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Research paper



Numerical Analysis of a Fractional Order Discrete Prey – Predator System with Functional Response

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Abstract

This study presents numerical examples of Discrete Fractional Order Prey Predator interactions with Functional Response. The process of discretization is applied and the version of discrete equations is obtained. Fixed points are determined and the stability around the fixed points is analyzed. Also the theoretical analysis has been verified from the numerical simulations, which help better understanding of the proposed system. Rich dynamics of system is exhibited by Bifurcation diagram and Periodic Oscillations for suitable parameters values.

Keywords: *Predator Prey System, Fractional Order, Functional Response, Discretization, Stability, Bifurcation.*

1. Introduction

In recent years, due to their importance, [2] population dynamic models have gained considerable interest among scientists and researchers. Models in population dynamics are studied with ordinary differential equations, fractional order differential equations, difference equations and diffusion equations. The fractional order differential equations has wide applications in mathematical biology [7, 8] and other interdisciplinary fields [4] due to their ability to accommodate memory effect.

2. Formulation of Model and Fixed Points

Classical basic Lotka - Volterra model [4] have been improved by many authors by introducing functional responses and harvesting. In the two species interactions, Holling type II functional response expressed by $f(x)=\frac{ax}{b+x}$ [3, 5] plays a vital role. This paper deals with the fractional order prey predator interactions with functional response described by differential equations.

$$\begin{aligned} D^{\alpha} p_1(t) &= rp_1(t)(1-p_1(t)) - \frac{ap_1(t)p_2(t)}{1+bp_1(t)} \\ D^{\alpha} p_2(t) &= \frac{cp_1(t)p_2(t)}{1+bp_1(t)} - ep_2(t) \end{aligned} \tag{1}$$

Where r, a, b, c, e are positive, $p_1(t)$ is prey density, $p_2(t)$ is the predator density, α is the fractional order.
Now, the following fractional order discrete predator prey model with functional response is considered by applying the discretization process outlined in [1, 6, 8].

$$\begin{aligned} p_1(t+1) &= p_1(t) + S\left[rp_1(t)(1-p_1(t)) - \frac{ap_1(t)p_2(t)}{1+bp_1(t)} \right] \\ p_2(t+1) &= p_2(t) + S\left[\frac{cp_1(t)p_2(t)}{1+bp_1(t)} - ep_2(t) \right] \end{aligned} \tag{2}$$

taking $S=\frac{h^{\alpha}}{\Gamma(1+\alpha)}$. This system has the following three fixed points namely;

1. The trivial state $E_0=(0,0)$.
2. The semi-trivial fixed point $E_1=(1,0)$.
3. The interior fixed point $E_2=\left(\frac{e}{c-eb}, \frac{cr}{c-eb}\left[1-\frac{e}{c-eb}\right]\right)$.

3. Analysis of Stability and Numerical Simulations

Stability properties of nonlinear systems can be studied qualitatively by linearization techniques. The stability is investigated from the eigen values of the Jacobin matrix associated with the fixed points of the model [9]. The following theorem [10] is helpful in the discussion of the nature of fixed points determined by the parameters.

Theorem 3.1. Let $\rho(m)=m^2-\mathrm{Tr}(J)m+\mathrm{Det}(J)$ and the roots of $\rho(m)=0$ are m_1, m_2. Suppose that $\rho(1)>0$ then we have the fixed point is

- $|m_{1,2}|<1$ if and only if $\rho(-1)>0$ and $\mathrm{Det}(J)<1$; (p_1^*, p_2^*) is a sink. The sink is locally asymptotically stable.

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- $|m_{1,2}|>1$ if and only if $\rho(-1)>0$ and $\mathrm{Det}(J)>1$; (p_1^*, p_2^*) is a source. The source is unstable.
- $|m_1|>1$ and $|m_2|<1$ (or $|m_1|<1$ and $|m_2|>1$) if and only if $\rho(-1)<0$; (p_1^*, p_2^*) is a saddle.
- $|m_1|=1$ or $|m_2|=1$; (p_1^*, p_2^*) is a non-hyperbolic.

We illustrate the results for the fixed points of the system (2) and present the numerical simulations for those fixed points. For the system (2), we have

$$J(p_1,p_2)=\begin{bmatrix} 1+S\left[r(1-2p_1)-\dfrac{ap_2}{(1+bp_1)^2}\right] & -\dfrac{Sap_1}{(1+bp_1)} \\ \dfrac{Scp_2}{(1+bp_1)^2} & 1+S\left[\dfrac{cp_1}{(1+bp_1)}-e\right] \end{bmatrix}$$

The characteristic polynomial is $p(\lambda)=\lambda^2-\lambda\mathrm{Tr}\,J+\mathrm{Det}\,J$.

We will give conditions under which the populations become extinct.

Theorem 3.2. The trivial fixed point E_0 is stable if $r>0$ and $e>0$ and unstable if $r<0$ and $e<0$.

At the trivial fixed point E_0, we have

$$J(E_0)=\begin{bmatrix} 1+Sr & 0 \\ 0 & 1-Se \end{bmatrix}$$

From the Eigen values and using theorem (1), we have

Corollary 3.3. The trivial fixed point E_0 is a saddle if $r>0$ and $e<0$.

Example 3.4. Consider $\alpha=0.9; h=2.5; r=0.004; a=0.32;$ $b=0.05; c=0.19$ and $e=0.05$ and $E_0=(0,0)$. The Eigen values are $m_1=0.9976$ and $m_2=0.8814$ so that $|m_{1,2}|<1$, then E_0 is stable (see Figure - 1).

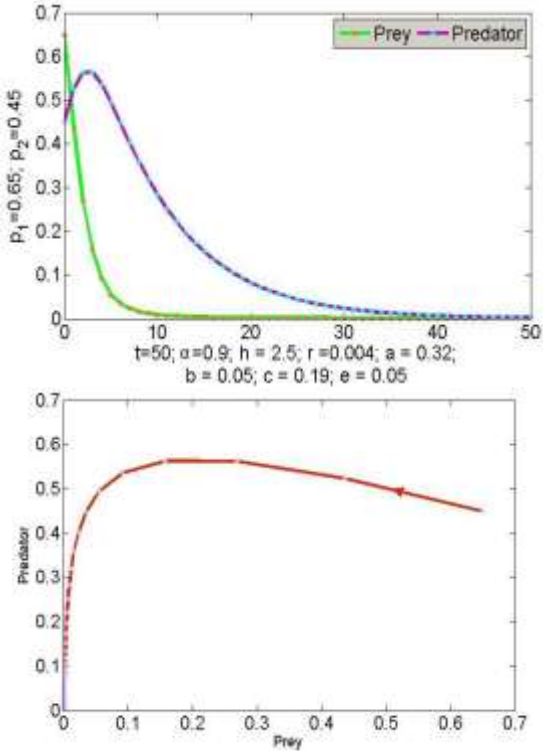


Fig. 1: Stable Solution of E_0 with the initial conditions (0.65, 0.45).

Theorem 3.5. The semi - trivial fixed point E_1 is a asymptotically stable if $c<e(1+b)$ & $r>0$ and unstable if $r<0$ & $c>e(1+b)$.

For the semi - trivial fixed point E_1 , we have

$$J(E_1)=\begin{bmatrix} 1-Sr & -\dfrac{Sa}{1+b} \\ 0 & 1+S\left[\dfrac{c}{1+b}-e\right] \end{bmatrix}$$

From the matrix we can find the Eigen values and using the theorem (1), we get the following result.

Corollary 3.6. The Semi-trivial fixed point E_1 is a saddle if $r>0$ and $c<e(1+b)$.

Example 3.7. For $r=0.4; \alpha=0.9; c=0.3;$ $a=0.2; h=0.1;$ $b=0.5$ and $e=0.3$ and the fixed point $E_1=(1,0)$. The Eigen values are $m_1=0.9476$ and $m_2=0.9869$ so that $|m_{1,2}|<1$. Stability of the semi – trivial fixed point E_1 is shown in Figure – 2.

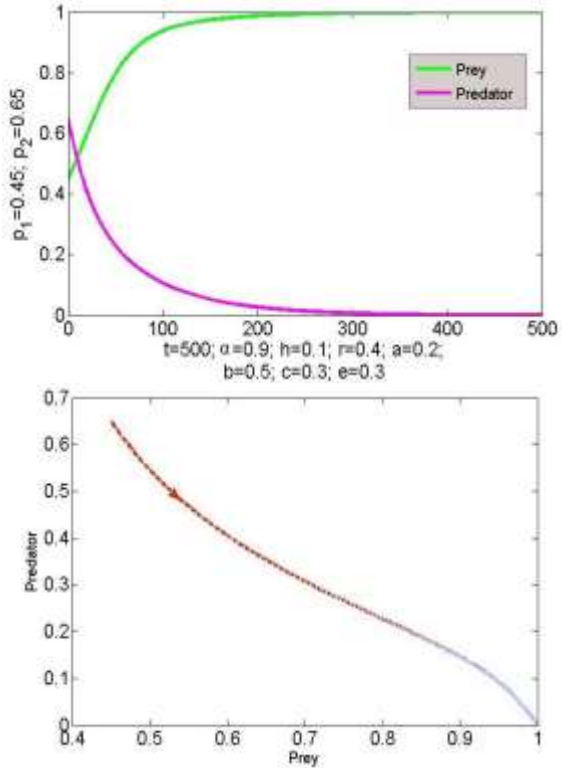


Fig. 2: Stable Solution of E_1 with the initial conditions (0.45, 0.65).

Theorem 3.8. The interior fixed point E_2 is globally stable if

$$r>\frac{4c(c-eb)}{Se\{(eb-c)[2b+S(c-eb-e)]+2(c+eb)\}}$$ and

$$S>\frac{1}{e(1+b)-c}\left[b+\frac{eb+c}{eb-c}\right]$$ and unstable if

$$r>\frac{4c(c-eb)}{Se\{(eb-c)[2b+S(c-eb-e)]+2(c+eb)\}}$$ and

$$S<\frac{1}{e(1+b)-c}\left[b+\frac{eb+c}{eb-c}\right].$$

At the interior fixed point E_2, we have

$$J(E_2)=\begin{bmatrix} 1+\frac{Sre}{c}\left[b+\frac{eb+c}{eb-c}\right] & -\frac{Sae}{c} \\ \frac{Sr}{a}[c-e(1+b)] & 1 \end{bmatrix}$$

From the matrix, we find Trace $(J)=2+\frac{Sre}{c}\left[b+\frac{eb+c}{eb-c}\right]$ and Determinant $(J)=1+\frac{Sre}{c}\left[b+\frac{eb+c}{eb-c}+S[c-e(1+b)]\right]$ of the matrix. Now using the theorem (1), we get the following result.

Corollary 3.9. The interior fixed point E_2 is saddle if $r<\frac{4c(c-eb)}{Se\{(eb-c)[2b+S(c-eb-e)]+2(c+eb)\}}$.

Example.3.10. Taking $\alpha=0.9; a=0.15;\ h=0.1;\ b=0.85;$ $r=0.03; c=0.45$ and $e=0.1,$ the interior fixed point E_2 is $(0.27,0.18)$. The Eigen values are $m_{1,2}=0.9975\pm i0.0294;$ so that $|m_{1,2}|=0.9979<1,$ hence by applying the theorem (1), the interior fixed point E_2 is Stable, see Figure - 3.

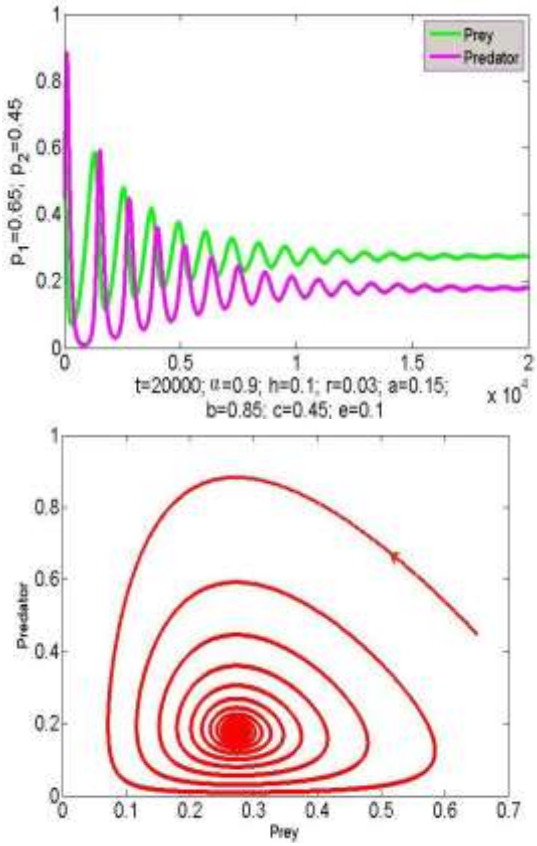


Fig. 3: Stable Solution of E_2 with the initial conditions (0.65; 0.45).

Example 3.11. Consider the values $r=0.5;\ \alpha=0.9; a=1.2;\ h=0.1;\ c=0.35;\ b=1.85; e=0.05.$ The Eigen values are $m_{1,2}=1.0006\pm i0.0159;$ so that $|m_{1,2}|=1.0007>1.$ Hence by using the theorem (1), the interior fixed point E_2 is Unstable, see Figure - 4.

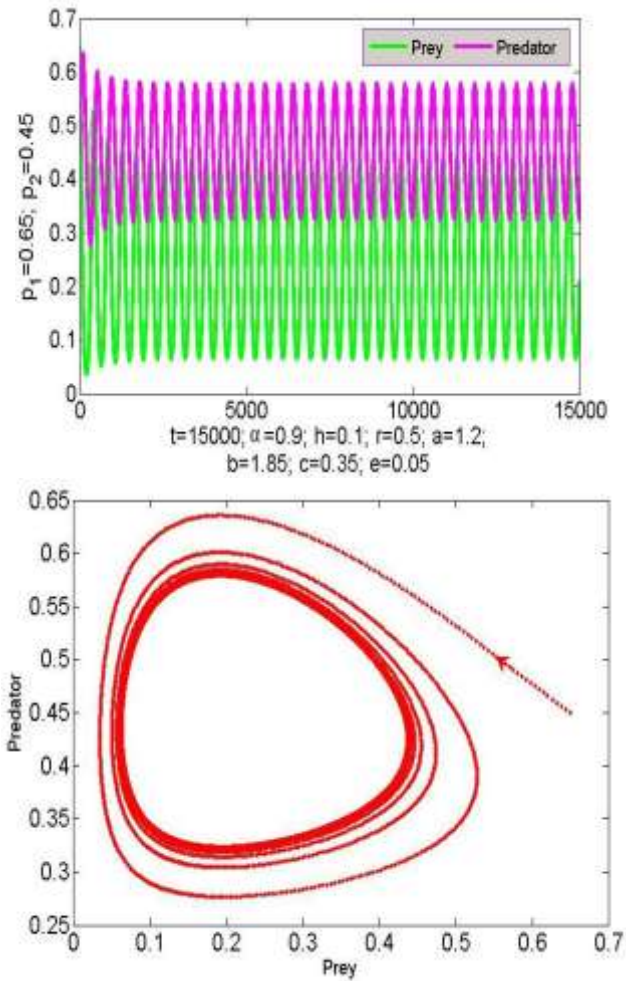


Fig. 4: Unstable Solution of E_2 with the initial conditions (0.65; 0.45).

4. Bifurcation Analysis

Studies in population dynamics aim at determining the changes in qualitative behavior of the model. The abrupt changes can occur in the dynamics for parameter values and such changes are called Bifurcation. In this section, we provide the bifurcation graph of the model (2) in selected ranges. The bifurcation diagram is plotted in the range $r_0<r<r_s;$ where bounded solution can occur. The parameters are assigned as $\alpha=0.9; a=0.6;\ h=2;\ b=0.1;$ $e=0.93;\ c=0.75;$ we plotted the bifurcation graph for the growth rate parameter value in the range $0-1.55.$ See figure - 5; the vertical lines from left to right show the beginning of (i) 2 orbit periodic (ii) 4 orbit periodic (iii) 8 orbit periodic.

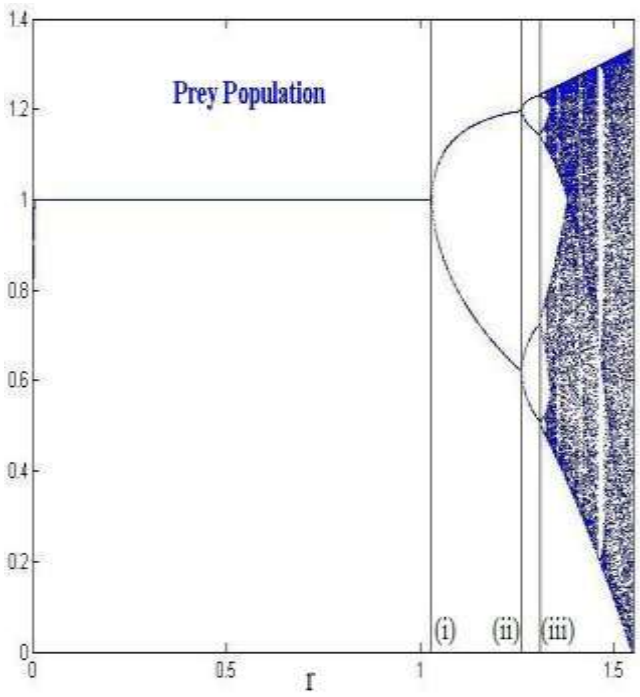


Fig. 5: Bifurcation Diagram of Prey Population

Figure - 6 shows
(a.) Periodic regime of Period - 1 (Uniform Oscillation) at $r = 1.2$;
(b.) Periodic regime of Period - 2 at $r = 1.3$;
(c.) Periodic regime of Period - 4 at $r = 1.32$;
(d.) Non - Periodic oscillation in chaotic region at $r = 1.5$.

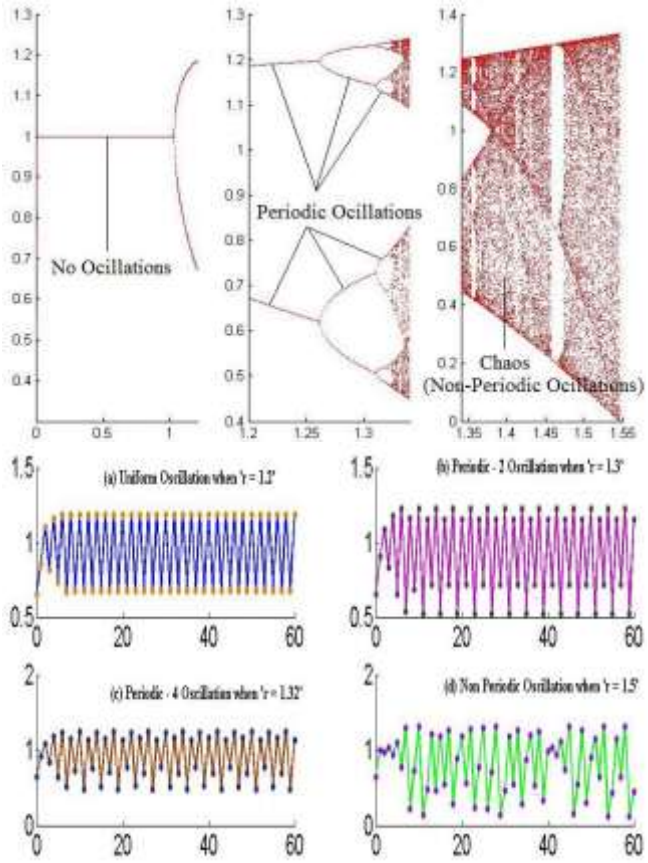


Fig. 6: Periodic Oscillations for the Prey Population

5. Conclusion

This paper dealt with some dynamical properties of Discrete Fractional Order Prey Predator interactions with Functional Response and suitable numerical examples are given. The process of discretization has been applied to get the discrete version. Also the stability at all three fixed points is discussed for the system. The theoretical analysis has been verified from the numerical simulations, which help for better understanding of the proposed system. Finally the rich dynamics of the system has been observed from the Bifurcation diagram and Periodic Oscillations.

References

[1] A. M. A. El-Sayed and S M Salman (2013), On a Discretization Process of Fractional-order Riccati Differential Equation, *Journal of Fractional Calculus and Applications*, Vol. 4, No.2, , pp.251-259.

[2] Berryman, A.A. (1992), The origins and evolution of predator prey theory, *Ecology*, Vol.73, No.5, pp.1530-1535. doi:https://doi.org/10.2307/1940005.

[3] B. S. N. Murthy, M. A. S. Srinivas and Y. Narasimhulu (2017), A Three Species Prey-Predator Holling Type-II Non-Autonomous Discrete Model, *Indian Journal of Science and Technology*, Vol.10, No.22, 14 pages. doi:10.17485/ijst/2017/v10i22/108930.

[4] Ching-Shan chou and Avner Friedman (2016), Introduction to Mathematical Biology: Modeling, Analysis, and Simulations, *Springer International Edition*. pp. 1-172. doi:10.1007/978-3-319-29638-8.

[5] H.N. Agiza, E.M. ELabbasy, H. EL-Metwally, A.A. Elsadany (2009), Chaotic dynamics of a discrete prey – predator model with Holling type II, *Nonlinear Analysis: Real World Applications*, Vol.10,No.1,pp.116-129. doi:https://doi.org/10.1016/j.nonrwa.2007.08.029.

[6] Ivo Petras (2011), Fractional order Nonlinear Systems - Modeling, Analysis and Simulation, *Higher Education Press, Springer International Edition*, pp.1-218. doi:https://doi.org/10.1007/978-3-642-18101-6.

[7] El-Sayed, AMA (1992), On the fractional differential equations, *J. Appl. Math. Comput*. Vol.49, No.(2-3), pp.205-213. doi:https://doi.org/10.1016/0096-3003(92)90024-u.

[8] Ravi P Agarwal, Ahmed MA El-Sayed and Sanaa M Salman (2013), Fractional-order Chuas system: discretization, bifurcation and chaos, *Agarwal et al. Advances in Difference Equations*, Vol.2013, No.1. doi:https://doi.org/10.1186/1687-1847-2013-320.

[9] Saber Elaydi(2008), An Introduction to Difference Equations, 3rd edn, *Springer International Edition*, First Indian Reprint. pp.1-539 doi:https://doi.org/10.1007/978-1-4757-9168-6.

[10] Xiaoli Liu, Dongmei Xiao (2007), Complex dynamic behaviors of a discrete-time predator - prey system, Chaos, Solutions and Fractals, Vol.32, No.1, , pp.80-94. doi:https://doi.org/10.1016/j.chaos.2005.10.081.