International Journal of Engineering & Technology, 7 (4.10) (2018) 702-705

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper



Infinite Fibonacci Series Arising from Generalized Second Order q - Difference Equations

G. Britto Antony Xavier [1]*, B. Mohan [2], , T. G. Gerly [3], R. Suganya[4]

[1]Department of Mathematics, Sacred Heart College(Autonomous), Tirupattur - 635 601, Vellore District, Tamil Nadu, S.India.
**Corresponding author E-mail: brittoshc@gmail.com, [2]mngbmohan@gmail.com*

Abstract

In this paper, we extend finite Second order q -Fibonacci formula to infinite Second order q -Fibonacci formula and also obtain the sum of infinite Second order q -Fibonacci multi-series formula. Suitable examples are inserted to illustrate our findings.

Keywords: *Fibonacci numbers, Second q -difference operator and Summation solution, Infinite Multi-series.Use.*

1. Introduction

The theory of Fibonacci sequence $0,1,1,2,3,5,\cdots$, where in each term is the sum of the two preceeding terms [4],[10],[11],[1], has found to fit a large variety of real life growth processes. Koshy [11] defined the generalized Fibonacci numbers as, for fixed a and b , the succeeding terms of the sequence A_n , with initial values $A_0 = 0$ and $A_1 = 1$, are determined by $A_{n+2} = aA_{n+1} + bA_n$.

$$\sum_{r=0}^{m} F_{a,r}\, u\left(\frac{k}{q^{r+2}}\right) = \overset{-1}{\underset{q,a}{\Delta}} u(k) - F_{a,m+1} \overset{-1}{\underset{q,a}{\Delta}} u\left(\frac{k}{q^{m+1}}\right) - a_2 F_{a,m} \overset{-1}{\underset{q,a}{\Delta}} u\left(\frac{k}{q^{m+2}}\right)$$

Here, we formulate higher order q -difference equation

$$\underset{q_1,a}{\Delta}\left(\underset{q_2,a}{\Delta}\left(\cdots \underset{q_t,a}{\Delta}(v(k))\cdots\right)\right) = u(k), k \in (-\infty,\infty), \tag{1}$$

and obtain some result on sum of infinite second order q -Fibonacci Multi-series by equating summation and closed form solutions of the equation (1).

2. Second Order q -Difference Operator

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be used for the subsequent discussions. Let $u(k)$ be a real valued function on $(-\infty,\infty)$, α , q and a are non-zero reals and m is a positive integer. Throughout this paper, we use the following notations:

(i) $\underset{q,a}{\Delta} = \underset{(a_1,a_2)}{\Delta_q}$ $\qquad (ii)$ $F_n = F_{a,n}$

(iii) $\sum_{(r)_{1\to i}}^{m} = \sum_{r_1=0}^{m}\sum_{r_2=0}^{m}\cdots\sum_{r_i=0}^{m}$ $\qquad (iv)$ $\underset{q_{1\to t},a}{\overset{-1}{\Delta}} = \overset{-1}{\underset{q_1,a}{\Delta}}\ \overset{-1}{\underset{q_2,a}{\Delta}}\cdots\overset{-1}{\underset{q_t,a}{\Delta}}$.

Definition *1* *[3] Let a_1 and a_2 be fixed reals, $a = (a_1, a_2) \in \mathrm{R}^2$ and $k \in (-\infty,\infty)$. Then the second order q-difference operator $\underset{q,a}{\Delta}$ is defined as*

$$\underset{q,a}{\Delta} u(k) = u(q^2k) - a_1 u(qk) - a_2 u(k) \tag{2}$$

and its inverse, denoted by $\underset{q,a}{\Delta^{-1}}$, is defined as below:

$$\text{if} \quad \underset{q,a}{\Delta}\, v(k) = u(k), \text{ then } v(k) = \overset{-1}{\underset{q,a}{\Delta}} u(k).$$

Lemma *2* *[3] If $q^{2n} - a_1 q^n - a_2 \neq 0$ for $n = 0,1,2,\cdots$, then*

$$\overset{-1}{\underset{q,a}{\Delta}} k^n = \frac{k^n}{q^{2n} - a_1 q^n - a_2} \quad \text{and} \quad \overset{-1}{\underset{q,a}{\Delta}}(1) = \frac{1}{1 - a_1 - a_2}. \tag{4}$$

Lemma 3 *[8] Let s_r^n be the Stirling numbers of first kind, $n \in N(1)$. If*

$k_q^{(n)} = \prod_{i=0}^{n-1}(k - iq)$ and $(\frac{1}{k})_q^{(n)} = \prod_{i=0}^{n-1}(\frac{1}{k} - iq)$ for $k, q \neq 0$, then

$$k_q^{(n)} = \sum_{r=1}^{n} s_r^n q^{n-r} k^r \quad \text{and} \quad (\frac{1}{k})_q^{(n)} = \sum_{r=1}^{n} s_r^n q^{n-r} (\frac{1}{k})^r. \tag{5}$$

3. Infinite Second Order q -Fibonacci Summation Formula

The purpose of this section is for obtaining the sum of infinite Second Order q -Fibonacci series by equating summation and closed form solutions of the Second order q -difference equation (1).

Copyright © 2018 Authors. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Definition 4 *[3] (Fibonacci Sequence) For each* $a=(a_1,a_2)\in \mathrm{R}^2$, *the Second order Fibonacci sequence is defined as*

$$F_a=\{F_{a,n}\}_{n=0}^{\infty}, \tag{6}$$

where $F_{a,0}=1, F_{a,1}=a_1$ and $F_{a,n}=a_1F_{a,n-1}+a_2F_{a,n-2}$ for $n\geq 2.$

When $a_1=a_2=1,$ (6) becomes the generalized Fibonacci sequence.

Theorem 5 *Let* $q,a\neq 0,$ $u(k)$ *be a real valued function defined on* $(-\infty,\infty)$ *and if*

$$\lim_{r\to\infty}\frac{1}{(-a_2)^{r+1}}\left\{F_{a,r+1}\Delta_{q,a}^{-1}u(q^{r+1}k)-F_{a,r}\overset{-1}{\underset{q,a}{\Delta}}u(q^{r+2}k)\right\}=0,$$

then

$$\Delta_{q,a}^{-1}u(k)=\frac{-1}{a_2}\sum_{r=0}^{\infty}\frac{(-1)^rF_{a,r}}{a_2^r}u(q^rk) \tag{7}$$

is an infinite Fibonacci series solution of the Second order q-difference equation (1) for $t=1.$

Proof. Taking $\Delta_{q,a}^{-1}u(k)=v(k)$ and by Definition 1, we arrive

$$v(k)=\frac{1}{a_2}v(q^2k)-\frac{a_1}{a_2}v(qk)-\frac{1}{a_2}u(k). \tag{8}$$

Replacing k by qk in (8), we get

$$v(qk)=\frac{1}{a_2}v(q^3k)-\frac{a_1}{a_2}v(q^2k)-\frac{1}{a_2}u(qk).$$

Therefore (8) becomes

$$v(k)=\frac{-1}{a_2}u(k)+\frac{a_1}{a_2}u(qk)+\frac{a_1^2+a_2}{a_2^2}v(q^2k)-\frac{a_1}{a_2^2}v(q^3k). \tag{9}$$

Again replacing k by q^2k and putting the resultant expressions in (9), we get

$$v(k)=\frac{-1}{a_2}\left\{u(k)-\frac{a_1}{a_2}u(qk)+\frac{a_1^2+a_2}{a_2^2}u(q^2k)-\frac{a_1^2+a_2}{a_2^2}v(q^4k)+\frac{a_1(a_1^2+a_2)+a_1a_2}{a_2^2}v(q^3k)\right\}.$$

Since $F_n\in F_a$, we have

$$v(k)=\frac{-1}{a_2}\left\{F_0u(k)-\frac{F_1}{a_2}u(qk)+\frac{F_2}{a_2^2}u(q^2k)-\frac{F_2}{a_2^2}v(q^4k)+\frac{F_3}{a_2^2}v(q^3k)\right\} \tag{10}$$

Now (7) is obtained by (10) replacing k by q^3k, q^4k, $\cdots$ in (8) repeatedly,

Corollary 6 *Let* $k\in(0,\infty)$ *and* $q,a\neq 0$. *If*

$$\lim_{r\to\infty}\frac{1}{(-a_2)^{r+1}}\left\{F_{a,r+1}\overset{-1}{\underset{q,a}{\Delta}}\frac{1}{(q^{r+1}k)^2}-F_{a,r}\overset{-1}{\underset{q,a}{\Delta}}\frac{1}{(q^{r+2}k)^2}\right\}=0,$$

then we have

$$\frac{-1}{a_2}\sum_{r=0}^{\infty}\frac{(-1)^rF_{a,r}}{a_2^r}\frac{1}{(q^rk)^2}=\frac{q^4}{(1-q^2a_1-q^4a_2)k^2}. \tag{11}$$

Proof. Let $u(k)=\frac{1}{k^2}$ in (7). Then we have

$$\frac{-1}{a_2}\sum_{r=0}^{\infty}\frac{(-1)^rF_r}{a_2^r}\frac{1}{(q^rk)^2}=\overset{-1}{\underset{q,a}{\Delta}}\left(\frac{1}{k^2}\right). \tag{12}$$

Now, $\underset{q,a}{\Delta}\left(\frac{1}{k^2}\right)=\frac{1}{(q^2k)^2}-\frac{a_1}{(qk)^2}-\frac{a_2}{k^2}$, yields

$$\Delta_{q,a}^{-1}\left(\frac{1}{k^2}\right)=\frac{q^4}{(1-q^2a_1-q^4a_2)k^2}. \tag{13}$$

Hence, the proof completes by substituting (13) in (12).

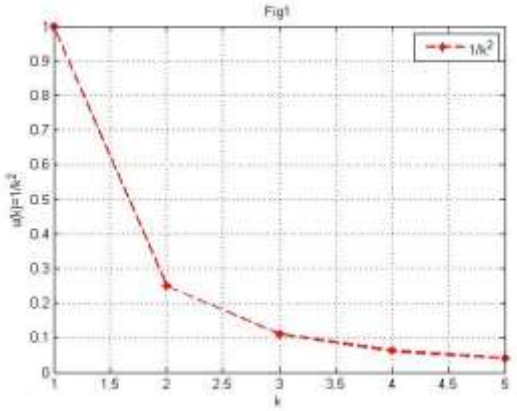
The following example is a verification of the Corollary 6.

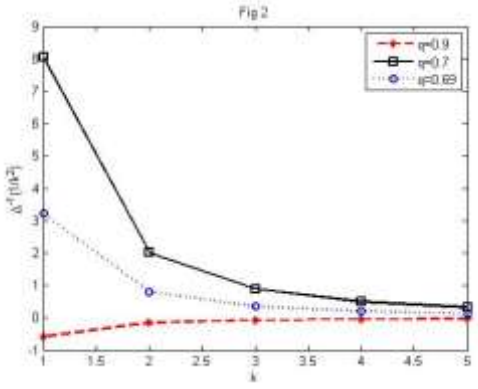
Example 7 *Taking* k=2, a_1=1, a_2=2 *and* q=3 *in (11), we get*

$$\frac{-1}{2}\sum_{r=0}^{\infty}\frac{(-1)^rF_{a,r}}{2^r}\frac{1}{(3^r2)^2}=\frac{3^4}{(1-3^2-3^4\times 2)2^2}=-0.1191176471.$$

Here $F_{a,0},F_{a,1},\cdots F_{a,n}\cdots$ are obtained from the Definition (4).

The portrait of the functions k^2 and $\log k$ is given in Fig1. Fig2 gives the portrait of $\Delta_{q,a}^{-1}(k^2\log k)$ for a fixed $a=(3,4,5)$ and $q=0.1,0.6,1.$





4. Infinite Second Order q -Fibonacci Multi-Series Formula

In this section, we obtain formula for sum of q -Fibonacci multi-series.

Theorem 8 *If*

$$\lim_{r_i\to\infty}\frac{1}{(-a_2)^{r_i+1}}\left\{F_{a,r_i+1}\underset{q_{1\to i},a}{\Delta^{-1}}u(q_i^{r_i+1}k)-F_{a,r_i}\underset{q_{1\to i},a}{\Delta^{-1}}u(q_i^{r_i+2}k)\right\}=0$$

for $i=1,2,\cdots,t,$ then we have

$$\sum_{(r)_{1\to t}}^{\infty}\prod_{p=1}^{t}\frac{F_{a,r_p}}{(-a_2)^{r_p}}u(\prod_{p=1}^{t}q_p^{r_p}k)=(-a_2)^t\underset{q_{1\to t},a}{\Delta^{-1}}u(k), \tag{14}$$

is a solution of the equation (1).

Proof. Replacing q,r by q_2,r_2 in (7), we get

$$F_0u(k)-\frac{F_1}{a_2}u(q_2k)+\frac{F_2}{a_2^2}u(q_2^2k)-\cdots+\infty=(-a_2)\underset{q_2,a}{\Delta^{-1}}u(k). \tag{15}$$

Replacing k by $q_1^{r_1}k$ and dividing by $\dfrac{a_2^{r_1}}{(-1)^{r_1}F_{r_1}}$ for $r_1=1,2,3,\cdots,\infty$ in (15), gives

$$\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\left\{F_0u(q_1^{r_1}k)-\frac{F_1}{a_2}u(q_1^{r_1}q_2k)+\frac{F_2}{a_2^2}u(q_1^{r_1}q_2^2k)\right.$$

$$\left.-\frac{F_3}{a_2^3}u(q_1^{r_1}q_2^3k)+\cdots+\infty\right\}=\frac{(-a_2)(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\underset{q_2,a}{\Delta^{-1}}u(q_1^{r_1}k),$$

for $r_1=1,2,3,\cdots,\infty.$

Summing the above equation with (15), we arrive

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1+r_2}F_{r_1}F_{r_2}u(q_1^{r_1}q_2^{r_2}k)}{a_2^{r_1}a_2^{r_2}(-a_2)}=\sum_{r_1=0}^{\infty}\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\underset{q_2,a}{\Delta^{-1}}u(q_1^{r_1}k). \tag{16}$$

Applying (7) in (16), we obtain

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}F_{r_1}F_{r_2}u(q_1^{r_1}q_2^{r_2}k)}{a_2^{r_1}a_2^{r_2}}=(-a_2)^2\underset{q_1,a\,q_2,a}{\Delta^{-1}\Delta^{-1}}u(k). \tag{17}$$

Replacing q_1,q_2,F_1,F_2,r_1,r_2 by q_2,q_3,F_2,F_3,r_2,r_3 in (17), we get

$$\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_2}(-1)^{r_3}F_{r_2}F_{r_3}u(q_2^{r_2}q_3^{r_3}k)}{a_2^{r_2}a_2^{r_3}}=(-a_2)^2\underset{q_2,a\,q_3,a}{\Delta^{-1}\Delta^{-1}}u(k). \tag{18}$$

Replacing k by $q_1^{r_1}k$ and dividing by $\dfrac{a_2^{r_1}}{(-1)^{r_1}F_{r_1}}$ for $r_1=1,2,3,\cdots,\infty$ in (18), yields

$$\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_2}(-1)^{r_3}F_{r_2}F_{r_3}u(q_1^{r_1}q_2^{r_2}q_3^{r_3}k)}{a_2^{r_2}a_2^{r_3}}$$

$$=\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}(-a_2)^2\underset{q_2,a\,q_3,a}{\Delta^{-1}\Delta^{-1}}u(q_1^{r_1}k). \tag{19}$$

Summing (19) with (18), gives

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(-1)^{r_3}F_{r_1}F_{r_2}F_{r_3}u(q_1^{r_1}q_2^{r_2}q_3^{r_3}k)}{a_2^{r_1}a_2^{r_2}a_2^{r_3}}$$

$$=(-a_2)^2\sum_{r_1=0}^{\infty}\frac{(-1)^{r_1}F_{r_1}}{a_2^{r_1}}\underset{q_2,a\,q_3,a}{\Delta^{-1}\Delta^{-1}}u(q_1^{r_1}k). \tag{20}$$

Applying (7) in (20), we obtain

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}(-1)^{r_3}F_{r_1}F_{r_2}F_{r_3}u(q_1^{r_1}q_2^{r_2}q_3^{r_3}k)}{a_2^{r_1}a_2^{r_2}a_2^{r_3}}$$

$$=(-a_2)^3\underset{q_1,a\,q_2,a\,q_3,a}{\Delta^{-1}\Delta^{-1}\Delta^{-1}}u(k).$$

Proceeding like this, we derive (14).

The following corollary gives formula for Infinite Fibonacci series involving Polynomial factorial.

Theorem 9 *Let* $k\in(-\infty,\infty)$ *and* $q,a\neq 0$. *If*

$$\lim_{r_i\to\infty}\frac{1}{(-a_2)^{r_i+1}}\left\{F_{a,r_i+1}\underset{q,a}{\Delta^{-i}}u(q^{r_i+1}k)-F_{a,r_i}\underset{q,a}{\Delta^{-i}}u(q^{r_i+2}k)\right\}=0$$

for $i=1,2,\cdots,t,$ then we have

$$\sum_{(r)_{1\to t}}^{\infty}\prod_{p=1}^{t}\frac{F_{a,r_p}}{(-a_2)^{r_p}}u(\prod_{p=1}^{t}q^{r_p}k)=(-a_2)^t\underset{q,a}{\Delta^{-t}}u(k). \tag{21}$$

5. Conclusion

In this paper, the multi-series solutions of Infinite Second Order q -Fibonacci Summation Formula have been obtained. Moreover, Infinite Second Order q -Fibonacci Multi-Series Formula also derived.

References

[1] Benjamin AT, Quinn JJ and Su F.E(2000), "*Generalized Fibonacci Identities through Phased Tilings*", The Fibonacci Quarterly, Vol. 38 No. 3, pp. 282-288.

[2] Xavier GBA, Gerly TG and Kumar SUV(2015), "*Multi-Series Solution of Generalized q-alpha Difference Equation*", International Journal of Applied Engineering Research, Vol. 10 No. 72, pp. 97-101.

[3] Xavier GBA and Gerly TG(2016), "*Fibonacci Sequence Generated From Two Dimensional q -Difference Equation*", International Journal Mathematics And its Applications, Vol. 4. No. 1-B, pp. 67-72.

[4] Horadam F(1961), "*A generalized Fibonacci sequence*", American Mathematical Monthly, Vol. 68, pp. 455-459.

[5] Jackson FH(1908), "*On q-functions and a Certain Difference Operator*", Trans. Roy.Soc.Edin, Vol. 46, pp. 64-72.

[6] Jackson FH, "*On q-definite integrals*", Qust.J. Pure Appl. Math. Vol. 41, pp. 193-203.

[7] Jerzy Popenda and Blazej Szmanda(1984), *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, Vol. 17 No. 1, pp. 153-164.
[8] Manuel MMS, Xavier GBA and Thandapani E, "*Theory of Generalized Difference Operator and Its Applications*", Far East Journal of Mathematical Sciences, Vol. 20 No. 2, pp. 163-171.
[9] Miller KS and Ross B(1989), *Fractional difference calculus, in "Univalent functions, fractional calculus and the applications(Koriyama, 1988)"*, pp. 139-152.
[10] Koshy T(2001), "*Fibonacci and Lucas Numbers with Applications*", Wiley, New York.
[11] Walton JE and Horadam AF(1974), "*Some further identities for the generalized Fibonacci sequence*", Fibonacci Quart. Vol. 12, pp. 272-280.