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***Research paper***

# Infinite Fibonacci Series Arising from Generalized Second Order $q$- Difference Equations

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**Abstract**

In this paper, we extend finite Second order $q$ -Fibonacci formula to infinite Second order $q$ -Fibonacci formula and also obtain the sum of infinite Second order $q$ -Fibonacci multi-series formula. Suitable examples are inserted to illustrate our findings.

***Keywords***: *Fibonacci numbers, Second $q$ -difference operator and Summation solution, Infinite Multi-series.Use.*

## 1. Introduction

The theory of Fibonacci sequence $0,1,1,2,3,5,\cdots$, where in each term is the sum of the two preceeding terms [4],[10],[11],[1], has found to fit a large variety of real life growth processes. Koshy [11] defined the generalized Fibonacci numbers as, for fixed $a$ and $b$ , the succeeding terms of the sequence $A_n$ , with initial values $A_0=0$ and $A_1=1$, are determined by $A_{n+2}=aA_{n+1}+bA_n$.

$$\sum_{r=0}^{m}F_{a,r}u(\frac{k}{q^{r+2}}) = \underset{q,a}{\overset{-1}{\Delta}}u(k)-F_{a,m+1}\underset{q,a}{\overset{-1}{\Delta}}u(\frac{k}{q^{m+1}})-a_2F_{a,m}\underset{q,a}{\overset{-1}{\Delta}}u(\frac{k}{q^{m+2}})$$

Here, we formulate higher order $q$ -difference equation

$$\underset{q_1,a}{\Delta}(\underset{q_2,a}{\Delta}(\cdots\underset{q_t,a}{\Delta}(v(k))\cdots)) = u(k), k\in(-\infty,\infty), \tag{1}$$

and obtain some result on sum of infinite second order $q$ -Fibonacci Multi-series by equating summation and closed form solutions of the equation (1).

## 2. Second Order $q$ -Difference Operator

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be used for the subsequent discussions. Let $u(k)$ be a real valued function on $(-\infty,\infty)$, $\alpha$ , $q$ and $a$ are non-zero reals and $m$ is a positive integer. Throughout this paper, we use the following notations:

$(i)$ $\underset{q,a}{\Delta} = \underset{(a_1,a_2)}{\Delta_q}$ $(ii)$ $F_n = F_{a,n}$

$(iii)$ $\sum\limits_{(r)_{1\to i}}^{m} = \sum\limits_{r_1=0}^{m}\sum\limits_{r_2=0}^{m}\cdots\sum\limits_{r_i=0}^{m}$ $(iv)$ $\underset{q_{1\to t},a}{\overset{-1}{\Delta}} = \underset{q_1,a}{\overset{-1}{\Delta}}\underset{q_2,a}{\overset{-1}{\Delta}}\cdots\underset{q_t,a}{\overset{-1}{\Delta}}.$

**Definition *1*** *[3] Let $a_1$ and $a_2$ be fixed reals, $a=(a_1,a_2)\in\mathrm{R}^2$ and $k\in(-\infty,\infty)$. Then the second order q-difference operator $\underset{q,a}{\Delta}$ is defined as*

$$\underset{q,a}{\Delta}u(k) = u(q^2k)-a_1u(qk)-a_2u(k) \tag{2}$$

and its inverse, denoted by $\underset{q,a}{\Delta^{-1}}$, is defined as below:

$$\text{if } \underset{q,a}{\Delta}v(k)=u(k), \text{ then } v(k)=\underset{q,a}{\overset{-1}{\Delta}}u(k).$$

**Lemma *2*** *[3] If $q^{2n}-a_1q^n-a_2\neq 0$ for $n=0,1,2,\cdots$, then*

$$\underset{q,a}{\overset{-1}{\Delta}}k^n = \frac{k^n}{q^{2n}-a_1q^n-a_2} \quad\text{and}\quad \underset{q,a}{\overset{-1}{\Delta}}(1) = \frac{1}{1-a_1-a_2}. \tag{4}$$

**Lemma 3** *[8] Let $s_r^n$ be the Stirling numbers of first kind, $n\in N(1)$. If*

$k_q^{(n)} = \prod\limits_{i=0}^{n-1}(k-iq)$ and $(\frac{1}{k})_q^{(n)} = \prod\limits_{i=0}^{n-1}(\frac{1}{k}-iq)$ for $k,q\neq 0$, then

$$k_q^{(n)} = \sum_{r=1}^{n}s_r^nq^{n-r}k^r \quad\text{and}\quad (\frac{1}{k})_q^{(n)} = \sum_{r=1}^{n}s_r^nq^{n-r}(\frac{1}{k})^r. \tag{5}$$

## 3. Infinite Second Order $q$ -Fibonacci Summation Formula

The purpose of this section is for obtaining the sum of infinite Second Order $q$ -Fibonacci series by equating summation and closed form solutions of the Second order $q$ -difference equation (1).

**Definition 4** *[3] (Fibonacci Sequence) For each* $a=(a_1,a_2)\in \mathrm{R}^2$, *the Second order Fibonacci sequence is defined as*

$$F_a=\{F_{a,n}\}_{n=0}^{\infty}, \tag{6}$$

where $F_{a,0}=1, F_{a,1}=a_1$ and $F_{a,n}=a_1F_{a,n-1}+a_2F_{a,n-2}$ for $n\geq 2.$

When $a_1=a_2=1,$ (6) becomes the generalized Fibonacci sequence.

**Theorem 5** *Let* $q,a\neq 0,$ $u(k)$ *be a real valued function defined on* $(-\infty,\infty)$ *and if*

$$\lim_{r\to\infty}\frac{1}{(-a_2)^{r+1}}\left\{F_{a,r+1}\underset{q,a}{\Delta^{-1}}u(q^{r+1}k)-F_{a,r}\overset{-1}{\underset{q,a}{\Delta}}u(q^{r+2}k)\right\}=0,$$

*then*

$$\underset{q,a}{\Delta^{-1}}u(k)=\frac{-1}{a_2}\sum_{r=0}^{\infty}\frac{(-1)^rF_{a,r}}{a_2^r}u(q^rk) \tag{7}$$

is an infinite Fibonacci series solution of the Second order $q$-difference equation (1) for $t=1$.

*Proof.* Taking $\underset{q,a}{\Delta^{-1}}u(k)=v(k)$ and by Definition 1, we arrive

$$v(k)=\frac{1}{a_2}v(q^2k)-\frac{a_1}{a_2}v(qk)-\frac{1}{a_2}u(k). \tag{8}$$

Replacing $k$ by $qk$ in (8), we get

$$v(qk)=\frac{1}{a_2}v(q^3k)-\frac{a_1}{a_2}v(q^2k)-\frac{1}{a_2}u(qk).$$

Therefore (8) becomes

$$v(k)=\frac{-1}{a_2}u(k)+\frac{a_1}{a_2}u(qk)+\frac{a_1^2+a_2}{a_2^2}v(q^2k)-\frac{a_1}{a_2^2}v(q^3k). \tag{9}$$

Again replacing $k$ by $q^2k$ and putting the resultant expressions in (9), we get

$$v(k)=\frac{-1}{a_2}\left\{u(k)-\frac{a_1}{a_2}u(qk)+\frac{a_1^2+a_2}{a_2^2}u(q^2k)\right.$$
$$\left.-\frac{a_1^2+a_2}{a_2^2}v(q^4k)+\frac{a_1(a_1^2+a_2)+a_1a_2}{a_2^2}v(q^3k)\right\}.$$

Since $F_n\in F_a$, we have

$$v(k)=\frac{-1}{a_2}\left\{F_0u(k)-\frac{F_1}{a_2}u(qk)+\frac{F_2}{a_2^2}u(q^2k)-\frac{F_2}{a_2^2}v(q^4k)+\frac{F_3}{a_2^2}v(q^3k)\right\} \tag{10}$$

Now (7) is obtained by (10) replacing $k$ by $q^3k$, $q^4k$, $\cdots$ in (8) repeatedly,

**Corollary 6** *Let* $k\in(0,\infty)$ *and* $q,a\neq 0$. *If*

$$\lim_{r\to\infty}\frac{1}{(-a_2)^{r+1}}\left\{F_{a,r+1}\overset{-1}{\underset{q,a}{\Delta}}\frac{1}{(q^{r+1}k)^2}-F_{a,r}\overset{-1}{\underset{q,a}{\Delta}}\frac{1}{(q^{r+2}k)^2}\right\}=0,$$

then we have

$$\frac{-1}{a_2}\sum_{r=0}^{\infty}\frac{(-1)^rF_{a,r}}{a_2^r}\frac{1}{(q^rk)^2}=\frac{q^4}{(1-q^2a_1-q^4a_2)k^2}. \tag{11}$$

*Proof.* Let $u(k)=\frac{1}{k^2}$ in (7). Then we have

$$\frac{-1}{a_2}\sum_{r=0}^{\infty}\frac{(-1)^rF_r}{a_2^r}\frac{1}{(q^rk)^2}=\overset{-1}{\underset{q,a}{\Delta}}(\frac{1}{k^2}). \tag{12}$$

Now, $\underset{q,a}{\Delta}(\frac{1}{k^2})=\frac{1}{(q^2k)^2}-\frac{a_1}{(qk)^2}-\frac{a_2}{k^2}$, yields

$$\underset{q,a}{\Delta^{-1}}(\frac{1}{k^2})=\frac{q^4}{(1-q^2a_1-q^4a_2)k^2}. \tag{13}$$

Hence, the proof completes by substituting (13) in (12).

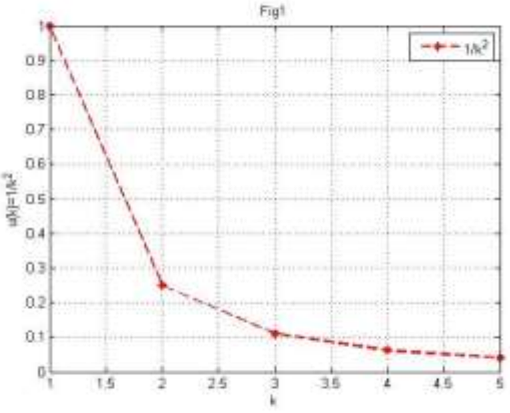
The following example is a verification of the Corollary 6.

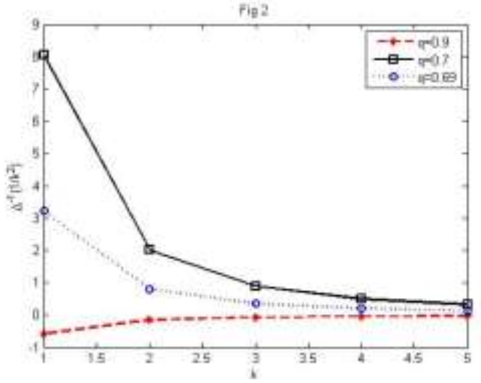
**Example 7** *Taking* $k=2, a_1=1, a_2=2$ *and* $q=3$ *in (11), we get*

$$\frac{-1}{2}\sum_{r=0}^{\infty}\frac{(-1)^rF_{a,r}}{2^r}\frac{1}{(3^r2)^2}=\frac{3^4}{(1-3^2-3^4\times 2)2^2}=-0.1191176471.$$

Here $F_{a,0}, F_{a,1},\cdots F_{a,n}\cdots$ are obtained from the Definition (4).

The portrait of the functions $k^2$ and $\log k$ is given in Fig1. Fig2 gives the portrait of $\underset{q,a}{\Delta^{-1}}(k^2\log k)$ for a fixed $a=(3,4,5)$ and $q=0.1,0.6,1.$





## 4. Infinite Second Order $q$-Fibonacci Multi-Series Formula

In this section, we obtain formula for sum of $q$-Fibonacci multi-series.

**Theorem 8** *If*

$$\lim_{r_i\to\infty}\frac{1}{(-a_2)^{r_i+1}}\left\{F_{a,r_i+1}\underset{q_{1\to i},a}{\Delta^{-1}}u(q_i^{r_i+1}k)-F_{a,r_i}\underset{q_{1\to i},a}{\Delta^{-1}}u(q_i^{r_i+2}k)\right\}=0$$

for $i=1,2,\cdots,t,$ then we have

$$\sum_{(r)_{1\to t}}^{\infty}\prod_{p=1}^{t}\frac{F_{a,r_p}}{(-a_2)^{r_p}}u\Big(\prod_{p=1}^{t}q_p^{r_p}k\Big)=(-a_2)^t\underset{q_{1\to t},a}{\Delta^{-1}}u(k),\qquad(14)$$

is a solution of the equation (1).

*Proof.* Replacing $q,r$ by $q_2,r_2$ in (7), we get

$$F_0u(k)-\frac{F_1}{a_2}u(q_2k)+\frac{F_2}{a_2^2}u(q_2^2k)-\cdots+\infty=(-a_2)\underset{q_2,a}{\Delta^{-1}}u(k).\qquad(15)$$

Replacing $k$ by $q_1^{r_1}k$ and dividing by $\frac{a_2^{r_1}}{(-1)^{r_1}F_{r_1}}$ for $r_1=1,2,3,\cdots,\infty$ in (15), gives

$$\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\left\{F_0u(q_1^{r_1}k)-\frac{F_1}{a_2}u(q_1^{r_1}q_2k)+\frac{F_2}{a_2^2}u(q_1^{r_1}q_2^2k)\right.$$

$$\left.-\frac{F_3}{a_2^3}u(q_1^{r_1}q_2^3k)+\cdots+\infty\right\}=\frac{(-a_2)(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\underset{q_2,a}{\Delta^{-1}}u(q_1^{r_1}k),$$

for $r_1=1,2,3,\cdots,\infty.$

Summing the above equation with (15), we arrive

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1+r_2}F_{r_1}F_{r_2}u(q_1^{r_1}q_2^{r_2}k)}{a_2^{r_1}a_2^{r_2}(-a_2)}=\sum_{r_1=0}^{\infty}\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\underset{q_2,a}{\Delta^{-1}}u(q_1^{r_1}k).\qquad(16)$$

Applying (7) in (16), we obtain

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}F_{r_1}F_{r_2}u(q_1^{r_1}q_2^{r_2}k)}{a_2^{r_1}a_2^{r_2}}=(-a_2)^2\underset{q_1,a}{\Delta^{-1}}\underset{q_2,a}{\Delta^{-1}}u(k).\qquad(17)$$

Replacing $q_1,q_2,F_1,F_2,r_1,r_2$ by $q_2,q_3,F_2,F_3,r_2,r_3$ in (17), we get

$$\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_2}(-1)^{r_3}F_{r_2}F_{r_3}u(q_2^{r_2}q_3^{r_3}k)}{a_2^{r_2}a_2^{r_3}}=(-a_2)^2\underset{q_2,a}{\Delta^{-1}}\underset{q_3,a}{\Delta^{-1}}u(k).\qquad(18)$$

Replacing $k$ by $q_1^{r_1}k$ and dividing by $\frac{a_2^{r_1}}{(-1)^{r_1}F_{r_1}}$ for $r_1=1,2,3,\cdots,\infty$ in (18), yields

$$\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_2}(-1)^{r_3}F_{r_2}F_{r_3}u(q_1^{r_1}q_2^{r_2}q_3^{r_3}k)}{a_2^{r_2}a_2^{r_3}}$$

$$=\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}(-a_2)^2\underset{q_2,a}{\Delta^{-1}}\underset{q_3,a}{\Delta^{-1}}u(q_1^{r_1}k).\qquad(19)$$

Summing (19) with (18), gives

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(-1)^{r_3}F_{r_1}F_{r_2}F_{r_3}u(q_1^{r_1}q_2^{r_2}q_3^{r_3}k)}{a_2^{r_1}a_2^{r_2}a_2^{r_3}}$$

$$=(-a_2)^2\sum_{r_1=0}^{\infty}\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\underset{q_2,a}{\Delta^{-1}}\underset{q_3,a}{\Delta^{-1}}u(q_1^{r_1}k).\qquad(20)$$

Applying (7) in (20), we obtain

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(-1)^{r_3}F_{r_1}F_{r_2}F_{r_3}u(q_1^{r_1}q_2^{r_2}q_3^{r_3}k)}{a_2^{r_1}a_2^{r_2}a_2^{r_3}}$$

$$=(-a_2)^3\underset{q_1,a}{\Delta^{-1}}\underset{q_2,a}{\Delta^{-1}}\underset{q_3,a}{\Delta^{-1}}u(k).$$

Proceeding like this, we derive (14).

The following corollary gives formula for Infinite Fibonacci series involving Polynomial factorial.

**Theorem 9** *Let* $k\in(-\infty,\infty)$ *and* $q,a\neq0$. *If*

$$\lim_{r_i\to\infty}\frac{1}{(-a_2)^{r_i+1}}\left\{F_{a,r_i+1}\underset{q,a}{\Delta^{-i}}u(q^{r_i+1}k)-F_{a,r_i}\underset{q,a}{\Delta^{-i}}u(q^{r_i+2}k)\right\}=0$$

for $i=1,2,\cdots,t,$ then we have

$$\sum_{(r)_{1\to t}}^{\infty}\prod_{p=1}^{t}\frac{F_{a,r_p}}{(-a_2)^{r_p}}u\Big(\prod_{p=1}^{t}q^{r_p}k\Big)=(-a_2)^t\underset{q,a}{\Delta^{-t}}u(k).\qquad(21)$$

## 5. Conclusion

In this paper, the multi-series solutions of Infinite Second Order $q$-Fibonacci Summation Formula have been obtained. Moreover, Infinite Second Order $q$-Fibonacci Multi-Series Formula also derived.

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