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Research paper

Behavior of a Discrete Fractional Order SIR Epidemic Model

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Abstract

In this paper we investigate the dynamical behavior of a SIR epidemic model of fractional order. Disease Free Equilibrium point, Endemic Equilibrium point and basic reproductive number are obtained. Time series plots, phase portraits and bifurcation diagrams are presented for suitable parameter values. Also some numerical examples are provided to illustrate the dynamics of the system.

Keywords: *Epidemic Model, Fractional Order, Stability, Bifurcation, Discretization Process.*

1. Introduction

Fractional Calculus is the study of integrals and derivatives of arbitrary order. The subject of Fractional Differential Equations found many applications in large areas of science and technology during the last two decades. The fractional order systems can be aptly used to describe and study the behaviors of many dynamical systems with memory and hereditary properties [3, 4, 5, 6].

2. Model Description

Mathematical modeling is an effective useful tool in analyzing the spread of infectious diseases and predicting the outbreak. Models help to formulate policies to control an epidemic in a population. In the compartmental model the total population N is divided into three segments denoted by $S(t)$, $I(t)$ and $R(t)$ [1, 2]. The model under study is the system of fractional differential equations given by

$$\begin{aligned} D^{\alpha} S(t) &= b-\beta I(t)S(t) - d\,S(t), \\ D^{\alpha} I(t) &= \beta\, I(t)S(t)-(\delta+d)I(t), \\ D^{\alpha} R(t) &= \delta I(t)-d\,R(t). \end{aligned} \tag{1}$$

The parameters, b is the birth rate or recruitment rate, β is contact rate, d is the death rate, δ is the infectious period and α is an arbitrary order $(0<\alpha<1)$. Now, applying the discretization process for a fractional order system described in [7], we obtain the discrete SIR epidemic model of fractional order as follows

$$\begin{aligned} S(t+1) &= S(t) + K\left(b-\beta S(t)I(t)-d\,S(t)\right), \\ I(t+1) &= I(t) + K\left(\beta S(t)I(t)-(\delta+d)I(t)\right), \\ R(t+1) &= R(t) + K\left(\delta I(t)-d\,R(t)\right), \end{aligned} \tag{2}$$

taking $K=\dfrac{h^{\alpha}}{\Gamma(1+\alpha)}$. In the next sections, we study dynamical behavior in a discrete form of SIR epidemic model of fractional order by computing equilibrium points and the corresponding Jacobian matrix with numerical simulations.

3. Equilibrium Points and Basic Reproduction Number

Now we analyze the stability of the model (2) which has two equilibrium points and Basic Reproductive Number as follows:

(i) Disease Free Equilibrium (DFE) point $E_0 = (b/d, 0, 0)$.

(ii) Endemic Equilibrium (EE) point

$$E_1=\left(\frac{\delta+d}{\beta}, \frac{d(R_0-1)}{\beta}, \frac{\delta(R_0-1)}{\beta}\right).$$

(iii) Basic Reproduction Number $R_0=\dfrac{b\beta}{d(\delta+d)}$.

The following lemma [7] helps in the study of the nature of equilibrium points.

Lemma 3.1. Let $P(\lambda)=\lambda^3+A\lambda^2+B\lambda+C$ where λ_1,λ_2 and λ_3 are an eigen values of the Jacobian matrix and the topological properties of the equilibrium points of the model (2) are given below

(i) $|\lambda_1|<1, |\lambda_2|<1$ and $|\lambda_3|<1$ then $E(S^*,I^*,R^*)$ is a sink (locally asymptotic stable) .

(ii) $|\lambda_1|>1, |\lambda_2|>1$ and $|\lambda_3|>1$ then $E(S^*,I^*,R^*)$ is a source (unstable).

(iii) $|\lambda_1|>1, |\lambda_2|<1$ and $|\lambda_3|<1$ (or $|\lambda_1|<1, |\lambda_2|>1$ and $|\lambda_3|>1$) then $E(S^*,I^*,R^*)$ is a saddle (unstable).

(iv) $|\lambda_1|=1, |\lambda_2|=1$ and $|\lambda_3|=1$ then $E(S^*,I^*,R^*)$ is non-hyperbolic.

4. Stability Analysis with Numerical Simulations

This section dscusses the dynamic behavior of the model (2). We compute the eigen values of the Jacobian matrix corresponding at each equilibrium point.

Theorem 4.1. The DFE point E_0 of the model (2) is locally asymptotically stable if $0 < R_0 < 1$ and E_0 is unstable if $R_0 > 1$.

Proof. At DFE point E_0 we have

$$J(E_0) = \begin{bmatrix} 1-Kd & -KE & 0 \\ 0 & 1-KF & 0 \\ 0 & K\delta & 1-Kd \end{bmatrix}$$

Linearizing the model (2) about E_0 yields the following characteristic equation

$$\begin{aligned} P(\lambda) &= \lambda^3 + \lambda^2\left[K(2d+F)-3\right] \\ &\quad + \lambda\left[3-2KF-4Kd+2K^2Fd+K^2d^2\right] \\ &\quad + \left[K^3d^2F-2K^2Fd-K^2d^2+K(2d+F)-1\right] \end{aligned}$$

where $E = \dfrac{\beta b}{d}$ and $F = \dfrac{1}{d}\left[d(\delta+d)-\beta b\right]$. Let

$$\begin{aligned} a_1 &= K(2d+F)-3, \\ a_2 &= 3-2KF-4Kd+2K^2Fd+K^2d^2, \\ a_3 &= K^3d^2F-2K^2Fd-K^2d^2+K(2d+F)-1. \end{aligned}$$

The eigen values associated to J evaluated at E_0 are

$$\lambda_{1,2} = 1-Kd \quad \text{and} \quad \lambda_3 = 1-KF.$$

From the Jury Test, if $P(1) > 0, P(-1) < 0$ and $a_3 < 1, |b_3| > b_1$, $c_3 > |c_2|$, where $b_3 = 1-a_3^2$, $b_2 = a_1 - a_3a_2$, $b_1 = a_2 - a_3a_1$, $c_3 = b_3^2 - b_1^2$ and $c_2 = b_3b_2 - b_1b_2$, then the roots of $P(\lambda)$ satisfy $|\lambda_j| < 1$ and $0 < R_0 < 1$ then DFE point E_0 is locally asymptotically stable. Suppose $P(1) < 0$ and $R_0 > 1$ then E_0 is unstable.

Proposition 4.2. The following properties hold for DFE point E_0 of the model (2),

(i) E_0 is a sink if $0 < Kd < 2$ and $0 < KF < 2$.

(ii) E_0 is a source if $Kd > 2$ and $KF > 2$.

(iii) E_0 is a saddle if $0 < Kd < 2$ and $KF > 2$ (or) $Kd > 2$ and $0 < KF < 2$.

(iv) E_0 is a non-hyperbolic if $Kd = 2$ and $KF = 2$.

Example 4.3. In the model (2), the following parameter values are considered $\beta = 0.1$, $b = 0.02$, $\delta = 0.2$, $d = 0.02$, $h = 0.1$ and the fractional derivative order $\alpha = 0.9$. with $S(0) = 0.95$, $I(0) = 0.03$ and $R(0) = 0.02$. Applying Jury test, we get $P(1) = 1.0765e-007 > 0$, $P(-1) = -7.9164 < 0$, $a_3 = -0.9791 < 1$ and $R_0 = 0.4545 < 1$, then the DFE point E_0 is locally asymptotically stable (See Fig.1). For $\beta = 0.1$, $b = 0.02$, $\delta = 0.2$, $d = 0.03$, $h = 0.1$, Jury Test yields $P(1) = 3.2969e-007 > 0$, $P(-1) = -7.8834 < 0$, $a_3 = -0.9709 < 1$ and $R_0 = 0.2899 < 1$, . Hence the DFE point E_0 is locally asymptotically stable (See Fig.2).

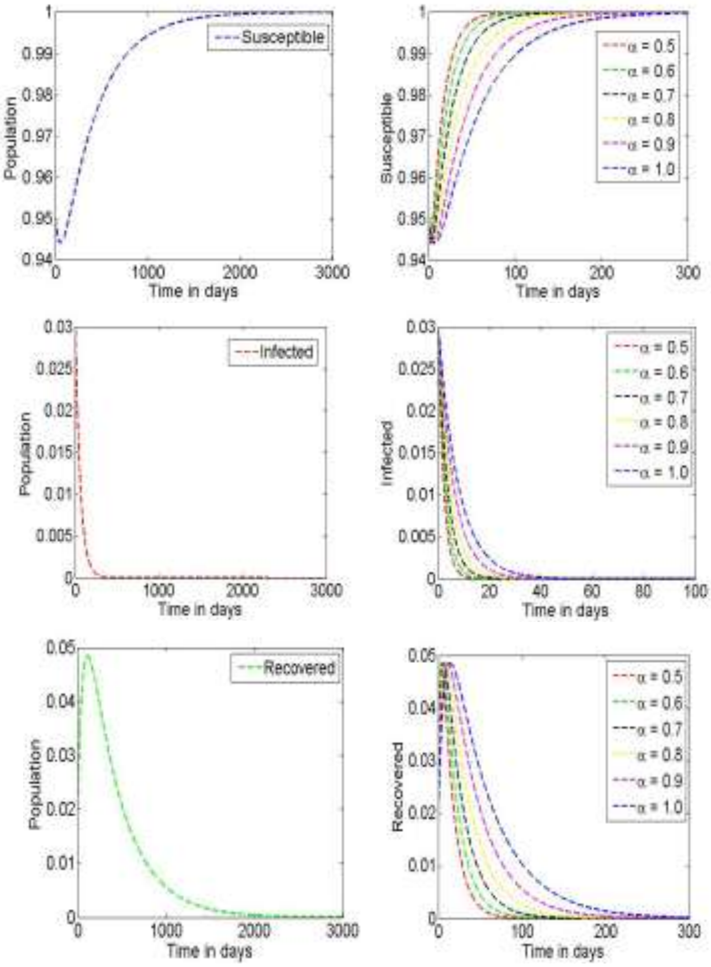
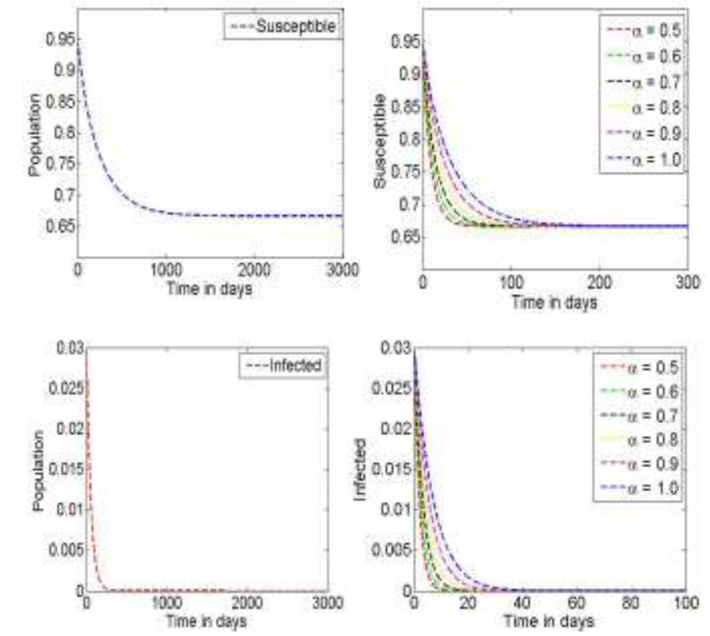


Fig. 1: Time series of the DFE E_0 for Different Fractional order derivatives (α's) and $R_0 < 1$.



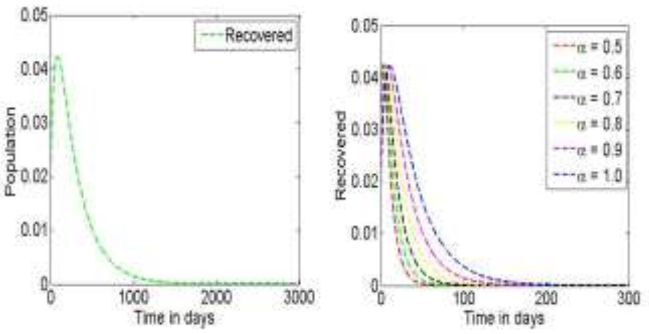


Fig. 2: Time series of the DFE E_0 for Different Fractional order derivatives $(\alpha's)$ and $R_0 < 1$.

Theorem 4.4. The EE point E_1 of the model (2) is locally asymptotically stable if $R_0 > 1$.

Proof. At EE point, E_1 we have

$$J(E_1) = \begin{bmatrix} 1-KdR_0 & -K(\delta+d) & 0 \\ Kd(R_0-1) & 1 & 0 \\ 0 & K\delta & 1-Kd \end{bmatrix}$$

Linearizing model (2) about E_1 yields the following characteristic equation

$$\begin{aligned} P(\lambda) = {} & \lambda^3 + \lambda^2\left[Kd(R_0+1)-3\right] \\ & + \lambda\left[3 - 2Kd(R_0+1) + K^2d^2(2R_0-1) + K^2d\delta(R_0-1)\right] \\ & + \left[\begin{matrix} K^3d^3(R_0-1) + K^3d^2\delta(R_0-1) - K^2d\delta(R_0-1) \\ - K^2d^2(R_0-1) + Kd(R_0+1) - 1 \end{matrix}\right] \end{aligned}$$

Where

$$\begin{aligned} a_{11} &= Kd(R_0+1)-3, \\ a_{22} &= 3 - 2Kd(R_0+1) + K^2d^2(2R_0-1) + K^2d\delta(R_0-1), \\ a_{33} &= K^3d^3(R_0-1) + K^3d^2\delta(R_0-1) - K^2d\delta(R_0-1) \\ &\quad - K^2d^2(R_0-1) + Kd(R_0+1) - 1. \end{aligned}$$

The eigen values associated to J evaluated at E_1 are

$\lambda_1 = 1 - Kd$ and

$$\lambda_{2,3} = \frac{1}{2}\left[2 - KdR_0\right] \pm \frac{1}{2}\sqrt{(2-KdR_0)^2 - 4\left[K^2d^2(R_0-1) + K^2d\delta(R_0-1) - KdR_0 + 1\right]}.$$

From the Jury Test, if $P(1) > 0, P(-1) < 0$ and $a_{33} < 1, |b_{33}| > b_{11}$, $c_{33} > |c_{22}|$, where $b_{33} = 1 - a_{33}^2$, $b_{22} = a_{11} - a_{33}a_{22}$, $b_{11} = a_{22} - a_{33}a_{11}$, $c_{33} = b_{33}^2 - b_{11}^2$ and $c_{22} = b_{33}b_{22} - b_{11}b_{22}$, then the roots of $P(\lambda)$ satisfy $|\lambda_j| < 1$ and $R_0 > 1$ then EE point E_1 is locally asymptotically stable. Suppose $P(1) < 0$ then E_0 is unstable.

Proposition 4.5. The following properties hold for EE point E_1 of the model (2),

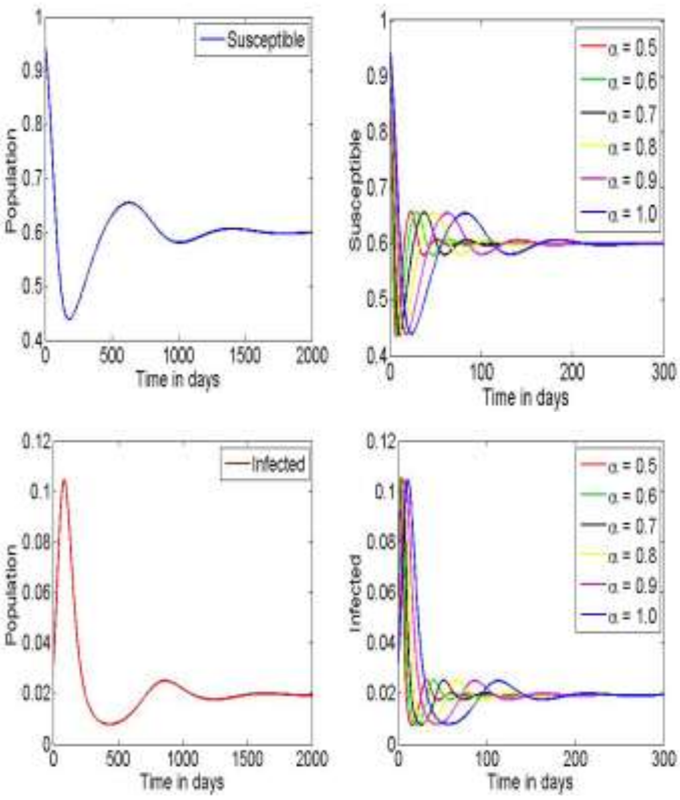
(i) E_1 is a sink if $4 + KdR_0\left[Kd + K\delta - 2\right] < K^2d(d+\delta)$, $0 < Kd < 2$ and $1 < R_0$.

(ii) E_1 is a source if $4 + KdR_0\left[Kd + K\delta - 2\right] > K^2d(d+\delta)$, $Kd > 2$ and $1 > R_0$.

(iii) E_1 is a saddle if $4 + KdR_0\left[Kd + K\delta - 2\right] > K^2d(d+\delta)$, $Kd > 2$ and $1 < R_0$ (or) $0 < Kd < 2$, $1 > R_0$ and $4 + KdR_0\left[Kd + K\delta - 2\right] > K^2d(d+\delta)$.

(iv) E_1 is a non-hyperbolic if $Kd = 2$, $R_0 = 1$ and $4 + KdR_0\left[Kd + K\delta - 2\right] = K^2d(d+\delta)$.

Example 4.6. We take $\beta = 0.6$, $b = 0.025$, $\delta = 0.3$, $d = 0.03$, $h = 0.1$ and the fractional derivative order $\alpha = 0.9$. and $S(0) = 0.95$, $I(0) = 0.03$ and $R(0) = 0.02$ with the fractional derivative order $\alpha = 0.9$. Applying Jury test, we get $P(1) = 2.3708e-005 > 0$, $P(-1) = -7.9607 < 0$, $a_{33} = -0.9902 < 1$ and $R_0 = 1.5152 < 1$, . hence the EE point E_1 is locally asymptotically stable (See Fig.3).



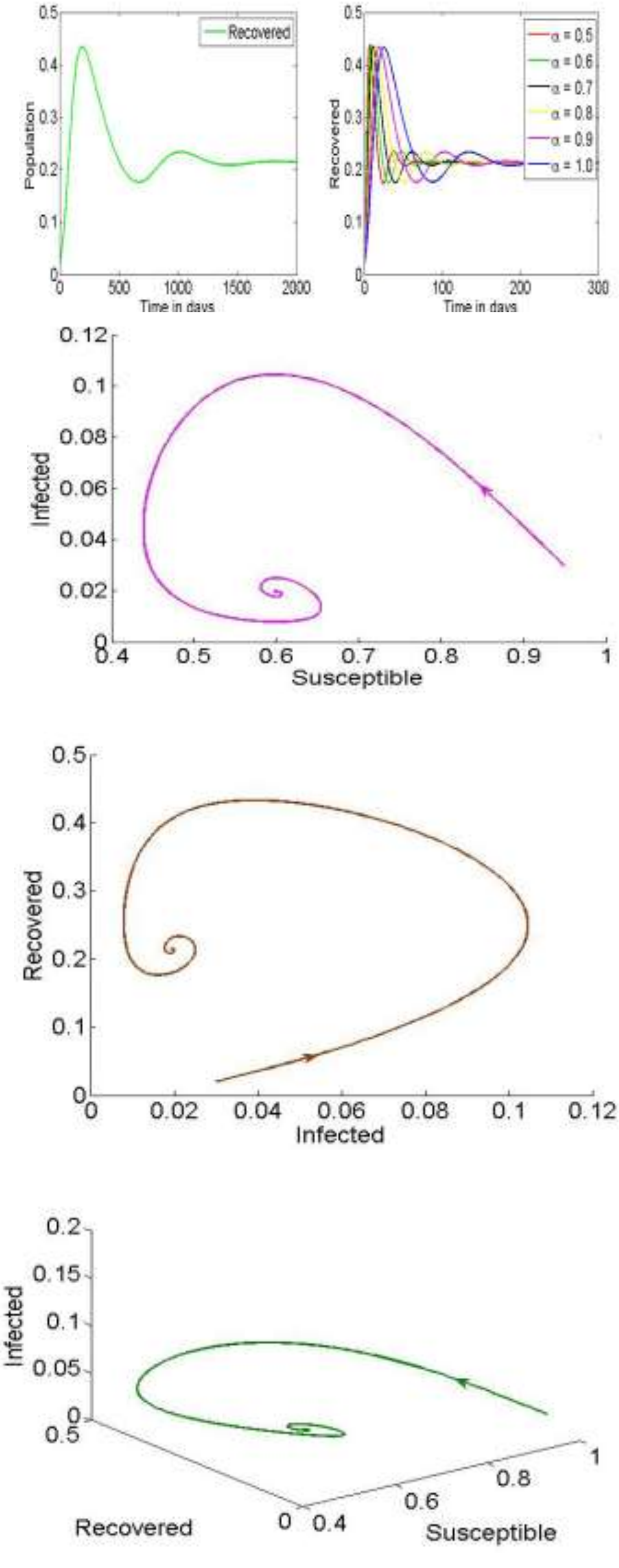
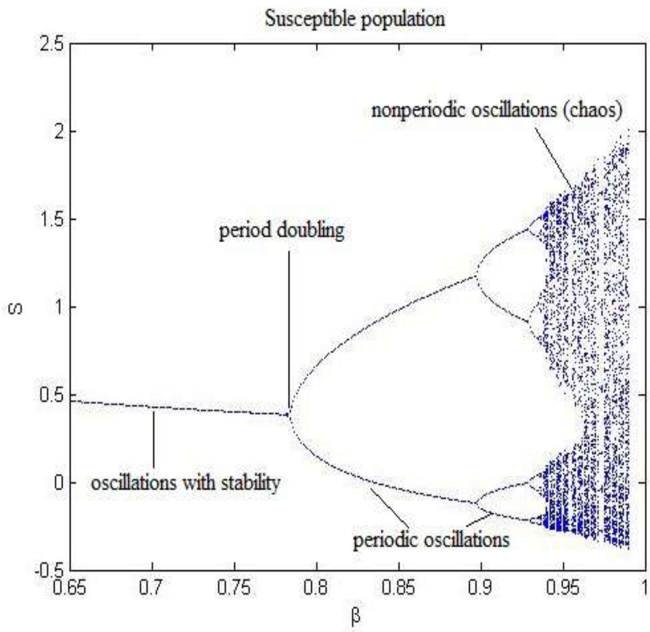
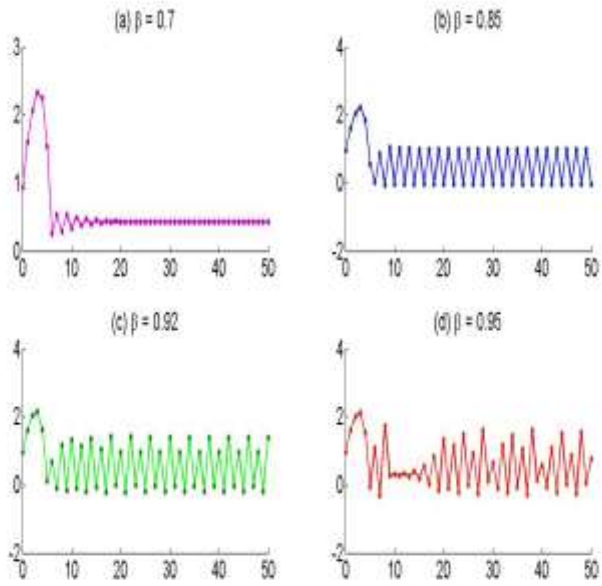


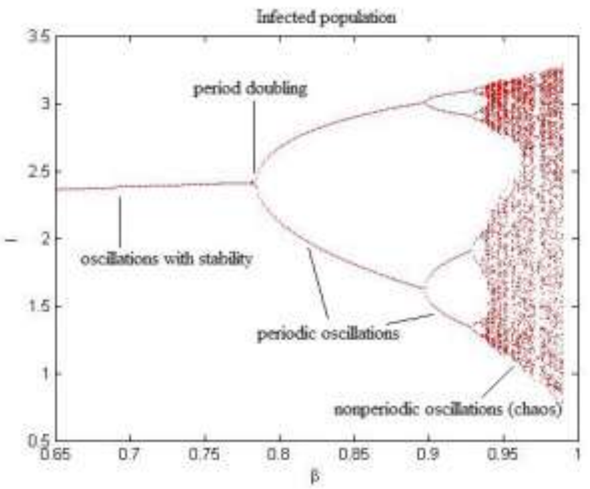
Fig. 3: Time series and Phase portraits of the EE point E_1 for Different Fractional order derivatives $(\alpha's)$ and $R_0 > 1$.

5. Bifurcation

In this section, bifurcation diagrams and time plots of the Model (2) are presented. The Bifurcation diagrams, time plots of Susceptible and Infected populations for $\beta = 0.7, 0.85, 0.92$ and 0.95 respectively provided. When $\beta \in (0.79, 0.89)$ there appear period-2 oscillations. In the range $\beta \in (0.89, 0.93)$ there appear period-4 oscillations and period-8 oscillations. At last, the 2^n period oscillations disappear and the dynamical behaviour change from non-periodic oscillations to the chaotic set with the increasing of β. when $\beta = 0.85$ there appear period-2 oscillations, when $\beta = 0.92$ there appear period-4 oscillations, when $\beta = 0.95$ there appear non-periodic oscillations-chaos (See Fig.4). The bifurcation diagrams, time plots of Susceptible and Infected and Recovered populations using $h = 4.0, 4.5, 4.66$ and $h = 4.9$ respectively.(a) oscillations with stability at $h = 4.0$, (b) periodic oscillations at $h = 4.5$, (c) periodic oscillations at $h = 4.66$, (d) non-periodic oscillations (chaos) at $h = 4.9$ (See Fig. 5).







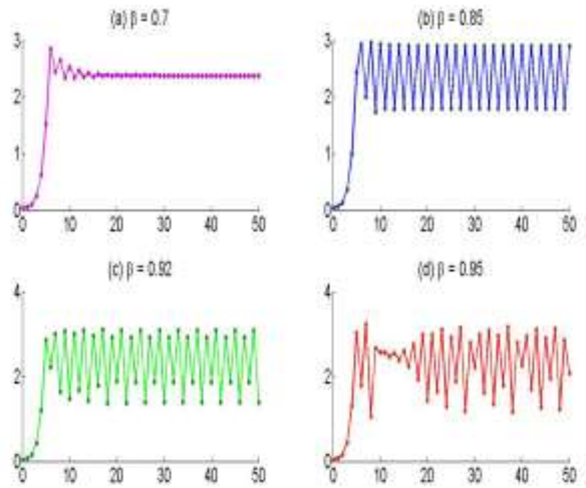
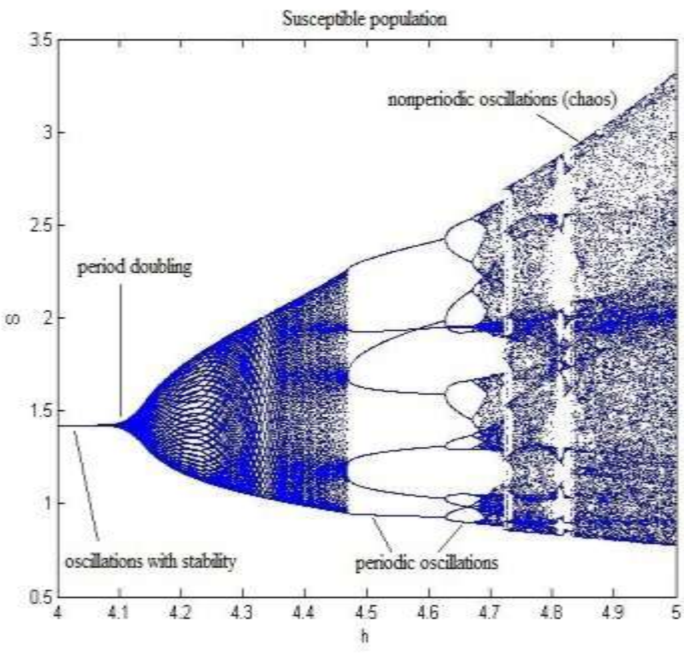
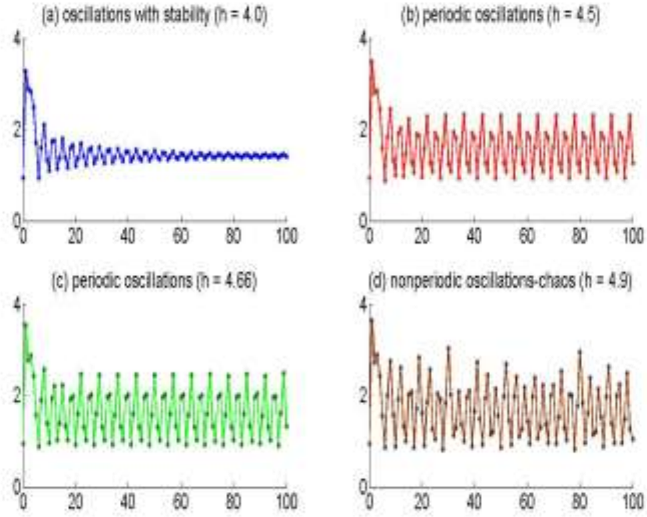
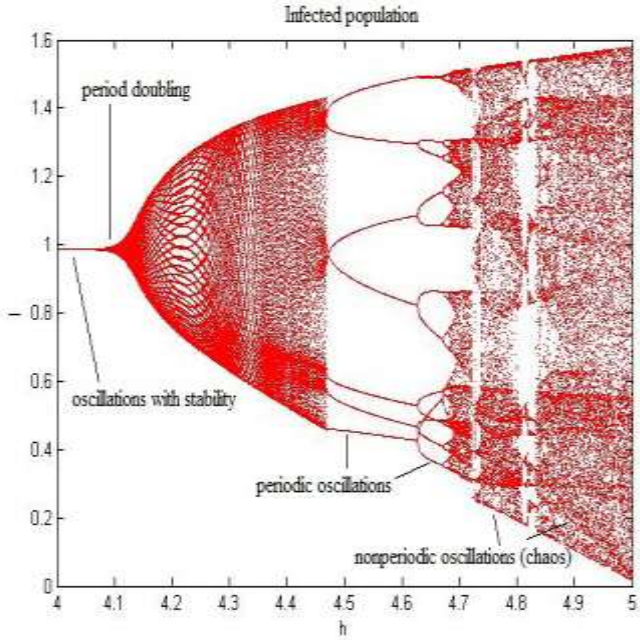
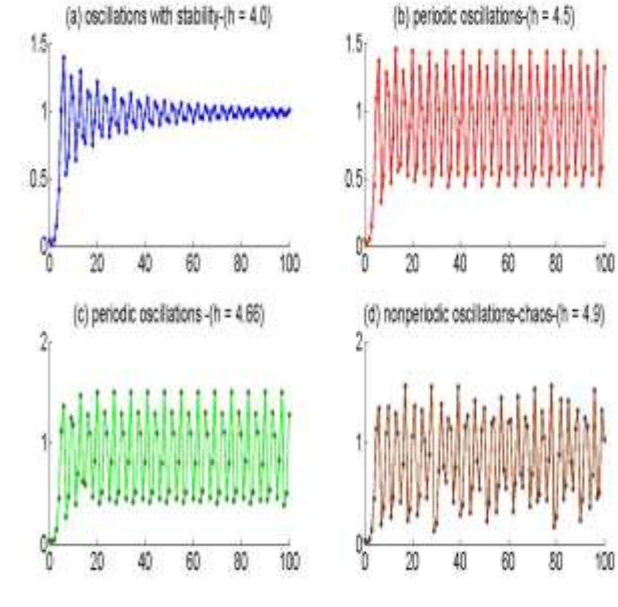


Fig. 4: Bifurcation diagrams of Susceptible and Infected populations with Initial values $(S(0), I(0), R(0))=(0.94, 0.04, 0.02)$, $b=0.8$, $d=0.2$, $\delta=0.1$, $h=1.0$, $\beta\in[0.65, 1.0]$ and $\alpha=0.5$. For (a) $\beta=0.7$; (b) $\beta=0.85$; (c) $\beta=0.92$; (d) $\beta=0.95$.









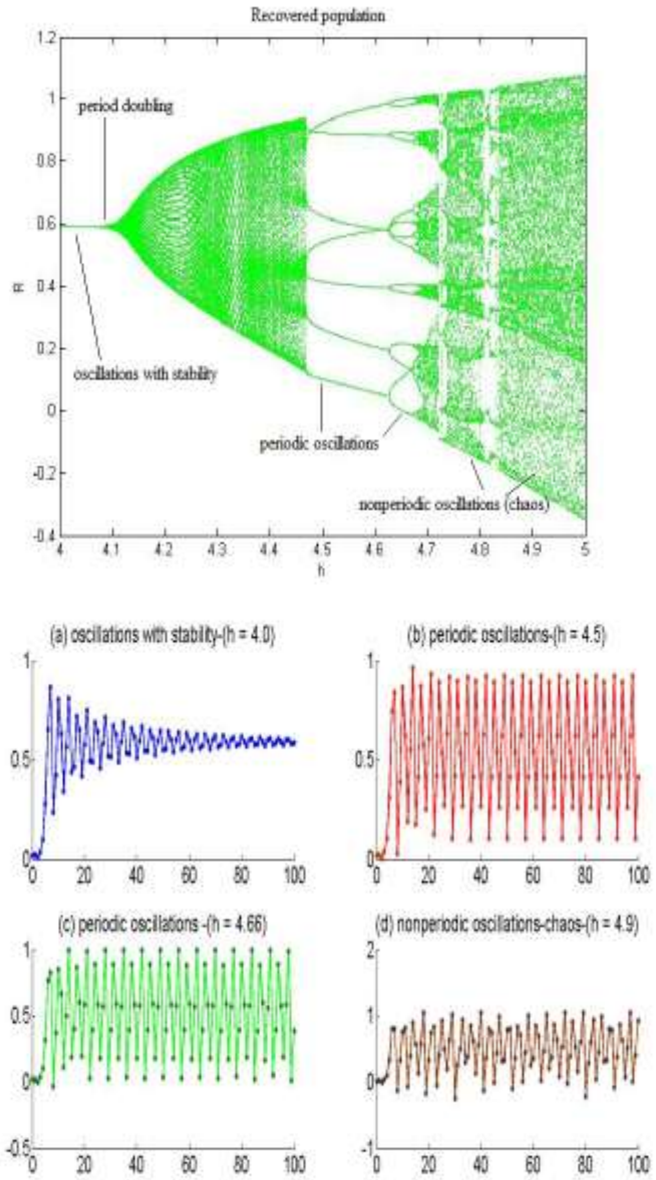


Fig. 5: Bifurcation diagrams**,** time plots of Susceptible, Infected and Recovered populations with Initial value $(S(0)=0.94)$, $(I(0)=0.04)$, $(R(0)=0.02)$, $b=1.2$, $d=0.4$, $\beta=0.45$, $\delta=0.24$, $h\in[4.0, 5.0]$ and $\alpha=0.7$. For (a) $h=4.0$; (b) $h=4.5$; (c) $h=4.66$; (d) $h=4.9$.

6. Conclusion

This paper considered a fractional order SIR epidemic model with a discretization process. Conditions for Stability at Disease Free and Endemic equilibrium points are discussed. Also the numerical examples for different values of fractional orders (α) are provided and the effects are illustrated. Changes in stability nature are demonstrated with Bifurcation diagrams. Time plots and phase portraits are presented for the Susceptible, Infected and Recovered populations.

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