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Research paper

A New Hybrid PRP-MMR Conjugate Gradient Methods with Exact Line Search

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Abstract

Conjugate gradient (CG) methods are an important class of methods for unconstrained optimization, especially for large-scale problems. Recently, they have been much studied. In this paper, we propose a new hybrid conjugate gradient method for solving unconstrained optimization problems, which is a convex combination of an earlier version of Polak- Ribiere and Polyak (PRP) and a recent modification of Mouiyad Bani Yousef (MMR) method. The proposed method is proved globally convergent under exact line search. This is supported by the results of the numerical tests. The numerical performance of the new hybrid CG method is reported to be more efficient compared with previous CG methods. Numerical experiments are made for two combinations of the new hybrid method and the PRP conjugate gradient method. The initial results show that one of the hybrid methods is especially effective for the given test problems.

Keywords*: hybrid conjugate gradient method; sufficient descent property; global convergence; unconstrained optimization; large-scale optimization.*

1. Introduction

It is well known that the conjugate gradient method plays a special role in nonlinear optimization today. Although a large number of optimization algorithms which are faster or more robust than the conjugate gradient method are available, many engineers and mathematicians still use the conjugate gradient method to solve the large-scale optimization problem for its simplicity and low memory requirement [1, 2]. The nonlinear conjugate gradient method is designed to solve the following unconstrained optimization problem

$$\min f(x),\ x \in R^n \tag{1}$$

where $f: R^n \to R$ is a continuously differentiable function. The iterative formula commonly used for solving (1) is given by:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0,1,2, \dots, \tag{2}$$

where x_k is the current iteration point, and $\alpha_k \geq 0$ is the stepsize obtained using the following exact line search formula:

$$f(x_k + \alpha_k d_k) = \min f(x_k + \alpha d_k) \tag{3}$$

The search direction d_k, is calculated using the CG formula which is defined by

$$d_k = \begin{cases} -g_k & if \quad k = 0, \\ -g_k + \beta_k d_{k-1} & if \quad k \geq 1, \end{cases} \tag{4}$$

where $\beta_k \in R$ is known as the CG coefficient that characterizes different CG methods. The parameter g_k denotes the gradient of $f(x)$ at the point x_k. Some examples of well-known formulas for β_k are Polak-Ribiere-Polyak (PRP) [3, 4], Fletcher-Reeves (FR) [5], Wei et al. [6], the 'Aini-Rivaie-Mustafa (ARM) method [7], Liu-Storey (LS) [8], Conjugate Descent (CD) by Fletcher [9] and Dai-Yuan (DY) [10]. Their formulas are given as follows:

$$\beta_k^{PRP} = \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2} \tag{5}$$

$$\beta_k^{FR} = \frac{g_k^T g_k}{\|g_{k-1}\|^2} \tag{6}$$

$$\beta_k^{LS} = \frac{g_k^T (g_k - g_{k-1})}{-d_{k-1} g_{k-1}} \tag{7}$$

$$\beta_k^{ARM} = -\frac{m_k \|g_k\|^2 - |g_k^T g_{k-1}|}{m_k g_{k-1}^T d_{k-1}}$$

where

$$m_k = \frac{\|d_{k-1} + g_k\|}{\|d_{k-1}\|} \tag{8}$$

$$\beta_k^{WYL} = \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} g_k^T g_{k-1}}{\|g_{k-1}\|^2} \tag{9}$$

$$\beta_k^{CD} = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}} \tag{10}$$

$$\beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}} \tag{11}$$

Numerous researchers have studied the convergence properties of CG method. Zoutendijk [11] proved that the FR method is globally convergent when used with the exact line search. Later, Al-Baali [12] and Touti-Ahmed and Storey [13] extended the proof for Wolfe line search. Gilbert and Nocedal [14] had further analyzed the global convergence of algorithm related to the FR method using the inexact line search with strong Wolfe conditions.

Recently, much effort has been made to design new formulas for CG coefficient that has good numerical performance and converge globally under various line searches. For some of the current references on nonlinear CG method, refer to Powell [15], Hager and Zhang [16], Andrei [17], Wei et al. [6], Sofi et al. [18]. Abashar et al. [19], Jusoh et al. [20] and Rivaie et al. [21, 22].

Hybrid CG methods hold an important role in solving large-scale unconstrained optimization. This is due to the part it plays in achieving better computational performance as well as retaining the strong global convergence properties of the various methods. Numerous researches and modifications have been done recently which focus mainly on hybridization of the different categories of the CG methods. These include Dai and Yuan [23], and Andrei [24-27]. The studies concentrate on the projection of various CG algorithms, usually with the aim of preventing the jamming phenomenon. Recently, Touati-Ahmed and Storey [13], suggested a hybrid method combining PRP and FR methods. This method possesses some nice properties of the PRP method, and the FR method define as

$$\beta_k^{HUS} = \max\left\{0, \min\left(\beta_k^{PRP}, \beta_k^{FR}\right)\right\} \tag{12}$$

Motivated by this idea, we proposed a new hybrid CG method between the PRP [3, 4] method, and MMR [28], where the PRP method is given in (5) and the MMR method is defined as

$$\beta_k^{MMR} = \frac{m_k \|g_k\|^2 - \left(g_k^T g_{K-1}\right)}{m_k \|g_{k-1}\|^2}$$

where

$$m_k = \frac{\|d_{k-1} + g_k\|}{\|d_{k-1}\|} \tag{13}$$

In this paper, we proposed two kinds of hybrid methods that is based on the method by Mouiyad-Mamat-Rivaie (MMR) [28] and the method of Polak-Ribiere-Polyak (PRP) [3, 4] in Section 2. The convergence analysis of the new algorithm under exact line search is shown in Section 3. In Section 4, we evaluate the numerical efficiency of our new hybrid CG method by comparing its performance with other hybrid CG methods. Finally, section 5 gives the conclusion.

2. New CG Coefficient Formula

In this section, we present the new coefficient as follows

$$\beta_k^{MMR-PRP} = \max\left\{0, \min\left(\beta_k^{MMR}, \beta_k^{PRP}\right)\right\} \tag{14}$$

where MMR-PRP stands for Mouiyad, MMR method, and PRP method. The new formula possesses some good properties of the PRP method and also the MMR method. The algorithm is given as follows:

Algorithm 2.1

1st Step: Initialization. Given $x_0 \in R^n$, $d_0 = -g_0, k = 0$.
2nd Step: Compute β_k based on (14).
3rd Step: Compute d_k based on (4). If $\|g_k\| = 0$, then stop.
4th Step: Compute stepsize based on (3).
5th Step: Update the new point based on (2)
6th Step: Convergence test and stopping criteria.
If $f(x_k) < f(x_{k+1})$ and $\|g_k\| \leq \varepsilon$, then stop.
Otherwise, set $k := k + 1$ and return to Step 1.

3. Convergence Analysis of MMR-PRP Method

In this section, the convergence properties of our new CG coefficient, $\beta_k^{MMR-PRP}$, will be studied. For the above algorithm to be convergent, it should fulfil the sufficient descent condition and possess global convergence properties.

3.1. Sufficient descent condition

The sufficient descent condition is given as follows:

$$g_k^T d_k \leq -c\|g_k\|^2 \quad \text{for } k \geq 0 \text{ and } c > 0 \tag{15}$$

Theorem 1
Consider a CG method with the search direction (4) and $\beta_k^{MMR-PRP}$, given as (14), then condition (15) holds for all $k \geq 0$ and $c > 0$.

Proof: If k = 0, then $g_0^T d_0 = -c\|g_0\|^2$. Hence, condition (15) holds true. We also need to show that for $k \geq 0$, condition (15) will also hold true. From (4), we have

$$d_{k+1} = -g_{k+1} + \beta_{k+1}^{MMR-PRP} d_k \tag{16}$$

Multiply both sides of (16) by g_{k+1}, then we have,

$$g_{k+1}^T d_{k+1} = g_{k+1}^T\left(-g_{k+1} + \beta_{k+1}^{MMR-PRP} d_k\right),$$

$$= -\|g_{k+1}\|^2 + \left(\beta_{k+1}^{MMR-PRP} g_{k+1}^T d_k\right) \tag{17}$$

For exact line search, $g_{k+1}^T d_k = 0$. Thus,

$$g_{k+1}^T d_{k+1} = -\|g_{k+1}\|^2$$

which implies that d_{k+1} is a sufficient descent direction. Hence, $g_k^T d_k \leq -c\|g_k\|^2$ holds. The proof is completed.

3.2. Global convergence properties

To analyze the global convergence of the CG method, the following assumption is often needed.

Assumption 1

i) $f(x)$ is bounded from below on the level set $\ell = \{x \in R^n | f(x) \leq f(x_0)\}$, where x_0 is the starting point and f is a continuously differentiable function in a neighborhood N of the level set ℓ.

ii) The gradient $g(x)$ is Lipschitz continuous in N, hence there exists a constant $l > 0$ such that $\|g(x) - g(y)\| \leq l\|x - y\|$ for any $x, y \in N$.

To make our convergence proof easier, we first simplify our new $\beta_k^{MMR-PRP}$ from (14). This implies that

$$0 \leq \beta_{k+1}^{MMR-PRP} \leq \frac{\|g_k\|^2}{\|g_{k-1}\|^2} = \beta_K^{FR} \tag{18}$$

It is obvious that when $\beta_k^{MMR} < \beta_k^{PRP}$, then the algorithm will follow MMR or otherwise. If either then β_k^{MMR} or β_k^{PRP} is negative then the $\beta_K = 0$ or the algorithm will respect as on SD-method.

Under Assumption 1, we have the following lemma, which was proven by Zoutendijk [11]. This lemma also holds for the exact minimization rule, the Goldstein rules and the Wolfe rules as shown in [29].

Lemma 1: Suppose that Assumption 1 holds true. Consider any CG method of the form (4), where d_k is a descent search direction and α_k satisfies the one-dimensional search direction condition. Then, the following condition holds, which is given by

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \tag{19}$$

The proof of this lemma can be seen in [9]. By using Lemma 1, we can obtain the following convergence theorem of the MMR-PRP CG method.

Theorem 2

Suppose that Assumption 1 holds true. Consider any CG method in the form of (2) and (4) where α_k is obtained by exact minimization rule. Also, suppose that Assumption 1 and the descent condition hold true. Then,

$$\lim_{k \to \infty} \|g_k\| = 0 \tag{20}$$

Proof: We prove Theorem 2 by contradiction. Suppose that Theorem 2 is not true, then a constant a > 0 exists such that

$$\|g_k\| \geq a \tag{21}$$

Case1: $\beta_k^{MMR} < \beta_k^{PRP}$

Rewriting (4) as

$$d_{k+1} + g_{k+1} = \beta_{k+1}^{MMR} d_k$$

and squaring both sides of the equation, we obtain

$$\|d_{k+1}\|^2 = (\beta_{k+1}^{MMR})^2 \|d_k\|^2 - 2g_{k+1}^T d_{k+1} - \|g_{k+1}\|^2 \tag{22}$$

Dividing both sides by $(g_{k+1}^T d_{k+1})^2$, then

$$\frac{\|d_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2} = \frac{(\beta_{k+1}^{MMMR})^2 \|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} - \frac{2}{g_{k+1}^T d_{k+1}} - \frac{\|g_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2}$$
$$= \frac{(\beta_{k+1}^{MMR})^2 \|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} - \left(\frac{1}{\|g_{k+1}\|} - \frac{\|g_{k+1}\|^2}{g_{k+1}^T d_{k+1}}\right) + \frac{1}{\|g_{k+1}\|^2}$$
$$\leq \frac{(\beta_{k+1}^{MMMR})^2 \|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} + \frac{1}{\|g_{k+1}\|^2} \tag{23}$$

Applying (18) yields

$$\leq \left(\frac{\|g_k\|^2}{\|g_{k-1}\|^2}\right)^2 \frac{\|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} + \frac{1}{\|g_{k+1}\|^2}$$

Since $(g_{k+1}^T d_{k+1})^2 = \|g_k\|^4$. Hence,

$$= \frac{\|d_k\|^2}{\|g_{k-1}\|^4} + \frac{1}{\|g_{k+1}\|^2}$$
$$\frac{\|d_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2} \leq \frac{\|d_k\|^2}{\|g_{k-1}\|^4} + \frac{1}{\|g_{k+1}\|^2} \tag{24}$$

Note that, $g_k^T d_k = -\|g_k\|^2$ and $(g_{k+1}^T d_{k+1})^2 = \|g_k\|^4$. Hence,

$$\frac{\|d_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2} = \frac{\|d_{k+1}\|^2}{\|g_{k+1}\|4} \leq \frac{\|d_k\|^2}{\|g_{k-1}\|4} + \frac{1}{\|g_{k+1}\|^2}$$

By noting that $\|d_0\|^2 = \|g_0\|^2$ and using recursive, we obtain

$$\frac{\|d_k\|^2}{\|g_k\|4} \leq \sum_{i=0}^{k} \frac{1}{\|g_i\|^2} \tag{25}$$

Therefore, from (19) and (23), it shows that

$$\frac{\|g_k\|4}{\|d_k\|^2} \geq \frac{a^2}{k}$$

The proof of MMR completed.

Case 2: $\beta_k^{PRP} < \beta_k^{MMR}$

Rewriting (4) as

$$d_{k+1} + g_{k+1} = \beta_{k+1}^{PRP} d_k$$

and squaring both sides of the equation, we obtain

$$\|d_{k+1}\|^2 = (\beta_{k+1}^{PRP})^2 \|d_k\|^2 - 2g_{k+1}^T d_{k+1} - \|g_{k+1}\|^2 \tag{24}$$

Dividing both sides by $(g_{k+1}^T d_{k+1})^2$, then

$$\frac{\|d_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2} = \frac{(\beta_{k+1}^{PRP})^2 \|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} - \frac{2}{g_{k+1}^T d_{k+1}} - \frac{\|g_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2}$$
$$= \frac{(\beta_{k+1}^{PRP})^2 \|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} - \left(\frac{1}{\|g_{k+1}\|} - \frac{\|g_{k+1}\|^2}{g_{k+1}^T d_{k+1}}\right) + \frac{1}{\|g_{k+1}\|^2}$$
$$\leq \frac{(\beta_{k+1}^{PRP})^2 \|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} + \frac{1}{\|g_{k+1}\|^2} \tag{25}$$

Applying (18) yields

$$\leq \left(\frac{\|g_k\|^2}{\|g_{k-1}\|^2}\right)^2 \frac{\|d_k\|^2}{(g_{k+1}^T d_{k+1})^2} + \frac{1}{\|g_{k+1}\|^2}$$

Since $(g_{k+1}^T d_{k+1})^2 = \|g_k\|^4$. Hence,

$$= \frac{\|d_k\|^2}{\|g_{k-1}\|^4} + \frac{1}{\|g_{k+1}\|^2}$$
$$\frac{\|d_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2} \leq \frac{\|d_k\|^2}{\|g_{k-1}\|^4} + \frac{1}{\|g_{k+1}\|^2} \tag{26}$$

Note that, $g_k^T d_k = -\|g_k\|^2$ and $(g_{k+1}^T d_{k+1})^2 = \|g_k\|^4$. Hence,

$$\frac{\|d_{k+1}\|^2}{(g_{k+1}^T d_{k+1})^2} = \frac{\|d_{k+1}\|^2}{\|g_{k+1}\|^4} \leq \frac{\|d_k\|^2}{\|g_{k-1}\|^4} + \frac{1}{\|g_{k+1}\|^2}$$

By noting that $\|d_0\|^2 = \|g_0\|^2$ and using recursive, we obtain

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \sum_{i=0}^{k} \frac{1}{\|g_i\|^2} \tag{27}$$

Therefore, from (21) and (27), it shows that

$$\frac{\|g_k\|^4}{\|d_k\|^2} \geq \frac{a^2}{k}$$

The proof of PRP completed. Hence,

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} = \infty$$

This contradicts the Zoutendijk condition in Lemma 1. Therefore, both cases proved.

4. Results and Discussion

In this section, we analyze the efficiency of $\beta_k^{MMR-PRP}$ by testing it on a set of unconstrained optimization test problems and comparing its performance with three hybrid CG methods (HDYZ, HUS, and LS-CD). These comparisons are based on number of iteration and CPU time. The stopping criterion is set at $\|g_k\| \leq 10^{-6}$. The test functions used are selected from [30] and tested with different dimensions $(2 \leq n \leq 10000)$. For each of the test problems, three initial points of varying distances from the solution point are used in order to test the global convergence properties of the new formula.
The list of tests and initial points are presented in Table 1.

Table 1: A list of problem functions

No.	Functions	n	Initial Points
1	Six Hump	2	(3,3)(-21,-21)(43,43)
2	Three Hump	2	(3,3)(22,22)(62,62)
3	Booth	2	(-8,-8)(49,49)(80,80)
4	Price	2	(1,1)(10,10)(-10,10)
5	TRECCANI	2	(20,20)(79,79)(-2.1 ,2)
6	Zettl	2	(6,6)(20,20)(-100,-100)
7	Colville	4	(99,…,99)(-20,..,-20) (-150,...,-150)
8	Raydan 2	2,4	(1,3)(-17,16)(2,24)
9	Dixon and Price	2,4	(-55,…,-55)(85,...,85) (101,…,101)
10	ARrwhead	2,4,10	(3,..,3)(23,..,23)(81,..,81)
11	Styblinski-Tank	2,4,10	(-0.1,0.2)(25,30)(80,80)
12	Extended penalty	2,4,10	(2,..,2)(19,..,19)(59,..,59)
13	Hager	2,4,10,100	(6,…,6)(-17,…,-17) (-78,…,-78)
14	Raydan 1	2,4,10,100	(1,..,1)(10,..,10)(20,..,20)
15	Freudestein and Roth	2,4,10,100	(2,...,2)(19.2,…,19.2) (0.5,30)
16	Extended Maratos	2,4,10,100	(18,…,18) (-84,-106) (-4.5,…,-4.5)
17	Generlized Tridiagonal 1	2,4,10,100	(1,1)(20,20)(40,40)
18	Extended Tridiagonal 1	2,4,10,100,500,1000	(3,3)(8,8)(24.6,24.7)
19	Generalized Quartic 1	2,4,10,100 500,1000	(10,…,10)(20,…,20) (80,…,80)
20	Extended Beale	2,4,10,100 500,1000	(-1.3,…,-1.3) (5,…,5) (11.3,…,11.3)
21	Extended Denschnb	2,4,10,100 500,1000	(3,…,3)(23,…,23) (200,…,200)
22	Extended Himmelblau	2,4,10,100 500,1000	(1,5)(10,…,10) (41,…,41)
23	Extended Shallow	2,4,10,100 500,1000	(11,..,11)(-1,..,-1) (-50,110)
24	Extended Strait	2,4,10,100 500,1000	(10,...,10)(50,…,50) (100,…, 100)
25	Sum Squares	2,4,10,100 500,1000	(3.7,...,3.7)(15,…,15) (35,…,35)
26	Extended White and Holst	2,4,10,100 500,1000	(-1.3,…,-1.3)(10,100) (11,...11)
27	NONDIA	2,4,10,100,500,1000	(-1,…,-1)(12.1,…,12.1) (25,…,25)
28	Qing	2,4,10,100 500,1000	(-2,…,-2)(7,…,7) (80,…,80)
29	Qudratic 2	2,4,10,100 500,1000	(0.5,…,0.5)(20,…,20) (80,…,80)
30	Diagonal 4	2,4,10,100 500,1000	(10,…,10)(50,...,50) (100,…,100)
31	Diagonal 2	2,4,10,100,500,1000	(1,…,1)(5,…,5) (15,…,15)
32	Extended Quadratic penalty2	2,4,10,100 500,1000	(0.5,…,0.5)(21,..,21) (50,…,50)
33	Extended Rosenbrock	2,4,10,100 500,1000	(2,…,2)(20,…,20) (80,…,80)
34	Extended Block Diagonal1	2,4,10,100	(0.1,...,0.1)(10,...,10)

		500,1000	(10.5,9)
35	TRIDIA	2,4,10,100 500,1000	(1,..,1)(10,…,10) (50,…,50)

We employ MATLAB version R2015a subroutine programming to execute the algorithms on a PC computer with Intel(R) Core™ i3-4005U CPU @ 1.70GHz processor, 4GB RAM, and Windows 10 Professional operating system. The Sigma Plot 10 programme is used to graph the data as shown in Figures 1 and 2.

We apply the method of performance profile proposed by Dolan and More [31] to study the performance of all tested CG methods. This approach evaluates and compares the performance of the set of interested solvers *S* on a whole set of test problems P.

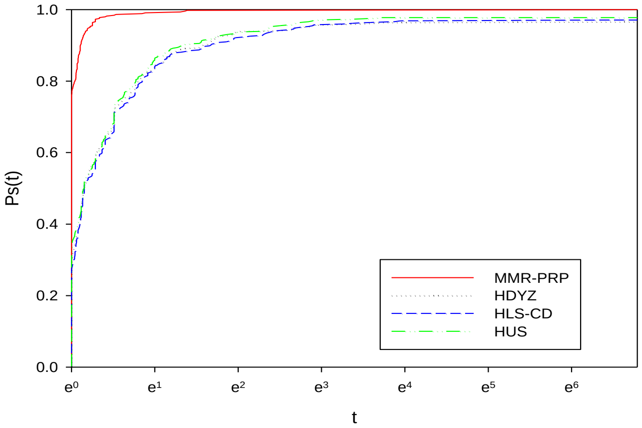

Fig. 1: Performance profile based on the number of iterations

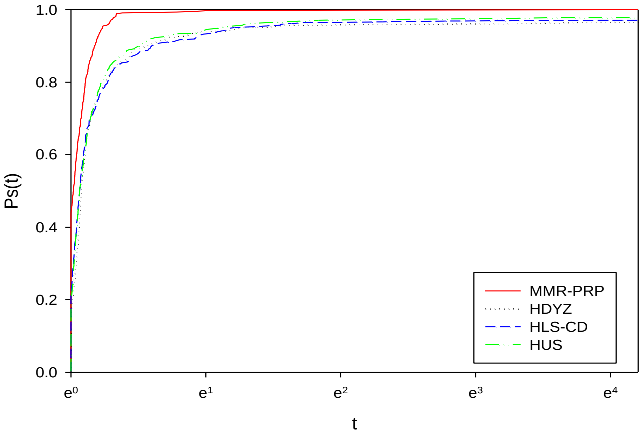

Fig. 2: Performance profile based on the CPU time

The value of   in the performance profile is the probability that the solver will win over the rest of the solvers. In general, a solver with high value of   or positioned at the top right of the figure is regarded as the best solver. Figures 1 and 2 plot the performance of our new method in comparison with the HDYZ, LS-CD, and HUS methods. By observing the top position in both figures, which are occupied by MMR-PRP, it is clear that MMR-PRP has the best performance regarding a number of iteration and CPU time. And CPU time. The HDYZ, LS-CD, and HUS methods are only able to solve 95%, 96 % and 97% of all the test problems, respectively. On the contrary, our new formula solves 100% of the test problem functions and reaches, which makes it the best solver for this test.

5. Conclusion

In this paper, we have examined a new hybrid method for solving unconstrained optimization and proved its convergence concerning sufficient descent and the global convergence properties under exact line search. The numerical results prove that the new modification is better than other tested CG parameters. In the future, we intend to test this new $\beta_k^{MMR-PRP}$ using the inexact line search.

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