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Research paper

Discrete Heat Equation model of Rod by Partial Fibonacci Difference Operator with shift values

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Abstract

Partial Fibonacci difference equation is introduced and subjected to investigation in discrete heat equation by having recourse to Fibonacci difference operator with shift values in this paper. By having Fourier law of cooling as its basis, the heat transfer in the long rod is investigated and the solutions obtained are validated by MATLAB.

Keywords: *Partial Difference Equation, Fibonacci Difference Operator and Discrete Heat Equation.*
AMS Subject classification: *39A70, 39A10, 47B39, 80A20.*

1. Introduction

The difference operator $\underset{\alpha}{\Delta}$ defined on $u(k)$ found its inception in 1984 by Jerzy Popenda [1, 4]. The above operator was further extended by M.M.S.Manuel, et.al,[6, 2] in 2011. The study gained its momentum by the contribution of G.B.A.Xavier, et.al,[3] when the k-difference operator $\underset{k(l)}{\Delta}$ with variable coefficients was introduced in 2016. The need for a comprehensive operator arose as the investigation was carried out on heat equation and thus an extension of the operator $\underset{\alpha}{\Delta}$ termed as $\underset{x(\ell)}{\Delta}$ [5] was introduced with inputs gained from the other operators mentioned in the above literature. Here $x=(x_1,x_2,...,x_n)$ and $\ell=(\ell_1,\ell_2,...,\ell_n)$.

2. Fibonacci Difference Operator on Two Variable

For $x=(x_1,x_2)$, the Fibonacci difference operator on two variable real valued function with shift values $\ell=(\ell_1,\ell_2)$, $k=(k_1,k_2)$ is defined as

$$\underset{x(\ell)}{\Delta} v(k)=v(k)-x_1v(k-\ell)-x_2v(k-2\ell). \tag{1}$$

The operator in (1) becomes partial Fibonacci difference operator if either ℓ_1 or ℓ_2 is zero but not both. The equations involving a first order linear partial fibonacci difference equation [7] is

$$\underset{x(\ell)}{\Delta} v(k)=u(k), \ell=(0,\ell_2) \text{ or } (\ell_1,0); \quad x=(x_1,x_2). \tag{2}$$

The equation (2) has a numerical solution of the form

$$v(k_1,k_2)-F_{n+1}v(k_1,k_2-(n+1)\ell_2) - x_2F_nv(k_1,k_2-(n+2)\ell_2)=\sum_{s=0}^{n}F_su(k_1,k_2-s\ell_2), \tag{3}$$

where $F_0=1$, $F_1=x_1$ and $F_n=x_1F_{n+1}+x_2F_n$ are the second order Fibonacci numbers. If the function $v(k_1,k_2)$ is assumed as temperature of a long rod at the position k_1 at time k_2, it will be influenced by certain quantity of heat values at the neighboring points $k_1-2\ell_1$, $k_1-\ell_1$, $k_1+\ell_1$, $k_1+2\ell_1$ etc. Hence we obtain the following heat equation model for a long rod.

3. Discrete Heat Equation Model of a Long Rod

Let $v(k_1,k_2)$ denote the temperature of long rod [3]. By (1) and Fourier's cooling law we have

$$\underset{x(0,\ell_2)}{\Delta} v(k_1,k_2)=\gamma \underset{x(\pm\ell_1,0)}{\Delta} v(k_1,k_2); \quad x=(x_1,x_2), \tag{4}$$

where $\underset{x(\pm\ell_1,0)}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta}$ which is a Fibonacci heat equation of rod. Our main aim is to analyze and discuss the solution of the heat equation (4). Here, we obtain the value of $v(k_1,k_2)$.

Theorem 3.1 *If* $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2)=\underset{x(\pm\ell_1)}{u}(k_1,k_2)$ *are known, then the heat equation* (4) *has a solution*

$$v(k_1,k_2)=F_{n+1}v(k_1,k_2-(n+1)\ell_2)$$

$$+x_2F_nv(k_1,k_2-(n+2)\ell_2)+\gamma\sum_{s=0}^{n}F_s\underset{x(\pm\ell_1)}{u}(k_1,k_2-s\ell_2) \quad (5)$$

Proof. By representing $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2)=\underset{x(\pm\ell_1)}{u}(k_1,k_2)$, (4) becomes

$$v(k_1,k_2)=F_{n+1}v(k_1,k_2-(n+1)\ell_2)$$

$$+x_2F_nv(k_1,k_2-(n+2)\ell_2)+\gamma\underset{x(0,\ell_2)}{\overset{-1}{\Delta}}\underset{x(\pm\ell_1)}{u}(k_1,k_2) \quad (6)$$

The proof of (5) follows from the relation,

$$\underset{x(0,\ell_2)}{\overset{-1}{\Delta}}\underset{x(\pm\ell_1)}{u}(k_1,k_2)=\sum_{s=1}^{n}F_s\underset{x(\pm\ell_1)}{u}(k_1-s(0),k_2-s\ell_2)$$ in (6).

Theorem 3.2 *Consider* (4) *and denote* $v(k_1\pm\ell_1,*)=v(k_1+\ell_1,*)+v(k_1-\ell_1,*)$ *and* $v(k_1\pm 2\ell_1,*)=v(k_1+2\ell_1,*)+v(k_1-2\ell_1,*)$. *Then, we derive the following relations:*

(a).
$$v(k_1,k_2)=\frac{x_1^{\,p}}{(1-2\gamma)^p}v(k_1,k_2-p\ell_2)$$
$$-\sum_{s=1}^{p}\frac{\gamma x_1^{\,s}}{(1-2\gamma)^s}v(k_1\pm\ell_1,k_2-(s-1)\ell_2)$$
$$+\sum_{s=1}^{p}\frac{x_2x_1^{\,s-1}}{(1-2\gamma)^s}[v(k_1,k_2-(s+1)\ell_2)$$
$$-\gamma v(k_1\pm 2\ell_1,k_2-(s-1)\ell_2)] \quad (7)$$

(b).
$$v(k_1,k_2)=\frac{(1-2\gamma)^p}{x_1^{\,p}}v(k_1,k_2+p\ell_2)$$
$$+\sum_{s=1}^{p}\frac{\gamma(1-2\gamma)^{s-1}}{x_1^{\,s-1}}v(k_1\pm\ell_1,k_2+s\ell_2)$$
$$-\sum_{s=1}^{p}\frac{x_2(1-2\gamma)^{s-1}}{x_1^{\,s}}[v(k_1,k_2+(s-2)\ell_2)$$
$$-\gamma v(k_1\pm 2\ell_1,k_2+s\ell_2)] \quad (8)$$

(c).
$$v(k_1,k_2)=\frac{1}{\gamma^p}v(k_1-p\ell_1,k_2-p\ell_2)$$
$$-\sum_{s=1}^{p}\frac{x_2}{x_1\gamma^{s-1}}[v(k_1-(s-2)\ell_1,k_2-(s-1)\ell_2)$$
$$+v(k_1-(s+2)\ell_1,k_2-(s-1)\ell_2)]$$
$$+\sum_{s=1}^{p}\frac{x_2}{x_1\gamma^s}v(k_1-s\ell_1,k_2-(s+1)\ell_2)$$
$$-\sum_{s=1}^{p}\frac{1}{\gamma^{s-1}}v(k_1-(s+1)\ell_1,k_2-(s-1)\ell_2)$$
$$\sum_{s=1}^{p}\frac{(1-2\gamma)}{x_1\gamma^s}v(k_1-s\ell_1,k_2-(s-1)\ell_2), \quad (9)$$

(d).
$$v(k_1,k_2)=\frac{1}{\gamma^p}v(k_1+p\ell_1,k_2-p\ell_2)$$
$$-\sum_{s=1}^{p}\frac{x_2}{x_1\gamma^{s-1}}[v(k_1+(s+2)\ell_1,k_2-(s-1)\ell_2)$$
$$+v(k_1+(s-2)\ell_1,k_2-(s-1)\ell_2)]$$
$$+\sum_{s=1}^{p}\frac{x_2}{x_1\gamma^s}v(k_1+s\ell_1,k_2-(s+1)\ell_2)$$
$$-\sum_{s=1}^{p}\frac{1}{\gamma^{s-1}}v(k_1+(s+1)\ell_1,k_2-(s-1)\ell_2)$$
$$-\sum_{s=1}^{p}\frac{(1-2\gamma)}{x_1\gamma^s}v(k_1+s\ell_1,k_2-(s-1)\ell_2), \quad (10)$$

Proof. (a). From (4), directly generates the relation

$$v(k_1,k_2)=\frac{x_1}{(1-2\gamma)}v(k_1,k_2-\ell_2)+$$
$$\frac{x_2}{(1-2\gamma)}v(k_1,k_2-2\ell_2)$$
$$-\frac{x_1\gamma}{(1-2\gamma)}v(k_1\pm\ell_1,k_2)-\frac{x_2\gamma}{(1-2\gamma)}v(k_1\pm 2\ell_1,k_2). \quad (11)$$

By replacing k_2 by $k_2-\ell_2,k_2-2\ell_2,\ldots,k_2-p\ell_2$ in (11), we obtain expressions for $v(k_1,k_2-s\ell_2)$ and $v(k_1\pm\ell_1,k_2-s\ell_2)$. Now proof of (a) follows by applying all these values in (11).

(b). The heat equation (4), we arrive the relation

$$v(k_1,k_2)=\frac{(1-2\gamma)}{x_1}v(k_1,k_2+\ell_2)+\gamma v(k_1\pm\ell_1,k_2+\ell_2)$$
$$-\frac{x_2}{x_1}[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm 2\ell_1,k_2+\ell_2)] \quad (12)$$

The proof of (b) follows by replacing k_2 by $k_2+\ell_2, k_2+2\ell_2,\ldots,k_2+p\ell_2$ repeatedly and substituting corresponding v-values in (12).

(c). A simple calculation on (4) gives the expression

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1-\ell_1,k_2-\ell_2)+$$
$$\frac{x_2}{x_1\gamma}v(k_1-\ell_1,k_2-2\ell_2)-v(k_1-2\ell_1,k_2)$$
$$-\frac{x_2}{x_1}[v(k_1+\ell_1,k_2)+v(k_1-3\ell_1,k_2)]$$
$$-\frac{(1-2\gamma)}{x_1\gamma}v(k_1-\ell_1,k_2)$$

The proof of (c) follows by replacing k_1 by $k_1-\ell_1,k_1-2\ell_1,\ldots,k_1-p\ell_1$ and k_2 by $k_2-\ell_2,k_2-2\ell_2,\ldots,k_2-p\ell_2$ repeatedly.

(d). (4) gives the expression

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1+\ell_1,k_2-\ell_2)+$$
$$\frac{x_2}{x_1\gamma}v(k_1+\ell_1,k_2-2\ell_2)-v(k_1+2\ell_1,k_2)$$
$$-\frac{x_2}{x_1}[v(k_1+3\ell_1,k_2)+v(k_1-\ell_1,k_2)]-$$
$$\frac{(1-2\gamma)}{x_1\gamma}v(k_1+\ell_1,k_2)$$

the proof of (d) follows by replacing k_1 by $k_1+\ell_1,k_1+2\ell_1,\ldots,k_1+p\ell_1$ and k_2 by $k_2-\ell_2,k_2-2\ell_2,\ldots,k_2-p\ell_2$ repeatedly.

Example 3.3 *Dissemination rate of rod is identified by the given example if the solution in* (4) *is known. If* $v(k_1,k_2)=e^{k_1+k_2}$ *is an exact solution of* (4)*, then*

$\underset{0,\ell_2(x)}{\Delta} e^{k_1+k_2}=\gamma[\underset{\ell_1(x)}{\Delta} e^{k_1+k_2}+\underset{-\ell_1(x)}{\Delta} e^{k_1+k_2}]$, which yields

$$e^{k_1+k_2}-x_1e^{k_1+k_2-\ell_2}-x_2e^{k_1+k_2-2\ell_2}=\gamma[e^{k_1+k_2}-x_1e^{k_1\pm\ell_1+k_2}-x_2e^{k_1\pm2\ell_1+k_2}]$$
.

Canceling $e^{k_1+k_2}$ on both sides derives

$$\gamma=\frac{1-x_1e^{-\ell_2}-x_2e^{-2\ell_2}}{2-x_1(e^{\ell_1}+e^{-\ell_1})-x_2(e^{2\ell_1}+e^{-2\ell_1})}. \quad (13)$$

For numerical verification, we give the MATLAB coding for (a) of Theorem (3.2) when $m=1$, $k_1=1$, $\ell_1=1$, $k_2=2$, $\ell_2=2$, $x_1=1$, $x_2=2$, $v(k_1,k_2)=e^{(k_1+k_2)}$ and γ is as given in (13).

For getting accuracy value of heat transmission, 1 can be replaced by,

$$\underset{x(0,\ell_2)}{\Delta} v(k_1,k_2)=v(k_1,k_2)-x_1v(k_1,k_2-\ell_2)$$
$$-x_2v(k_1,k_2-2\ell_2))-x_3v(k_1,k_2-3\ell_2). \quad (14)$$

In this case, the corresponding heat equation model is

$$\underset{x(0,\ell_2)}{\Delta} v(k_1,k_2)=\gamma \underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2);\quad x=(x_1,x_2,x_3) \quad (15)$$

Theorem 3.4 *Let us take an integer* $n>0$*, and a real number* $\ell_2>0$ *such that* $v(k_1,k_2-n\ell_2)$ *and* $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2)=\underset{x(\pm\ell_1)}{u}(k_1,k_2)$ *are known. Then the heat equation* (15) *has a solution* $v(k_1,k_2)$ *of the form*

$$v(k_1,k_2)=F_{n+1}v(k_1,k_2-(n+1)\ell_2)$$
$$+(x_2F_n+x_3F_{n-1})v(k_1,k_2-(n+2)\ell_2)$$
$$+x_3F_nv(k_1,k_2-(n+3)\ell_2)+\gamma\sum_{s=0}^{n}F_s\underset{\pm\ell_1(x)}{u}(k_1,k_2-s\ell_2) \quad (16)$$

where $F_0=1$, $F_1=x_1$ and $F_n=x_1F_{n+2}+x_2F_{n+1}+x_3F_n$.

Let us denote the following notations in the below theorem:

$v(k_1\pm\ell_1,*)=v(k_1+\ell_1,*)+v(k_1-\ell_1,*)$,
$v(k_1\pm2\ell_1,*)=v(k_1+2\ell_1,*)+v(k_1-2\ell_1,*)$,
$v(k_1\pm3\ell_1,*)=v(k_1+3\ell_1,*)+v(k_1-3\ell_1,*)$.

The following theorem gives more accuracy values of heat transmission.

Theorem 3.5 *Consider the equation* (15) *then, we derive the following relations:*

(a).
$$v(k_1,k_2)=\frac{x_1^{\,p}}{(1-2\gamma)^p}v(k_1,k_2-p\ell_2)$$
$$-\sum_{s=1}^{p}\frac{\gamma x_1^{\,s}}{(1-2\gamma)^s}v(k_1\pm\ell_1,k_2-(s-1)\ell_2)$$
$$+\sum_{s=1}^{p}\frac{x_2x_1^{\,s-1}}{(1-2\gamma)^s}[v(k_1,k_2-(s+1)\ell_2)$$
$$-\gamma v(k_1\pm2\ell_1,k_2-(s-1)\ell_2)]$$
$$+\sum_{s=1}^{p}\frac{x_3x_1^{\,s-1}}{(1-2\gamma)^s}[v(k_1,k_2-(s+2)\ell_2)$$
$$-\gamma v(k_1\pm3\ell_1,k_2-(s-1)\ell_2)] \quad (17)$$

(b).
$$v(k_1,k_2)=\frac{(1-2\gamma)^p}{x_1^{\,p}}v(k_1,k_2+p\ell_2)$$
$$+\sum_{s=1}^{p}\frac{\gamma(1-2\gamma)^{s-1}}{x_1^{\,s-1}}v(k_1\pm\ell_1,k_2+s\ell_2)$$
$$-\sum_{s=1}^{p}\frac{x_2(1-2\gamma)^{s-1}}{x_1^{\,s}}[v(k_1,k_2+(s-2)\ell_2)$$
$$-\gamma v(k_1\pm2\ell_1,k_2+s\ell_2)]$$

$$-\sum_{s=1}^{p}\frac{x_3(1-2\gamma)^{s-1}}{x_1^{\ s}}[v(k_1,k_2+(s-3)\ell_2) - \gamma v(k_1\pm 3\ell_1,k_2+s\ell_2)] \quad (18)$$

(c).

$$v(k_1,k_2)=\frac{1}{\gamma^p}v(k_1-p\ell_1,k_2-p\ell_2) + \sum_{s=1}^{p}\frac{x_2}{x_1\gamma^i}v(k_1-s\ell_1,k_2-(s+1)\ell_2) + \sum_{s=1}^{p}\frac{x_3}{x_1\gamma^s}v(k_1-s\ell_1,k_2-(s+2)\ell_2) - \sum_{s=1}^{p}\frac{1}{\gamma^{s-1}}v(k_1-(s+1)\ell_1,k_2-(s-1)\ell_2) - \sum_{s=1}^{p}\frac{x_2}{x_1\gamma^{s-1}}[v(k_1-(s-2)\ell_1,k_2-(s-1)\ell_2) + v(k_1-(s+2)\ell_1,k_2-(s-1)\ell_2)]$$

$$-\sum_{s=1}^{p}\frac{x_3}{x_1\gamma^{s-1}}[v(k_1-(s-3)\ell_1,k_2-(s-1)\ell_2) + v(k_1-(s+3)\ell_1,k_2-(s-1)\ell_2)] - \sum_{s=1}^{p}\frac{(1-2\gamma)}{x_1\gamma^s}v(k_1-s\ell_1,k_2-(s-1)\ell_2), \quad (19)$$

(d).

$$v(k_1,k_2)=\frac{1}{\gamma^p}v(k_1+p\ell_1,k_2-p\ell_2) + \sum_{s=1}^{p}\frac{x_2}{x_1\gamma^s}v(k_1+s\ell_1,k_2-(s+1)\ell_2) + \sum_{s=1}^{p}\frac{x_3}{x_1\gamma^s}v(k_1+s\ell_1,k_2-(s+2)\ell_2) - \sum_{s=1}^{p}\frac{1}{\gamma^{s-1}}v(k_1+(s+1)\ell_1,k_2-(s-1)\ell_2) - \sum_{s=1}^{p}\frac{x_2}{x_1\gamma^{s-1}}[v(k_1+(s+2)\ell_1,k_2-(s-1)\ell_2) + v(k_1+(s-2)\ell_1,k_2-(s-1)\ell_2)] - \sum_{s=1}^{p}\frac{x_3}{x_1\gamma^{s-1}}[v(k_1+(s+3)\ell_1,k_2-(s-1)\ell_2). + v(k_1+(s-3)\ell_1,k_2-(s-1)\ell_2)]$$

$$-\sum_{s=1}^{p}\frac{(1-2\gamma)}{x_1\gamma^s}v(k_1+s\ell_1,k_2-(s-1)\ell_2), \quad (20)$$

Proof. The proof of the theorem is identical to the Theorem 3.2.

Example 3.6 *If* $v(k_1,k_2)=e^{k_1+k_2}$ *is a exact solution, then we get the relation*

$\underset{0,\ell_2(x)}{\Delta} e^{k_1+k_2}=\gamma[\underset{\ell_1(x)}{\Delta} e^{k_1+k_2}+\underset{-\ell_1(x)}{\Delta} e^{k_1+k_2}]$, which yields

$$e^{k_1+k_2}-x_1e^{k_1+k_2-\ell_2}-x_2e^{k_1+k_2-2\ell_2}-x_3e^{k_1+k_2-3\ell_2}$$

.

Cancelling $e^{k_1+k_2}$ on both sides derives

$$\gamma=\frac{1-x_1e^{-\ell_2}-x_2e^{-2\ell_2}-x_3e^{-3\ell_2}}{2-x_1(e^{\ell_1}+e^{-\ell_1})-x_2(e^{2\ell_1}+e^{-2\ell_1})-x_3(e^{3\ell_1}+e^{-3\ell_1})} \quad (21)$$

For numerical verification, we give the MATLAB coding for (a), (b), (c), (d) as similar as two variables.

4. Conclusion

The partial Fibonacci difference operator provides great possibility through the main results (3.2) and (3.5) to study the various aspects of heat equation: the transfer of heat, nature of material used and prediction of temperature having the knowledge of the present values as the basis. The results provide a mathematical model for the conservation of energy and wisdom in the selection of right material to save the economy involved. Simulations supported by MATLAB are inserted at relevant sections.

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