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Research paper

A Backward Difference Formulation for Solving Doffing-Van Der Pol Type Oscillators

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Abstract

The study of chaotic motions in periodic self-excited oscillators are an area of interest in science and engineering. The current research proposes a numerical solution in backward difference form for solving these chaotic motions in periodic- self excited oscillators. The research conducted in this study focuses on chaotic motions in the form of Duffing-Van Der Pol Oscillators because of its various applications. A backward difference formulation in predictor-corrector (PeCe) mode is introduced for solving these Duffing-Van Der Pol directly. Numerical simulations provided will show the accuracy of the PeCe backward difference formulation compared against known viable methods. Results will also show that the PeCe backward formulation is a feasible alternative for solving Duffing-Van Der Pol oscillators.

Keywords: *Backward difference; ODEs; Duffing-Van Der Pol oscillators.*

1. Introduction

The Duffing-Van Der Pol oscillator is commonly used to describe electrical circuits with many applications in science and engineering. Research conducted in [1] show that the study of chaotic phenomena in the area of nonlinear periodic self-excited oscillators has been an area of attraction of researchers. The focus of attraction is the Duffing-Van Der Pol oscillator. The governing equation of the Duffing-Van Der Pol oscillator is the following second order ordinary equation (ODE)

$$y'' - \mu\left(1-t^2\right)y' + \omega_0 t + \beta t^3 = f\left(t\right) \tag{1}$$

where $\mu > 0$, ω_0 and β are constants and $f(t)$ is a function of time.

When dealing with higher order ODEs, including second order ODEs, a popular approach for obtaining a numerical solution is the multistep method. Multistep methods were popularized by authors such as [2-4]. Suleiman [4], designed a multistep method using a divided difference code for solving stiff and non-stiff higher order ODEs directly. The divided difference code was referred as the Direct Integration (DI) method. The DI method proposed by in [4] provides an algorithm for solving ODEs that can be programmed for parallel computing. Because of its applicability in parallel computing, fundamental ideas adopted from [4] was continued by author [5-11] and more.

The only downside of the DI method is the demanding calculations required by the divided differences formulation which needs re-computing the integration coefficients at every step change.

The current research implements a backward difference rather than a divided difference formulation because it produces a more elegant error formulae which provides an advantage when formulating and coding the method. As an added advantage, a backward difference formulation requires calculating the integrations only once.

In the next section, the formulation of the backward difference coefficient is provided.

2. Derivation of the Explicit and Implicit Backward Differences Method for Second Order ODEs

For the specific purpose of this study, we consider an initial value problem (IVP) in the form of a second order ODE,

$$y'' = f\left(t, \tilde{Y}\right), \tag{2}$$

with $\tilde{Y}(\alpha) = \tilde{\eta}$ in the interval $\alpha \le t \le \beta$, where $\tilde{Y}(t) = (y, y')$ and $\tilde{\eta}(t) = (\eta, \eta')$

Integrating the second order ODE (2) once gives

$$y'(t_{n+1}) = y'(t_n) + \int_{t_n}^{t_{n+1}} f(y, y')\,dt.$$

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Newton-Gregory backward difference interpolation polynomial, $P_n(t)$ as

$$P_n(t)=\sum_{i=0}^{k-1}(-1)^i\binom{-s}{i}\nabla^i f_n, \qquad s=\frac{t-t_n}{h}.$$

which interpolates $f(y,y')$ at k back values.

Next, $f(y,y')$ is approximated by $P_n(t)$ and by changing the limit of integration

$$y'(t_{n+1})=y'(t_n)+h\sum_{i=0}^{k-1}\gamma_{1,i}\nabla^i f_n\, ds, \quad \gamma_{1,i}=(-1)^i\int_0^1\binom{-s}{i}ds. \quad (3)$$

Let $G_1(t)$ be the generating function for the coefficients $\gamma_{1,i}$ and defined as follows:

$$G_1(t)=\sum_{i=0}^{\infty}\gamma_{1,i}t^i$$

By substituting $\gamma_{1,i}$ from (3) in $G_1(t)$ and through mathematical induction, the generating function can be denoted as

$$G_1(t)=-\left[\frac{(1-t)^{-1}}{log(1-t)}-\frac{1}{log(1-t)}\right].$$

This gives the backward difference coefficients formulation, $\gamma_{1,k}$

$$\gamma_{1,0}=1, \quad \gamma_{1,k}=1-\sum_{i=0}^{k-1}\left(\frac{\gamma_{1,i}}{k-i+1}\right), \quad k=1,2,\ldots.$$

Then, integrating (2) twice and repeating the same mathematical process as above, we obtain

$$y(t_{n+1})=y(t_n)+hy'(t_n)+h^2\sum_{i=0}^{k-1}\gamma_{2,i}\nabla^i f_n\, ds.$$

Thus, it can be perceived that the second order generating functions, $G_2(t)$ and $G_2^*(t)$ can be written similar as in [6], where

$$G_2(t)=\left[\frac{1}{log(1-t)}-\frac{G_1(t)}{log(1-t)}\right],$$

$$G_2^*(t)=\left[\frac{(1-t)}{log(1-t)}-\frac{G_1^*(t)}{log(1-t)}\right].$$

and has the following relationship

$$G_2^*(t)=(1-t)G_2(t), \quad d=1,2.$$

Using the relationship between the predictor-corrector generating function, we obtain the following second order explicit and implicit coefficients

$$\sum_{i=0}^{k}\gamma_{d,i}^{*}=\gamma_{d,k}, \quad d=1,2.$$

The following are examples a few explicit and implicit integration coefficients and their coding algorithm

Table 1: Explicit and Implicit coefficients for k from 0 to 5

k	**0**	**1**	**2**	**3**	**4**	**5**
$\gamma_{1,k}$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$
$\gamma_{2,k}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{19}{180}$	$\frac{3}{32}$	$\frac{683}{10080}$
$\gamma_{3,k}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{7}{240}$	$\frac{17}{720}$	$\frac{41}{2016}$	$\frac{731}{40320}$
$\gamma_{1,k}^*$	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$
$\gamma_{2,k}^*$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{24}$	$-\frac{7}{360}$	$-\frac{17}{1440}$	$-\frac{41}{5040}$
$\gamma_{3,k}^*$	$\frac{1}{6}$	$-\frac{1}{8}$	$-\frac{1}{80}$	$-\frac{1}{180}$	$-\frac{11}{3360}$	$-\frac{89}{40320}$

Algorithm 1: Integration coefficients.

```
for(I=0;I<=12;I++)
{
G[0][I]=pow(1,I);
}
for(M=1;M<=D[1];M++)
{
for (I= 0; I<= 15-M;I++)
{
if (M==1)
{
TEMP = 1.0;
}
else
{
TEMP=G[M-1][I+1];
}
for (T=0;T<=I-1;T++)
{
TEMP=TEMP-(G[M][T]/(I+1.0-T));
}
G[M][I]=TEMP;
if(I==0) G1[M][0]=G[M][0];
else G1[M][I]=G[M][I]- G[M][I-1];
}
}
```

3. Numerical Simulations

The possible applications of Duffing-Van Der Pol equation are immense, especially in the field of science and engineering. Problems relating to electrical circuits are among the reason it has been studied by authors such as [1,12-15]. The current research propose a backward difference method (BDF) to numerically approximate a few popular Duffing-Van Der Pol type equations. The accuracy of the BDF method is tested against known methods that have been proven to be not only viable but also efficient. Here are a few abbreviations that will be use in upcoming section

H: step size,
LM: Lindsteds method
ADM: Adomians Decomposition method,
HPM: Homotopy Perturbation Method,
DTM: Differential Transform method,
RK4: Fourth Order Runge-Kutta method,
DI: Direct Integration method,
1PBDF: 1 Point Backward Difference method.

Duffing-Van Der Pol Problem 1: [13]

$$y''(t)=-\left(\frac{4}{3}-3y^2(t)\right)y'(t)-\frac{1}{3}y(t)-y(t)^3,$$

with initial conditions

$$y(0) = -0.28868, \quad y'(0) = 0.12.$$

Table 2: Numerical Results for Duffing-Van Der Pol Problem 1.

t	RK4	HPM	DTM	1PBDF
0.1	-0.287483	-0.287483	-0.287485	**-0.287483**
0.2	-0.286293	-0.286293	-0.286306	**-0.286293**
0.3	-0.285111	-0.285111	-0.285144	**-0.285111**
0.4	-0.283935	-0.283935	-0.283998	**-0.283935**
0.5	-0.282765	-0.280447	-0.282869	**-0.282765**
0.6	-0.281603	-0.279297	-0.281756	**-0.281603**
0.7	-0.280447	-0.278145	-0.280660	**-0.280447**
0.8	-0.279297	-0.277018	-0.279518	**-0.279297**
0.9	-0.278154	-0.278154	-0.278154	**-0.278154**
1.0	-0.277018	-0.277018	-0.277476	**-0.277018**

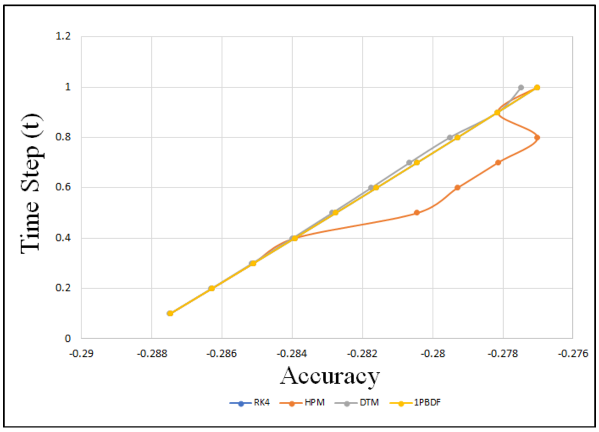

Fig. 1: Time Step versus Accuracy for Problem 1

Duffing-Van Der Pol Problem 2: [14]

$$y''(t) = 0.1\left(1 - y^2(t)\right)y'(t) - y(t) - 0.01y(t)^3,$$

with initial conditions

$$y(0) = 2, \quad y'(0) = 0.$$

Table 3: Numerical Results for Duffing-Van Der Pol Problem 2

t	LM	ADM	1PBDF
0.0	2.00000	1.99750	**2.00000**
0.1	1.98971	1.98724	**1. 98971**
0.2	1.95936	1.95697	**1.95936**
0.3	1.90980	1.90758	**1.90980**
0.4	1.84202	1.84008	**1.84202**
0.5	1.75702	1.75552	**1.75702**
0.6	1.65586	1.65493	**1.65586**
0.7	1.53958	1.53937	**1.53958**
0.8	1.40922	1.40982	**1.40923**
0.9	1.26581	1.26726	**1.26586**
1.0	1.11033	1.11267	**1.11054**

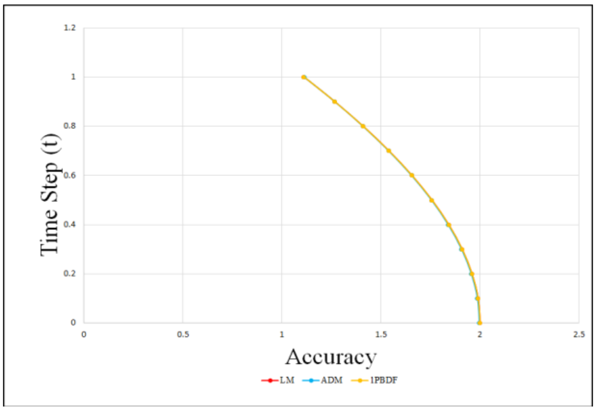

Fig. 2: Time Step versus Accuracy for Problem 2

Duffing-Van Der Pol Problem 3: [15]

$$y''(t) = \cos(0.7t) + \left(1 - y^2(t)\right)y'(t) + y(t) - y(t)^3,$$

with initial conditions

$$y(0) = 0.1, \quad y'(0) = -0.2.$$

Table 4: Numerical Results for Duffing-Van Der Pol Problem 3.

t	DI	1PBDF		
	$H = 0.1$	$H = 0.1$	$H = 0.001$	$H = 0.00001$
0.5	0.13038	**0.13041**	**0.13047**	**0.13047**
1.0	0.52399	**0.52395**	**0.52428**	**0.52428**
1.5	1.33649	**1.33602**	**0.52428**	**0.52428**
2.0	1.74809	**1.74955**	**1.74814**	**1.74814**
2.5	1.42639	**1.42773**	**1.42613**	**1.42613**
3.0	0.88625	**0.88643**	**0.88594**	**0.88594**
3.5	0.16497	**0.16388**	**0.16456**	**0.16456**
4.0	-1.12614	**-1.12754**	**-1.12676**	**-1.12676**
4.5	-2.08061	**-2.08416**	**-2.08006**	**-2.08006**

4. Discussion and Conclusion

The Duffing-Van Der Pol oscillator is a second order ODE with high frequency oscillations. Problem 1-3 consist of known application problems in the form of Duffing-Van Der Pol equations. TABLE 2 and FIGURE 1 presents the numerical results for Problem 1. They show the accuracy of the 1PBDF method compared to Lindsted's method (LM) and a modified Adomians Decomposition method (ADM). The results show that the 1PBDF method provides a better accuracy than the ADM but is as competitive as the LM. In TABLE 3 and FIGURE 2, the accuracy of the 1PBDF method is tested against the fourth order Runge-Kutta method (RK4), Homotopy Perturbation method (HPM) and Differential Transform method (DTM) and proven to be as accurate. Finally, TABLE 4 provides a comparison between the numerical results of the 1PBDF method with another multistep method, the Direct Integration (DI) method. For Problem 3, the 1PBDF method was tested three different step sizes to show the consistency of the method. Unfortunately, we were only able to provide numerical results for the DI method using only a single step size. This was attributed to the failure of the DI method in computing the approximated solution when using smaller step sizes. This set back was most likely due to the division component of the DI method. This issue had previously been addressed in [16] when attempting to solve a type Van Der Pol equation.

By justification of the numerical results, the 1PBDF has shown to be a viable alternative method for solving Duffing-Van Der Pol type oscillatory problems.

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