrangement and subsequently a numerical technique must be utilized. A general numerical method for their answer has not already existed. The improvement of non-straight particular problems has been considered by a few scientists [5,6]. However, no were near solutions accessible in that article. In some examination of neural systems, both particular problems [7] and non-linear problems [8] have been utilized. Lewis et al. [9] have been talked about broadly in the writing. In any case, Lewis et al. [10] applies for the time-invariant case. As of later, Balachandran and Murugesan [11], Murugesan et al. [12] acquired the numerical arrangement of a particular non-problem problem from Fluid dynamics. The investigation of non-linear problem from Fluid dynamics utilizing broadened Runge-Kutta techniques. S. Sekar et al. [13] examine the same Singular non-Linear problem from Fluid dynamics utilizing single-term Haar wavelet arrangement strategy. Murugesan et al. [14] explained about the nonlinear particular problems from Fluid dynamics utilizing the Runge Kutta techniques in light of assortment of means. In this article, the same problem from Fluid dynamics.

2. Leapfrog Method

The Eulers Method for the derivative is of the form

$$f(t,y)=y',\quad y_0=y(t_0),\quad y\in R^d$$

$\frac{[y(t+h)-y(t)]}{h}\approx y'$. with $t_n+h=t_{n+1},\quad n=0,1,...t_0$ .Hence, $N=1$.

Modifying the distinction remainder gives

$$y_n+hf(t_n,y_n)=y_{n+1},\ n=0,1,...t_0\ .$$

The proposed method we define t_n as

$$t_n+h=t_{n+1},\quad n=0,1,..t_0\ ,\ hy'\left(t+\frac{h}{2}\right)\approx y(t+h)-y(t)$$

$$\frac{[y(t+2h)..y(t)]}{h}\approx y'(t+h)$$

and

$$y_{n-1}+2hf(t_n,y_n)=y_{n+1},\ n=0,1,..,t_0\ .$$

is a linear $m=2-$step method, with

$a_0=0,\ b_0=2\ \ a_1=1,\ b_1=0$ and $b_{-1}=-1$,

This circumstance recommends a potential instability present in multistep strategies, which must be tended to when we examine them two qualities y_0 and y_1, are

$$y_{n-1}+2hf(t_n,y_n)=y_{n+1},\ n=0,1,2,...,t_0$$

$$f(t,y)=y',y_0=y(t_0),\quad y\in R^d$$

3. Structure of Flow Equations as a Non-Linear

Consider figure 1, Here, m_i denote to the hub mass stream rate in sub channel i and w denote to the cross stream rate per unit length, accepted positive if the stream is from first sub channel to second sub channel.

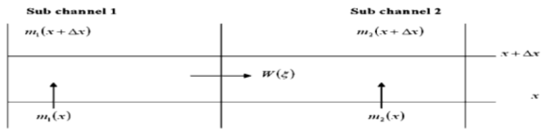

Fig. 1: Flow Variables

Continuity:

$$\frac{dm_i}{dx} = -w$$

Axial momentum:

$$\frac{d}{dx}(m_1u_1) + w\left[H(w)u_1 + H(-w)u_2\right] = -F_1 - A_1\frac{dp_1}{dx}$$

Energy:

$$\frac{d}{dx}(m_1h_1) = q_1 - w\left[H(w)h_1 + H(-w)h_2\right]$$

The second sub channel can be obtained from these by substituting w and w and by interchanging subscripts 1 and 2. In this equation set, H is heavy side unit step function. F represents pressure loss per unit length due to friction, A is the cross-sectional area, q represents the heat energy added per unit length, and the variables, u, p and h stand for particle velocity, pressure and enthalpy respectively.

$$Cw|W| = p_1 - p_2,$$

where C is a cross-flow friction factor.

Consider, The incompressible coolant, steady Cross-sectional, no change in enthalpy; and the frictional pressure loss

$$m_1u_1F = F_1,$$

where F is a Steady. With the above assumptions:

$$-w = \frac{dm_1}{dx},$$

$$\varepsilon^{-1}\left\{\frac{1-2m_1}{2} + 2w\left[1 - H(w)m_1 + H(-w)(m_1 - 1)\right]\right\} = \frac{d}{dx}(w|w|)$$

To make the above derived system into the symmetric form, take

$$m_1 - \frac{1}{2} = x,\ \frac{w}{2} = y,\ x = t.$$

Hence we get,

$$-2y = \frac{dx}{dt},\ (4\varepsilon)^{-1} = \left[-x + 2(y - 2x|y|)\right]\frac{d}{dt}(y|y|)$$

Replacing x by x_1 and y by x_2, we have

$$-2x_2 = \dot{x}_1,\ (4\varepsilon)^{-1}\left[-x_1 + 2(x_2 - 2x_1|x_2|)\right] = \frac{d}{dt}(x_2|x_2|)$$

An experiment is made the values of x_2 and ε given below

$(i)\, x_2 > 0$ and $\varepsilon \neq 0$,

$(ii)\, x_2 < 0$ and $\varepsilon \neq 0$,

$(iii)\, x_2 > 0$ and $\varepsilon = 0$ and

$(iv)\, x_2 < 0$ and $\varepsilon = 0$.

From first two cases ε has been varied from $10^0, 10^1, 10^2, \ldots 10^7$

Case (i)

if $x_2 > 0$ $\varepsilon \neq 0$, we have

$$\dot{x}_1 = -2x_2 \text{ and } 8\varepsilon x_2\dot{x}_2 = -x_1 + 2x_2 - 4x_1x_2.$$

$$i.e.,\ \begin{pmatrix}0 & -2\\ -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ -4x_1x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 8\varepsilon x_2\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix},$$

The above equation is of the form (Evans et al. [15]).

$$K\left[x(t)\right]\dot{x}(t) = Ax(t) + f\left[x(t)\right].$$

when $x_2 > 0$ and $\varepsilon \neq 0$,

$$\frac{1}{2\varepsilon}\left[\frac{-x_1}{\dot{x}_1} - 1 + 2x_1\right] = \ddot{x}_1, \text{ where } \frac{-\dot{x}_1}{2} = x_2.$$

Hence the above system

$$\phi(\varepsilon)\, f(t, x_1, \dot{x}_1) = \ddot{x}_1, \text{ Where } \frac{1}{2\varepsilon} = \phi(\varepsilon).$$

Case (ii) When $x_2 < 0$, $\varepsilon \neq 0$, In this equation becomes

$$2x_2 = \dot{x}_1 \text{ and } x_1 + 2x_2 + 4x_1x_2 = 8\varepsilon x_2\dot{x}_2$$

$$i.e.,\ \begin{pmatrix}0 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 4x_1x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 8\varepsilon x_2\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix},$$

This is of the form (Evans et al. [15]).

$$K\left[x(t)\right]\dot{x}(t) = Ax(t) + f\left[x(t)\right].$$

when $x_2 < 0$ and $\varepsilon \neq 0$, $\frac{1}{2\varepsilon}\left[\frac{x_1}{\dot{x}_1} + 1 + 2x_1\right] = \ddot{x}_1$, where $\frac{\dot{x}_1}{2} = x_2$.

i.e., $\phi(\varepsilon)\, f(t, x_1, \dot{x}_1) = \ddot{x}_1$, Where $\frac{1}{2\varepsilon} = \phi(\varepsilon)$.

Case(iii) When $x_2 > 0$ $\varepsilon = 0$, In this equation becomes

$$-2x_2 = \dot{x}_1 \text{ and } -x_1 + 2x_2 - 4x_1x_2 = 8\varepsilon x_2\dot{x}_2$$

$$i.e.,\ \begin{pmatrix}0 & -2\\ -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ -4x_1x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix},$$

$$i.e.,\ K\left[x(t)\right]\dot{x}(t) = Ax(t) + f\left[x(t)\right].$$

$$i.e.,\ \frac{x_1}{2x_1 - 1} = \dot{x}_1, \text{ Where } \frac{-\dot{x}_1}{2} = x_2.$$

Make it as a second order differential equations as

$$\frac{-\dot{x}_1}{(2x_1 - 1)^2} = \ddot{x}_1, \text{ Where } \frac{-\dot{x}_1}{2} = x_2.$$

Hence $f(t, x_1, \dot{x}_1) = \ddot{x}_1$.

Case(vi) When $\varepsilon = 0$ and $x_2 < 0$, In this equation becomes

$$2x_2 = \dot{x}_1 \text{ and } 0 = -x_1 - 2x_2 - 4x_1x_2$$

$$i.e.,\ \begin{pmatrix}0 & 2\\ -1 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ -4x_1x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix},$$

$$i.e.,\ K\left[x(t)\right]\dot{x}(t) = Ax(t) + f\left[x(t)\right].$$

$$i.e.,\ \frac{x_1}{2x_1 + 1} = \dot{x}_1, \text{ Where } \frac{\dot{x}_1}{2} = x_2.$$

Now, $\frac{-\dot{x}_1}{(2x_1 + 1)^2} = \ddot{x}_1$, Where $\frac{-\dot{x}_1}{2} = x_2$.

Hence $f(t, x_1, \dot{x}_1) = \ddot{x}_1$.

3.1. Discrete Solutions for Non-Linear Singular System from Fluid Dynamics

The motivation behind this paper is to discover discrete solution for the rearranged two channel model of an atomic reactor problem from Fluid dynamics under each of the four cases mentioned about in the part 3. On account of (*i*) and (*ii*), the problem has been diminished to a solitary problem, which is additionally changed over into a second request condition. Consequently it has been broke down by the accompanying numerical strategies by the method for deciding the discrete solutions at various time interims:

(i) Single-term Haar wavelet series

(ii) Leapfrog method

The above method has been applied nuclear reactor core problem explained in part 3 as a subsequent cases

(i) If $\varepsilon \neq 0$ and $x_2 < 0$

(ii) If $\varepsilon \neq 0$ and $x_2 > 0$,

by varying the parameter ε from 10^0 to 10^7 with $x_1(0) = 1$, $\dot{x}_1(0) = 1$ and the results are given in the Tables 1 - 4 and the discrete solution for the cases

(iii) If $\varepsilon = 0$ and $x_2 > 0$

(iv) If $\varepsilon = 0$ and $x_2 < 0$

have been determined using single-term Haar wavelet series method with $x_1(0)=1$, $\dot{x}_1(0)=1$ and the results are given in the Tables 5 - 6.

The Leapfrog method, discussed in section 2, has been applied to determine the approximate solutions for all four cases of the nuclear reactor core problem discussed in section 3.

(i) when $x_2 > 0$ and $\varepsilon \neq 0$,

(ii) when $x_2 < 0$ and $\varepsilon \neq 0$

have been determined using the Leapfrog method by varying the parameter ε from 10^0 to 10^7 with $x_1(0)=1$, $\dot{x}_1(0)=1$ and the results are given in the Tables 7 – 10 and the discrete solution for the cases

(iii) when $x_2 > 0$ and $\varepsilon = 0$

(iv) when $x_2 < 0$ and $\varepsilon = 0$ [i.e., singular systems]

have been determined using Leapfrog method with $x_1(0)=1$, $\dot{x}_1(0)=1$ and the results are given in the Tables 11 - 12.

Discussions

The problem has been studied under different cases (specified part 3) for different time *t* using the single-term Haar wavelet series method and Leapfrog method. The solution was obtained by using Leapfrog method and it was mentioned that the single-term Haar wavelet series method failed when $\varepsilon \geq 10^3$. This proposed technique, it has been established that the single-term Haar wavelet series method are sufficient enough for ε .

If $\varepsilon = 0$ in *(iii)* and *(iv)* the problem decreases to a non-linear problem for both $x_2 > 0$ and $x_2 < 0$. (refer Tables 1 - 12). The solution obtained by single-term Haar wavelet series method and Leapfrog method, coincide with each other (refer Tables 1 - 12). When $\varepsilon \geq 10^6$, the solution obtained for the given problem converges and remains stable. Hence, the proposed method is more suitable for the given problem.

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Table 1: Solutions of Equation Case (i) and STHWS Method for x_1

Time *t*	$\varepsilon=10^0$	$\varepsilon=10^1$	$\varepsilon=10^2$	$\varepsilon=10^3$	$\varepsilon=10^4$	$\varepsilon=10^5$	$\varepsilon=10^6$	$\varepsilon=10^7$
0	1	1	1	1	1	1	1	1
0.5	0.5	1.50105	1.5001	1.5	1.5	1.5	1.5	1.5
1	2.103502	2.00852	2.00084	2.00008	2.00001	2	2	2
1.5	2.907169	2.5293	2.50282	2.50028	2.50003	2.5	2.5	2.5
2	4.13519	3.07107	3.00671	3.00067	3.00007	3	3	3
2.5	6.119246	3.64267	3.51314	3.5013	3.50013	3.50001	3.5	3.5
3	9.381907	4.25438	4.02279	4.00225	4.00022	4.00002	4	4
3.5	14.767	4.91829	4.53632	4.53632	4.50036	4.50003	4.50001	4.50001
4	23.65592	5.6486	5.05446	5.05446	5.00053	5.00004	5.00002	5.00002
4.5	38.32171	6.46194	5.57791	5.57791	5.50076	5.50006	5.50003	5.50003
5	62.5103	7.37769	6.10744	6.10744	6.00104	6.0001	6.00004	6.00004

Table 2: Solutions of Equation Case (i) and STHWS Method for x_2

Time *t*	$\varepsilon=10^0$	$\varepsilon=10^1$	$\varepsilon=10^2$	$\varepsilon=10^3$	$\varepsilon=10^4$	$\varepsilon=10^5$	$\varepsilon=10^6$	$\varepsilon=10^7$
0	1	1	1	1	1	1	1	1
0.5	-0.535368	-0.50316	-0.50031	-0.500003	-0.50003	-0.5	-0.5	-0.5
1	-0.670201	-0.51292	-0.50125	-0.500125	-0.50001	-0.5	-0.5	-0.5
1.5	-0.972082	-0.52991	-0.50283	-0.500281	-0.50003	-0.5	-0.5	-0.5
2	-1.538084	-0.5551	-0.50505	-0.500501	-0.50005	-0.50005	-0.5	-0.5
2.5	-2.515008	-0.5898	-0.50793	-0.500782	-0.50008	-0.50001	-0.5	-0.5
3	-4.147465	-0.63563	-0.51148	-0.501127	-0.50011	-0.50001	-0.5	-0.5

3.5	-6.847252	-0.69456	-0.51572	-0.501535	-0.50015	-0.50001	-0.5	-0.5
4	-11.30009	-0.76884	-0.52067	-0.502007	-0.5002	-0.50002	-0.5	-0.5
4.5	-18.64044	-0.86107	-0.52636	-0.502542	-0.50025	-0.50003	-0.5	-0.5
5	-30.74066	-0.97417	-0.53282	-0.503141	-0.50031	-0.50003	-0.5	-0.5

Table 3: Solutions of Equation Case (ii) and STHWS Method for x_1

Time *t*	$\varepsilon = 10^0$	$\varepsilon = 10^1$	$\varepsilon = 10^2$	$\varepsilon = 10^3$	$\varepsilon = 10^4$	$\varepsilon = 10^5$	$\varepsilon = 10^6$	$\varepsilon = 10^7$
0	1	1	1	1	1	1	1	1
0.5	1.770253	1.527938	1.50281	1.500281	1.500028	1.500003	1.5	1.5
1	3.227253	2.123737	2.012484	2.00125	2.000125	2.000012	2	2
1.5	5.743905	2.806125	2.530888	2.503093	2.50031	2.500031	2.5	2.5
2	9.967638	3.595289	3.059891	3.005599	3.0006	3.00006	3	3
2.5	16.98114	4.51351	3.601371	3.510154	3.501016	3.500101	3.5	3.5
3	28.578028	5.585829	4.15721	4.015746	4.001575	4.000157	4	4
3.5	47.720726	6.840786	4.729303	4.522964	4.502297	4.500229	4.50001	4.50001
4	79.296951	8.311201	5.319559	5.031993	5.0032	5.000318	5.00002	5.00002
4.5	131.36757	10.035072	5.929907	5.543022	5.504303	5.5000427	5.50003	5.50003
5	217.22435	12.056558	6.562268	6.056237	6.005626	6.000559	6.00004	6.00004

Table 4: Solutions of Equation Case (ii) and STHWS Method for x_2

Time *t*	$\varepsilon = 10^0$	$\varepsilon = 10^1$	$\varepsilon = 10^2$	$\varepsilon = 10^3$	$\varepsilon = 10^4$	$\varepsilon = 10^5$	$\varepsilon = 10^6$	$\varepsilon = 10^7$
0	1	1	1	1	1	1	1	1
0.5	1.069232	0.55884	0.505932	0.500594	0.500059	0.500006	0.5	0.5
1	1.905654	0.6333587	0.513728	0.501375	0.500138	0.500014	0.5	0.5
1.5	3.232944	0.732259	0.523392	0.502343	0.500234	0.5000234	0.5	0.5
2	5.390528	0.849773	0.534929	0.503499	0.50035	0.500035	0.5	0.5
2.5	8.928052	0.99083	0.548344	0.504843	0.500484	0.500049	0.5	0.5
3	14.747292	1.158551	0.563649	0.506373	0.500637	0.500064	0.500002	0.5
3.5	24.332659	1.356814	0.580855	0.508092	0.500809	0.500081	0.500005	0.5
4	40.130196	1.590331	0.599979	0.509998	0.501	0.5001	0.500008	0.5
4.5	66.171783	1.86475	0.621044	0.512091	0.501209	0.500121	0.500011	0.5
5	109.10431	2.186782	0.644074	0.514372	0.501437	0.500144	0.500014	0.5

Table 5: Solutions of Equation Case (iii) and STHWS Method for x_1 and x_2

`Time *t*	x_1	x_2
0	1	1
0.5	1.428211	-0.38467
1	1.791537	-0.34678
1.5	2.127465	-0.32681
2	2.447541	-0.31418
2.5	2.757087	-0.30538
3	3.059052	-0.29885
3.5	3.355264	-0.29378
4	3.646944	-0.2872
4.5	3.934948	-0.28639
5	4.219906	-0.2836

Table 6: Solutions of Equation Case (iv) and STHWS Method for x_1 and x_2

`Time *t*	x_1	x_2
0	1	1
0.5	1.488729	0.479521
1	1.961758	0.467443
1.5	2.424938	0.459403
2	2.881311	0.453635
2.5	3.332675	0.449281
3	3.780187	0.445871
3.5	4.224638	0.443124
4	4.666598	0.44086
4.5	5.106484	0.438962
5	5.544618	0.437346

Table 7: Solutions of Equation Case (i) and Leapfrog Method for x_1

Time t	$\varepsilon = 10^0$	$\varepsilon = 10^1$	$\varepsilon = 10^2$	$\varepsilon = 10^3$	$\varepsilon = 10^4$	$\varepsilon = 10^5$	$\varepsilon = 10^6$	$\varepsilon = 10^7$
0	1	1	1	1	1	1	1	1
0.5	0.500001	1.50105	1.50019	1.50009	1.50009	1.5	1.5	1.5
1	2.103502	2.00852	2.00084	2.00008	2.00001	2	2	2
1.5	2.907169	2.52939	2.50282	2.50028	2.50003	2.5	2.5	2.5
2	4.135199	3.07107	3.00671	3.00067	3.00007	3	3	3
2.5	6.119246	3.64267	3.51314	3.5013	3.50013	3.50001	3.5	3.5
3	9.381907	4.25438	4.02279	4.00225	4.00022	4.00002	4	4
3.5	14.76799	4.91829	4.53632	4.53632	4.50036	4.50003	4.50001	4.50001
4	23.65592	5.64869	5.05446	5.05446	5.00053	5.00004	5.00002	5.00002
4.5	38.32171	6.46194	5.57791	5.57791	5.50076	5.50006	5.50003	5.50003
5	62.51039	7.37769	6.10744	6.10744	6.00104	6.00019	6.00004	6.00004

Table 8: Solutions of Equation Case (i) and Leapfrog Method for x_2

Time *t*	$\varepsilon = 10^0$	$\varepsilon = 10^1$	$\varepsilon = 10^2$	$\varepsilon = 10^3$	$\varepsilon = 10^4$	$\varepsilon = 10^5$	$\varepsilon = 10^6$	$\varepsilon = 10^7$
0	1	1	1	1	1	1	1	1
0.5	-0.535368	-0.50316	-0.50031	-0.500003	-0.50003	-0.5	-0.5	-0.5

1	-0.670201	-0.51292	-0.50125	-0.500125	-0.50001	-0.5	-0.5	-0.5
1.5	-0.972082	-0.52991	-0.50283	-0.500281	-0.50003	-0.5	-0.5	-0.5
2	-1.538084	-0.55519	-0.50505	-0.500501	-0.50005	-0.50005	-0.5	-0.5
2.5	-2.515008	-0.58989	-0.50793	-0.500782	-0.50008	-0.50001	-0.5	-0.5
3	-4.147465	-0.63563	-0.51148	-0.501127	-0.50011	-0.50001	-0.5	-0.5
3.5	-6.847252	-0.69456	-0.51572	-0.501535	-0.50015	-0.50001	-0.5	-0.5
4	-11.30009	-0.76884	-0.52067	-0.502007	-0.50029	-0.50002	-0.5	-0.5
4.5	-18.64044	-0.86107	-0.52636	-0.502542	-0.50025	-0.50003	-0.5	-0.5
5	-30.74066	-0.97417	-0.53282	-0.503141	-0.50031	-0.50003	-0.5	-0.5

Table 9: Solutions of Equation Case (ii) and Leapfrog Method for x_1

Time *t*	$\varepsilon = 10^0$	$\varepsilon = 10^1$	$\varepsilon = 10^2$	$\varepsilon = 10^3$	$\varepsilon = 10^4$	$\varepsilon = 10^5$	$\varepsilon = 10^6$	$\varepsilon = 10^7$
0	1	1	1	1	1	1	1	1
0.5	1.7702536	1.5279384	1.50281	1.500281	1.500028	1.5000035	1.5	1.5
1	3.2272538	2.1237375	2.012484	2.001255	2.000125	2.0000122	2	2
1.5	5.7439054	2.8061257	2.530888	2.503093	2.500319	2.5000317	2.5	2.5
2	9.9676389	3.5952898	3.059891	3.005599	3.000699	3.0000649	3	3
2.5	16.981142	4.5135187	3.601371	3.510154	3.501016	3.5001018	3.5	3.5
3	28.578028	5.5858293	4.15721	4.015746	4.001575	4.0001572	4	4
3.5	47.720726	6.8407866	4.729303	4.522964	4.502297	4.5002298	4.50001	4.50001
4	79.296951	8.3112018	5.319559	5.031993	5.003299	5.0003189	5.00002	5.00002
4.5	131.36757	10.035072	5.929907	5.543022	5.504303	5.5000427	5.50003	5.50003
5	217.22435	12.056558	6.562268	6.056237	6.005626	6.0005599	6.00004	6.00004

Table 10: Solutions of equation Case (ii) and Leapfrog method for x_2

Time *t*	$\varepsilon = 10^0$	$\varepsilon = 10^1$	$\varepsilon = 10^2$	$\varepsilon = 10^3$	$\varepsilon = 10^4$	$\varepsilon = 10^5$	$\varepsilon = 10^6$	$\varepsilon = 10^7$
0	1	1	1	1	1	1	1	1
0.5	1.0692324	0.5588483	0.505932	0.5005946	0.500059	0.5000066	0.5	0.5
1	1.9056584	0.6333587	0.513728	0.5013750	0.500138	0.5000141	0.5	0.5
1.5	3.2329445	0.7322594	0.523392	0.5023438	0.500234	0.5000234	0.5	0.5
2	5.3905289	0.8497732	0.534929	0.5034992	0.500356	0.5000355	0.5	0.5
2.5	8.9280521	0.9908322	0.548344	0.5048437	0.500484	0.5000491	0.5	0.5
3	14.747292	1.1585519	0.563649	0.5063731	0.500637	0.5000646	0.500002	0.5
3.5	24.332659	1.3568148	0.580855	0.5080929	0.500809	0.5000811	0.500005	0.5
4	40.130196	1.5903316	0.599979	0.5099982	0.501384	0.5001683	0.500008	0.5
4.5	66.171783	1.8647593	0.621044	0.5120917	0.501209	0.5001218	0.500011	0.5
5	109.10431	2.1867828	0.644074	0.5143724	0.501437	0.5001449	0.500014	0.5

Table 11: Solutions of equation Case (iii) and Leapfrog method for x_1 and x_2

Time t	x_1	x_2
0	1	1
0.5	1.428211	-0.38467
1	1.791537	-0.34678
1.5	2.127465	-0.32681
2	2.447541	-0.31418
2.5	2.757087	-0.30538
3	3.059052	-0.29885
3.5	3.355264	-0.29378
4	3.646944	-0.2872
4.5	3.934948	-0.28639
5	4.219906	-0.2836

Table 12: Solutions of equation Case (iv) and Leapfrog method for x_1 and x_2

Time t	x_1	x_2
0	1	1
0.5	1.488729	0.479521
1	1.961758	0.467443
1.5	2.424938	0.459403
2	2.881311	0.453635
2.5	3.332675	0.449281
3	3.780187	0.445871
3.5	4.224638	0.443124
4	4.666598	0.44086
4.5	5.106484	0.438962
5	5.544618	0.437346