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Research paper

Bounds of Laplacian Energy of a Hypercube Graph

K.Ameenal Bibi[1], B.Vijayalakshmi[2] and R.Jothilakshmi[3 *]

[1,2] PG and Research Department of Mathematics,
D.K.M. College for women (Autonomous), Vellore – 632 001.
[3] PG and Research Department of Mathematics Mazharul Uloom College, Ambur.
**Corresponding author E-mail:jothilakshmiphd@gmail.com*

Abstract

Let Q_n denote the n – dimensional hypercube with order 2^n and size $n2^{n-1}$. The Laplacian L is defined by $L = D - A$ where D is the degree matrix and A is the adjacency matrix with zero diagonal entries. The Laplacian is a symmetric positive semidefinite. Let $\mu_1 \geq \mu_2 \geq \ldots \mu_{n-1} \geq \mu_n = 0$ be the eigen values of the Laplacian matrix. The Laplacian energy is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. In this paper, we defined Laplacian energy of a Hypercube graph and also attained the lower bounds.

Keywords: *Hypercube graph, Laplacian Energy, Regular graph*
AMS Classification: *05C50, 05C69*

1. Introduction

Let G be a graph with order n and size m and A be the adjacency matrix of the graph $A = (a_{ij})_{nxn}$ as

$$a_{ij} = \begin{cases} \sigma_{ij}, & if\ v_i\ is\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of A are assumed in the non increasing order of the graph G [3,7,12]. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e.,
$E(G) = \sum_{i=1}^{n} |\lambda_i|$. [1,2]

I.Gutman and B.zhou defined the Laplacian energy of a graph G in the year 2006 [6,8,10]. Let G be a finite, simple and connected graph with order n and size m [5,11]. The Laplacian matrix of the graph G denoted by L(G) = D(G) - A(G) is a square matrix of order n, where D(G) is the diagonal matrix of vertex degrees of the graph G and A(G) is the adjacency matrix. Let $\mu_1, \mu_2, \ldots, \mu_n$ form the Laplacian spectrum of its Laplacian matrix G then the Laplacian energy LE(G) of G is defined as $LE(G) = \sum_{i=0}^{n} \left|\mu_i - \frac{2m}{n}\right|$. [4,9,12]
The eigen values of the Laplacian graph satisfy the following relations:
$\sum_{i=1}^{n} \lambda_i = 0$, $\sum_{i=1}^{n} \lambda_i^2 = 2m$, $\sum_{i=1}^{n} \mu_i = 2m$, $\sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2$.
The n-dimensional hypercube Q_n is the simple graph whose vertices are the n-tuples with entries in {0,1} and whose edges are the pairs of n-tuples that differ in exactly one position, and its order 2^n and size $n2^{n-1}$.The hypercube graph Q_n is defined in terms of the Cartesian product of the two graphs as follows
$Q_1 = K_2$, $Q_n = K_2 \times Q_{n-1}$.[16]

2. Basic Definitions

Definition: 2.1
Let G be a graph with order n and size m. The Laplacian matrix of the graph G is denoted by $L = (L_{ij})$ is a square matrix defined by

$$L_{ij} = \begin{cases} -1, & if\ v_i\ is\ adjacent\ to\ v_j \\ 0, & if\ v_i\ is\ not\ adjacent\ to\ v_j \\ d_i, & if\ v_i = v_j \end{cases}$$

Where d_i is the degree of the vertex v_i.

Definition: 2.2
Let $\mu_1, \mu_2, \ldots \mu_n$ be the eigenvalues of LE (G), which are called Laplacian eigenvalues of G. TheLaplacian energy LE(G) of G is defined $E(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, where $\frac{2m}{n}$ is the average degree of the graph G.

3. Basic Properties of Hypercube

Some of the basic properties of the hypercubes are as follows:

Regularity: 3.1
Hypercubes are regular graphs and the degree of each vertex of Q_n is equal to n.

Bipartitenss: 3.2
Hypercubes are also bipartite, that is the vertex set of the graph can be partitioned into two subsets, where within each set no vertices are adjacent. In the hypercube Q_n, the cardinality of these sets are equal, so exactly half of the vertices are in each bipartite set.

Vertex Transitivity: 3.3

Hypercubes are vertex transitivity graphs, that is given any two vertices in Q_n, there is an automorphism mapping from one vertex to the other while maintaining vertex adjacency.

Hamiltonicity:3.4

Every hypercube Q_n with n>1 has a Hamiltonian cycle, a cycle that visits each vertex exactly once.

4. Laplacian Matrix of Q_n

To construct the Laplacian matrix of a hypercube graph Q_n, we use the fact that $L(Q_n) = D(Q_n) - A(Q_n)$.

If n=1,

$L(Q_1) = D(Q_1) - A(Q_1)$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

If n=2, $L(Q_2) = D(Q_2) - A(Q_2)$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2.I & 0 \\ 0 & 2.I \end{pmatrix} - \begin{pmatrix} A(Q_1) & I \\ I & A(Q_1) \end{pmatrix}$$

Proceeding like this, $L(Q_n) = D(Q_n) - A(Q_n)$.

$$= \begin{pmatrix} n.I & 0 \\ 0 & n.I \end{pmatrix} - \begin{pmatrix} A(Q_n) & I \\ I & A(Q_n) \end{pmatrix}$$

$$= \begin{pmatrix} n.I - A(Q_{n-1}) & -I \\ -I & n.I - A(Q_{n-1}) \end{pmatrix}$$

Since Q_n is regular graph of degree n, then $D(Q_n) = n.I$

Hence, $L(Q_n) = D(Q_n) - A(Q_n)$.

$= n.I - A(Q_n)$

It follows that,

$L(Q_{n-1}) = (n-1)I - A(Q_{n-1}) = n.I - I - A(Q_{n-1})$

$L(Q_{n-1}) + I = n.I - A(Q_{n-1})$

Hence the Laplacian matrix of the hypercube graph Q_n is

$$L(Q_n) = \begin{pmatrix} L(Q_{n-1}) + I & -I \\ -I & L(Q_{n-1}) + I \end{pmatrix}.$$

4.1Eigen Values of the Laplacian matrix Q_n

Using MATLAB, we found the eigen values of Q_2, Q_3 and Q_4 and their multiciplities as follows

Table 1:

N	Eigen Values	Multiciplities
1	0,2	1,1
2	0,2,4	1,2,1
3	0,2,4,6	1,3,3,1
4	0,2,4,6,8	1,4,6,4,1
...	....	....
N	0,n+2,n+4...... 2n	nC_0, nC_1, nC_2 .... nC_n

5. Laplacian Energy of a Hypercube Graph Q_n.

Theorem:5.1

Let Q_n be the n-dimensional hypercube with order 2^n and degree $n2^{n-1}$, then theLaplacian energy of the hypercube graph Q_n is

$\frac{n}{2} + \left(\frac{n+4}{2}\right) nC_1 + \left(\frac{n+8}{2}\right) nC_2 + \cdots \ldots \ldots . . + \left(\frac{3n-2}{2}\right) nC_{n-1} + \left(\frac{3n}{2}\right)$approximately.

Proof:

From the above table, the eigen values of the n- dimensional hypercube Q_n is

$0, (n+2)^{nC_1}, (n+4)^{nC_2}, \ldots, (2n-2)^{nC_{n-1}}, 2n^{nC_n}$.

The characteristic equation of the hypercube graph Q_n is

$$\mu(\mu - (n+2))^{nC_1}(\mu - (n+4))^{nC_2} \ldots (\mu - (2n-2)^{nC_{n-1}}(\mu - 2n) = 0.$$

Average degree of the hypercube graph Q_n is $\frac{2m}{n}$, where n is the order and m is the size of the hypercube graph Q_n.Here $n = 2^n$ and $m = n2^{n-1}$.

Therefore, Average degree of $Q_n$is$\frac{n}{2}$.

The Laplacian Energy of the hypercube graph Q_n is

$$LE(Q_n) = \left|0 - \frac{n}{2}\right| + \left|n + 2 - \frac{n}{2}\right| nC_1 + \left|n + 4 - \frac{n}{2}\right| nC_2 + \ldots\ldots\ldots + \left|2n - \frac{n}{2}\right|$$

$$= \frac{n}{2} + \left(\frac{n+4}{2}\right) nC_1 + \left(\frac{n+8}{2}\right) nC_2 + \ldots + \left(\frac{3n-2}{2}\right) nC_{n-1} + \left(\frac{3n}{2}\right).$$

We utilize the following known results for attaining the lower bounds for Q_n:

Lemma 5.2

The hypercube graph Q_n has order 2^n and size $n2^{n-1}$, then it is a n- regular graph.

Lemma 5.3

If G is k- regular graph, then LE(G) = E(G).

Lemma 5.4

Let Q_n be n- dimensional hypercube, then Q_n has n+1 distinct eigenvalues. They are given by $q_k = -n+2k$, and the eigenvalue q_k has multiciplity$\binom{n}{k}$, k = 0,1,2 ..... n where $n \geq 1$, $\binom{n}{k}$ is binomial coefficient.

Lemma 5.5

Let n be a non-negative integer then

(1)$\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(2)$\sum_{k=0}^{n} k\binom{n}{k} = n\, 2^{n-1}$

(3)$\sum_{k=0}^{n} k\binom{2n}{k} = n\, 2^{2n-1}$

(4)$\sum_{k=0}^{n} k\binom{2n+1}{k} = (2n+1)2^{2n-1} - \frac{2n+1}{2}\binom{2n}{n}$

(5)$\sum_{j=0}^{n} \binom{j}{k} = \binom{n+1}{k+1}$

Lemma 5.6

Let G be an undirected simple and connected k- regular graph with n, $n \geq 3$ vertices and m edges, $1 \leq k \leq n-1$, then

$$LE(G) \geq k + \frac{2}{n-1}\sqrt{nk(n-k-1)}.$$

Lemma 5.7

Let G be an undirected simple and connected k- regular graph with n, $n \geq 3$ vertices and m edges, $1 \leq k \leq n-1$, then

$LE(G) > \sqrt{\frac{2nk(n-k-1)}{n-1}}$.

6. Lower Bounds for the Laplacian Energy of the Hypercube Graph Q_n

Here, we established the lower bounds for the Laplacian Energy of Q_n: [10,12,15]

Theore: 6.1
For the Hypercube Q_n,

$$LE(Qn) \geq n + \frac{2}{2^n - 1}\sqrt{2^n n(2^n - n - 1)}.$$

$$LE(G) > \sqrt{\frac{n2^{n+1}(2^n-n-1)}{n-1}}.$$

Proof:
By lemma 1 & 2, the n- dimensional hypercube graph is a n- regular graph so it follows the bounds of laplacian energy of regular graph and it is also laplacian energy of Q_n is equal to energy of Q_n. By lemma 5, the lower bound of laplacian energy of hypercube Q_n is

$LE(G) \geq k + \frac{2}{n-1}\sqrt{nk(n-k-1)}$.

Here $k = n$, $n = 2^n$

Hence $LE(Q_n) \geq n + \frac{2}{2^n-1}\sqrt{2^n n(2^n - n - 1)}$.

Similarly by lemma 6, the lower bound of laplacian energy of hypercube Q_n is

$$LE(G) > \sqrt{\frac{2nk(n-k-1)}{n-1}}.$$

$$LE(G) > \sqrt{\frac{n2^{n+1}(2^n - n - 1)}{n-1}}.$$

Conclusion

In this paper, we obtained the Laplacian Energy of the Hypercube Q_n and attained the lower bounds for Q_n. We extended the study of finding the Laplacian Energy for some more special classes of graphs.

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