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Research paper

Existence of Ψ-bounded Solutions for System of Linear Dynamic equations on Time Scales

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Abstract

In this work, we develop the criteria for existence of Ψ- bounded solutions of system of linear dynamic equations on time scales. The advantage of results in this dynamical system is it unifies discrete as well as continuous systems. Initially, we develop if and only if conditions for the existence of at least one Ψ-bounded solution for linear dynamic equation $y^{\Delta}(\tau) = P(\tau)y+g(\tau)$, for each Ψ- delta integrable Lebesgue function g, on time scale T^{+}. Later, we obtain asymptotic nature of Ψ-bounded solutions of dynamical system. Also we provided the examples for supporting the results.
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1. Introduction

The time scale(measure chains) calculus was initially developed and introduced by Stefan Hilger [9]. Coppel [5] developed the results of this model, for nonlinear and linear differential equations and Agarwal [1] developed for difference equations .

Many authors [3, 4, 7, 10] developed the concept of Ψ-bounded solutions for the systems of linear difference equations also an ordinary differential equations. Now we present the results unify the existence of Ψ - bounded solutions of linear difference equations [8] and linear differential equations [6]. Now we consider the non-homogenous vector dynamic linear equation

$$y^{\Delta}(\tau) = P(\tau)y+g(\tau) . \qquad (1)$$

Later, we present result connected to the asymptotic nature of solutions of (1).

2. Preliminaries

Here, we review basic notations, standard results and definitions on time scales, for detailed study regarding time scales refer [2]. Here we assumed the time scale T is unbounded above and below in the entire study , then we get

$$T^{k} = T .$$

Definition 1. [2] The forward jump operator is represented by $\sigma : T \longrightarrow T$, and the backward jump operator is denoted by $\rho : T \longrightarrow T$, also the graininess function denoted by $\mu : T \longrightarrow R^{+}$ and are defined as follows
$\sigma(\tau) = \inf\{s \in T : s > \tau\}, \rho(\tau) = \sup\{s \in T : s < \tau\}, \mu(\tau) = \sigma(\tau)-\tau$, for $\tau \in T$ respectively.

Definition 2.[2] When the time scale T has a left-scattered maximum n, implies $T^{k} = T\text{-}\{n\}$. If T is not a left-scattered maximum n, then $T^{k} = T$.

Definition 3. [2] Let the matrix valued function P on time scale T of order $m \times n$. We call P is delta differentiable on T *if* every element of P is delta differentiable on T and in this situation we substitute

$$P^{\Delta} = [p^{\Delta}{}_{kl}]_{1\le k\le m,\ 1\le l\le n} \; here \;\; P = [p_{kl}]_{1\le k\le m,\ 1\le l\le n}.$$

The homogenous dynamic system corresponding to (1) is

$$y^{\Delta}(\tau) = P(\tau)y \qquad (2)$$

Also X is a fundamental matrix of (2) satisfying $X(v) = I_d$.
The subspace of R^d represented by $\mathbf{Y}_1$ which contains each vector. These values are Ψ-bounded solution of system (2) on T^{+} at $\tau = v$ also let $\mathbf{Y}_2$ be an arbitrary fixed subspace of R^d, supplementary to $\mathbf{Y}_1$. The corresponding projections are Q_1, Q_2 of $R^{d\times d}$ onto $\mathbf{Y}_1$, $\mathbf{Y}_2$ respectively.

Let $\Psi_i : T^{+} \rightarrow (0,\infty)$, i=1,2 ,3, . . . , d denote continuous right dense and regressive functions, now we define

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \ldots \Psi_d].$$

Thus $\Psi(\tau)$ is rd-continuous, regressive and non-singular on T^+.

Definition 4. [7] Let $\varphi : T^+ \to R^d$ be a function is called Ψ-bounded on T^+ provided $\Psi(\tau)\varphi(\tau)$ is bounded on T^+ (i.e., $\exists\ M_1 > 0$ such that $\|\Psi(\tau)\varphi(\tau)\| \le M_1$, for each $\tau \in T^+$).

Definition 5. [7] Let $u : T^+ \to R^d$ be a function is said to be Lebesgue Ψ- deltaintegrable on T^+ provided u is delta measurable and u is Lebesgue deltaintegrable on T^+

i.e., $\int_{\nu}^{\infty} \|\psi(s)u(s)\|\Delta s < \infty.$

3. Vector Dynamic Equations on Time Scales

Here, we develop the main results along with numerical examples.

Theorem 6. *Let* $P \in R$ *be a square matrix of order* $d\times d$ *, and* (1) *has at least one* Ψ*-bounded solution on time scale* T^+ *for each Lebesgue* Ψ*-deltaintegrable function* g *on* T^+ *if and only if there exists a constant* $L > 0$ *such that*

$$\left|\psi(\tau)X(\tau)Q_1X^{-1}(\sigma(s))\psi^{-1}(s)\right| \le L, \text{if } \nu \le \sigma(s) \le \tau, \quad (3)$$

$$\left|\psi(\tau)X(\tau)Q_2X^{-1}(\sigma(s))\psi^{-1}(s)\right| \le L, \text{if } \nu \le \tau \le s.$$

Proof. Initially, we assume that the non homogenous dynamic equation (1) has at least one Ψ-bounded solution on T^+ for each Ψ-deltaintegrable Lebesgue function g on T^+. We define the sets.
$H_\Psi = \{y : T^+ \to R^d : y$ is rd-continuous and Ψ-bounded on $T^+\}$,
$J_\Psi = \{y : T^+ \to R^d : y$ is Lebesgue Ψ-deltaintegrable on $T^+\}$,
$K_\Psi = \{y : T^+ \to R^d : y$ is right dense continuous on all intervals $J \in T^+$, Ψ-bounded solution on T^+, $y(\nu) \in Q_2$,
$y^\Delta(\tau) - P(\tau)y(\tau) \in J_\Psi$. Clearly H_Ψ and J_Ψ are Banach spaces with the following norms

$$\|y\|H_\psi = \sup_{r \ge \nu}\|\psi(\tau)y(\tau)\|,$$

$$\|y\|J_\psi = \int_{\nu}^{\infty}\|\psi(\tau)y(\tau)\|\Delta\tau$$

respectively. Clearly known that the set K_Ψ is a normed linear space , defined by

$$\|y\|k_\psi = \sup_{r\ge\nu}\|\psi(\tau)y(\tau)\| + \left\|y^\Delta(\tau) - P(\tau)y(\tau)\right\|J_\psi.$$

C **Claim:** $(k_\psi, \|.\|k_\psi)$ is a Banach space. Choose any sequence

$\{y_n\}_{n=1}^{\infty}$ in K_Ψ. Implies the sequence $\{x_n\}_{n=1}^{\infty}$is in H_Ψ and $y^\Delta(\tau) - P(\tau)y_n(\tau)\ J_\Psi$. Since H_Ψ is a Banach space, a right dense continuous and Ψ-bounded function $y : T^+ \to R^d$ with the following property

$$\lim_{n\to\infty} \Psi(\tau)y_n(\tau) = y(\tau) \text{ on } T^+.$$

represent $\bar{x}(\tau) = \psi^{-1}(\tau)x(\tau) \in H_\psi$.

Consider

$$\bar{y}(\tau) - \bar{y}(\nu) = \lim_{n\to\infty}(y_n(\tau)) - y_n(\nu).$$

It implies

$[(y_n(\tau)) - \bar{y}(\tau)]_{K_\psi} = 0$ as $n \to \infty$ Therefore K_Ψ is a Banach space on T^+.

Clearly that the solution of (1) on T^+ is

$$y(\tau) = \int_{\nu}^{\infty} K(\tau,\sigma(s))g(s)\Delta s,$$

Where

$$K(\tau,s) = \begin{cases} X(\tau)Q_1X^{-1}(s), \text{ for } \nu \le s \le \tau \\ X(\tau)Q_2X^{-1}(s), \text{ for } \nu \le \tau < s \end{cases}.$$

A Any fixed point $s \in T^+$, now we discuss the following three cases.

C **case(1)**: Suppose s is a right-dense. Let $\delta_k = s_k - s$ and define g as

$$g(t) = \begin{cases} \psi^{-1}(\tau)\xi, & s \le \tau \le s + \delta_k \\ 0, & \text{otherwise} \end{cases}$$

Where ξ is any fixed constant vector , $s \ge \nu$, then $g(\tau) \in J_\psi$ and $\|g(\tau)\|J_\psi = \delta_k\xi$. It implies from the arbitrariness of ξ that $\left|\psi(\tau)K(\tau,\sigma(s))\psi^{-1}(s)\right| \le L.$

Case(2): If s is right as well as left scattered. Now we define g as

$$g(\tau) = \begin{cases} \psi^{-1}(\tau)\xi, & \rho(s) \le \tau \le s \\ 0, & \text{otherwise} \end{cases}$$

Observe that $\|g(\tau)\|J_\psi = \mu\rho(s)|\xi|$, we get

$$\left|\psi(\tau)K(\tau,\sigma(s))\psi^{-1}(s)\right| \le L.$$

Case(3): If s is left dense and right scattered . Now the function g defined as follows

$$g(\tau) = \begin{cases} \psi^{-1}(\tau)\xi, & s \le \tau \le \sigma(s) \\ 0, & \text{otherwise} \end{cases}$$

Since $\|g(\tau)\|J_\psi = \mu s\|\xi\|$, then we get

$$\left|\psi(\tau)K(\tau,\sigma(s))\psi^{-1}(s_n)\right| \le L.$$

therefore (3) follows.

Conversely assume that condition (3) holds. We have to show that dynamic system (1) has at least one Ψ-bounded solution on T^+. Consider the function

$$y(t) = \int_{\nu}^{\tau} X(\tau)Q_1X^{-1}(\sigma(s))g(s)\Delta s - \int_{\tau}^{\infty} X(\tau)Q_2X^{-1}(\sigma(s))g(s)\Delta s$$

for $\tau \geq \nu$, where g is a Lebesgue Ψ-deltaintegrable function on T^+. Clearly we observe that the Ψ-bounded solution $y(\tau)$ of (1) on T^+.

Remark 7. When time scale $T = R$, then $T^+ = R_+ = [0,\infty)$ and Theorem 6 converts Theorem 3.1 of [6]. Suppose the time scale $T = Z$, then $T^+ = N = 0, 1, 2, \ldots$ and Theorem 6 becomes Theorem 2.1 of [8]. Thus Theorem 6 unify Theorem 3.1 of [6] and Theorem 2.1 of [8].

Theorem 8. *Assume that:*

(1) $X(\tau)$ the fundamental matrix of (2) obey the following properties:

(a) $\lim_{\tau\to\infty} \Psi(\tau)X(\tau)Q_1 = 0$;

(b) $|\Psi(\tau)X(\tau)Q_1X^{-1}(\sigma(s))\Psi^{-1}(s)| \leq L$, *for* $v \leq \sigma(s) \leq \tau$,
$|\Psi(\tau)X(\tau)Q_2X^{-1}(\sigma(s))\Psi^{-1}(s)| \leq L$, *for* $v \leq \tau \leq s$

where L is a +ve constant.

(2) The function $g : T^+ \to R^d$ *is Lebesgue Ψ-deltaintegrable on* T^+.

Then each Ψ-bounded solution $y(\tau)$ of (1) satisfies

$$\lim_{\tau\to\infty} \|\Psi(\tau)y(\tau)\| = 0.$$

Proof: Suppose that $y(\tau)$ is the ψ-bounded solution of (1), then there exists a constant L > 0 such that $\|\psi(\tau)y(\tau)\| \leq L$, for each $\tau \geq v$. now we take the function

$$x(\tau) = y(\tau) - X(\tau)Q_1y(\nu) - \int_{\nu}^{\tau} X(\tau)Q_1X^{-1}(\sigma(s))g(s)\Delta s + \int_{\tau}^{\infty} X(\tau)Q_2X^{-1}(\sigma(s))g(s)\Delta s, \forall \tau \geq \nu.$$

From the given data, the function $x(\tau)$ is a Ψ-bounded solution of (2). Then $x(v) \in \mathbf{Y}_1$. otherwise, $Q_1 x(v) = 0$. Thus, $x(v) = Q_2 x(v) \in \mathbf{Y}_2$. Therefore, $x(v) = 0$ and then $x(\tau) = 0$ for $\tau \geq \nu$. Therefore, for $\tau \geq \nu$ we get $y(\tau)$. Since g is a Lebesgue Ψ-deltaintegrable function on T^+, for a given $\in > 0, \exists \tau_1 \geq \nu$ satisfies the following condition

$$\int_{\tau}^{\infty} \|\psi(s)g(s)\|\Delta s < \frac{\varepsilon}{2L}, \text{ for } \tau \geq \tau_1.$$

Further $\exists \tau_2 > \nu_1$ satisfies

$$|\psi(\tau)X(\tau)Q_1| \leq \frac{\varepsilon}{2}[\|y(\nu)\| + \int_{\tau}^{\nu_1} \|X^{-1}(\sigma(s))g(s)\|\Delta s]^{-1}$$

for $\tau \geq \tau_2$, we get

$$\|\psi(\tau)y(\tau)\| \leq |\psi(\tau)X(\tau)Q_1| + [\|y(\nu)\| + \int_{\nu}^{\tau_1} \|X^{-1}(\sigma(s))g(s)\|\Delta s] + L\int_{\tau_1}^{\infty} \|\psi(s)g(s)\|\Delta s < \varepsilon.$$

Which implies that $\lim_{\tau\to\infty} \|\Psi(\tau)g(\tau\| = 0$.

Remark 9. Theorem 8 is never true provided that the function g is Ψ-bounded on T^+, in place of property(2). Theorem 8 does not apply, even though g satisfies with the following property

$$\lim_{\tau\to\infty}\|\Psi(\tau)g(\tau\| = 0.$$

We will show how to utilize our criteria by considering the

numerical example.

Example 10. The linear dynamic equation (1) by taking $P(\tau) = O_2$, then the fundamental matrix solution of (2) is $X(\tau) = I_2$. Now choose the function

$$\psi(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & t+1 \end{bmatrix},$$

Then there exists projections

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Satisfies condition(1) with L=1, if we take

$$g(t) = \begin{bmatrix} \sqrt{t+1} \\ \frac{1}{(1+t)^2} \end{bmatrix},$$

then $\lim_{\tau\to\infty} \|\Psi(\tau)g(\tau\| = 0$. Otherwise , the solutions of (1) are

$$y(t) = \begin{bmatrix} \int\sqrt{(t+1)}\Delta t + c_1 \\ \int\frac{1}{(t+1)^2}\Delta t + c_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{2}{3}(t+1)^{\frac{3}{2}} + c_1 \\ \frac{-1}{t+1} + c_2 \end{bmatrix} \text{when T = R} \\ \begin{bmatrix} \sum_{k=1}^{t}\sqrt{k} + c_1 \\ \sum_{k=1}^{t}\frac{1}{k^2} + c_2 \end{bmatrix} \text{when T = Z.} \end{cases}$$

It implies that the solution of the dynamic equation (1) are

Ψ-unbounded on T^+. If we choose

$$g(t)=\begin{bmatrix}\dfrac{1}{(t+1)}\\ \dfrac{1}{(1+t)^3}\end{bmatrix},$$

then we get

$$\int_{T^+}\|\psi(\tau)g(\tau)\| = \begin{cases}\int_0^\infty \|\psi(t)g(t)\|dt = \int_0^\infty \dfrac{1}{(t+1)^2} = 1, T = R\\ \sum_{k=0}^\infty \|\psi(k)g(k)\| = \sum_{k=1}^\infty \dfrac{1}{k^2} = \dfrac{\pi^2}{6}, T = Z.\end{cases}$$

Thus g(t) is **Ψ**-delta integrable on ***T*** $^+$**.** Otherwise, the solutions of (1) are

$$y(t)=\begin{cases}\begin{bmatrix}\log(t+1)+c_1\\ \dfrac{-1}{2(t+1)^2}+c_2\end{bmatrix}, \text{ when T = R}\\ \begin{bmatrix}\sum_{k=1}^{t}\dfrac{1}{k}+c_1\\ \sum_{k=1}^{t}\dfrac{1}{k^3}+c_2\end{bmatrix}, \text{ when T = Z.}\end{cases}$$

It is clearly observe that the above solutions are **Ψ**-unbounded on **T** $^+$ if and only if $c_2 = 0$. This shows that,

$$\lim_{\tau\to\infty} \|\Psi(\tau)y(\tau)\| = 0.$$

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