International Journal of Engineering & Technology, 7 (4.10) (2018) 552-557

International Journal of Engineering & Technology

Website: www.sciencepubco.com/index.php/IJET

Research paper

Analysis of M/M/1 Queuing System with Customer Reneging During Server Vacations Subject To Server Breakdown and Delayed Repair

Ch. Swathi, V. Vasanta Kumar*

Dept.of Mathematics, Koneru Lakshmaiah Education Foundation,
Vaddeswaram, Guntur, Andhra Pradesh, India.
*Corresponding author E-mail:vvkumar@kluniversity.in

Abstract

In this paper, we consider an M/M/1 queuing system with customer reneging for an unreliable sever. Customer reneging is assumed to occur due to the absence of the server during vacations. Detailed analysis for both single and multiple vacation models during different states of the server such as busy, breakdown and delayed repair periods is presented. Steady state probabilities for single and multiple vacation policies are obtained. Closed form expressions for various performance measures such as average number of customers in the system, proportion of customers served and reneged per unit time during single and multiple vacations are obtained.

Keywords: *breakdown, delayed repair ,impatient customers, multiple vacation ,reneging, single vacation.*

1. Introduction

The customer reneging phenomena has been treated by many researchers under different assumptions. For the queuing models with impatient customers during vacations and for the impatient behavior of the customers during server breakdown and repair period readers may refer [2,4,5,6,7 and 8].Altman and Yechiali[1]analyzed the M/M/1,M/M/C,M/G/1,M/G/∞ queuing models with impatient customers during single and multiple vacation cases and proved that the number of customers abandoned during single vacation is smaller than multiple vacation. Dequan Yue et al[3] analyzed customers' impatience in an M/M/1 queuing system under server vacations, where the assumption is that the "impatience timers" of customers depend on the servers' states. They analyzed both multiple and single vacation scenarios and derived the probability generating functions of the number of customers in the system when the server is on vacation period and busy period.

In this paper, we consider M/M/1 queuing system with reneging during the server vacations (single and multiple).While serving the customers the server may not be available due to breakdown or delay in repair. The rest of the paper is organized as follows. In section 2,we describe the mathematical model with multiple vacations. In section 3, we derive the probability generating functions of the steady state probabilities for the number of customers in the system and obtain the closed form expressions for some performance measures like mean system size, average rate of customers served for the multiple vacation model. Sections 4 and 5 deals with the single vacation model. Conclusions are given in section 6.

2. Multiple Vacation Model with Impatient Customers.

We consider an M/M/1 queue where the server takes multiple vacations. At the end of every busy period the server takes vacation. After completion of vacation if the queue line is empty the server will take another vacation. Otherwise, if customers are waiting for service in queue then it starts busy period until the queue line exhaust.

2.1 Mathematical Model

The following assumptions and notations are used to study the steady state behavior of the system.

i. Customers arrive individually with Poisson arrival rate λ , (λ >0).
ii. Customers are served individually and the service times are exponential with mean 1/ μ ,(μ >0).
iii. When there are no customers in the system the server leaves for vacation and the length of the vacation period is exponential with mean 1/ θ , (θ >0).
iv. Whenever a customer arrives to the system realized that the server is on vacation, he activates a timer 'T' exponentially distributed with parameter ξ , which is independent of queue size at that moment. The customer stays in the system, if the server return from vacation before time T expires and leaves the system if the server did not return before T.
v. While in service, the server may breakdown at any instant, and the server breakdowns follow Poisson distribution with parameter α.

vi. Due to non-availability of the repair facility broken down server may not be repaired immediately and there may be delay. Delay times are exponential with mean $1/\gamma$ (γ >0).

vii. Repair times are exponential with mean repair time $1/\eta$,(η >0)

Let L denote the state of the server (L=0 implies the sever is on vacation, L=1 implies the server is working, L=2 implies the server is breakdown and is waiting for repair, L=3 implies the server is under repair) and N denotes the total number of customers in the system.

We define

$$P_{l,n} = \Pr(L = l, N = n), (l = 0,1,2,3, n = 0,1,2,3,...)$$

Then the set of steady sate difference equations are as follows:

$$\lambda P_{0,0} = \xi P_{0,1} + \mu P_{1,1} \quad (2.1)$$

$$(\lambda + n\xi + \theta) P_{0,n} = \lambda P_{0,n-1} + (n+1)\xi P_{0,n+1} \quad , n \geq 1 \quad (2.2)$$

$$(\lambda + \mu + \alpha) P_{1,1} = \mu P_{1,2} + \theta P_{0,1} + \eta P_{3,1} \quad (2.3)$$

$$(\lambda + \mu + \alpha) P_{1,n} = \mu P_{1,n+1} + \theta P_{0,n} + \eta P_{3,n} + \lambda P_{1,n-1}, n \geq 2 \quad (2.4)$$

$$(\lambda + \gamma) P_{2,1} = \alpha P_{1,1} \quad (2.5)$$

$$(\lambda + \gamma) P_{2,n} = \alpha P_{1,n} + \lambda P_{2,n-1} \quad , n \geq 2 \quad (2.6)$$

$$(\lambda + \eta) P_{3,1} = \gamma P_{2,1} \quad (2.7)$$

$$(\lambda + \eta) P_{3,n} = \gamma P_{2,n} + \lambda P_{3,n-1} \quad , n \geq 2 \quad (2.8)$$

3. The Steady State Solution.

Define the following generating functions to solve the steady state equations

$$G_0(z) = \sum_{n=0}^{\infty} P_{0,n} z^n \, , G_1(z) = \sum_{n=1}^{\infty} P_{1,n} z^n ,$$
$$G_2(z) = \sum_{n=2}^{\infty} P_{2,n} z^n \quad , \quad G_3(z) = \sum_{n=3}^{\infty} P_{3,n} z^n \quad (3.1)$$

Multiply equation (2.2) by z^n , sum over n from 1 to ∞, use (2.1) and simplify.
Then we have

$$\xi (1-z) G_0^1(z) - (\lambda(1-z) + \theta) G_0(z) = -(\mu P_{1,1} + \theta P_{0,0}) \quad (3.2)$$

Let $A = (\mu P_{1,1} + \theta P_{0,0})$.
Then (3.2) can be written as

$$G_0^1(z) - \left(\frac{\lambda}{\xi} + \frac{\theta}{\xi(1-z)} \right) G_0(z) = -\frac{A}{\xi(1-z)} \quad (3.3)$$

Multiply this equation by $e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\theta}{\xi}}$ on both sides and integrate from 0 to z.
Then we have

$$e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\theta}{\xi}} G_0(z) = -\frac{A}{\xi}\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx + c \quad (3.4)$$

Put z=0 in this equation to get $c = G_0(0)$.
Substitute the value of c in (3.4) .Then we have

$$G_0(z) = e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}} G_0(0) - \frac{A}{\xi} e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}} \int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx \quad (3.5)$$

Since $G_0(1) > 0$ and $\underset{Z \to 1}{Lt}(1-z)^{\frac{\theta}{\xi}} = 0$, we must have

$$G_0(0) = \frac{A}{\xi}\int_0^1 (1-x)^{\frac{\theta}{\xi}-1} e^{-\frac{\lambda}{\xi}x} dx \quad . \quad (3.6)$$

Let $K = \int_0^1 (1-x)^{\frac{\theta}{\xi}-1} e^{-\frac{\lambda}{\xi}x} dx$. (3.7)

Then

$$G_0(0) = \frac{A}{\xi}K = \left(\frac{\mu P_{1,1} + \theta P_{0,0}}{\xi} \right) K \, . \quad (3.8)$$

Since $G_0(0) = P_{0,0}$, from equation (3.8) we get

$$P_{0,0} = \frac{(\mu P_{1,1})K}{\xi - \theta K} \quad (3.9)$$

From the equation (3.5),

$$G_0(z) = e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}} G_0(0) \left[1 - \frac{\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx}{\int_0^1 e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx} \right] . \quad (3.10)$$

The probability that the sever is on vacation is given by $\sum_{n=0}^{\infty} P_{0,1} = G_0(1)$

Put z=1 in equation (3.10) and use L'Hospital's rule .Then

$$G_0(1) = e^{\frac{\lambda}{\xi}} G_0(0) \underset{z \to 1}{Lt} \frac{\left[1 - \frac{\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx}{\int_0^1 e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx} \right]}{(1-z)^{\frac{\theta}{\xi}}} .$$

$$= e^{\frac{\lambda}{\xi}} G_0(0) \left[\frac{\xi e^{-\frac{\lambda}{\xi}}}{\theta} \frac{1}{K} \right] = \frac{\xi}{\theta K} G_0(0) = \frac{\xi}{\theta K} P_{0,0} . \quad (3.11)$$

Use (3.8), in (3.11) to get

$$G_0(1)=\frac{A}{\theta}=\frac{\mu P_{1,1}+\theta P_{0,0}}{\theta}. \quad (3.12)$$

Multiply equation (2.3) by z and equation (2.4) by z^n, sum over n from 1 to ∞. Then we get

$$[\lambda z(1-z)-\mu(1-z)+\alpha z]G_1(z)=\theta zG_0(z)+\eta zG_3(z)-Az. \quad (3.13)$$

Multiply equation (2.5) by z and equation (2.6) by z^n, $(n\geq 2)$, sum over n from 1 to ∞. Then we get

$$[\lambda(1-z)+\gamma]G_2(z)=\alpha G_1(z). \quad (3.14)$$

Multiply equation (2.7) by z and equation (2.8) by z^n, sum over n from 1 to ∞. Then we get

$$[\lambda(1-z)+\eta]G_3(z)=\gamma G_2(z) \quad (3.15)$$

Substitute $G_2(z)$ from (3.14) in (3.15) and then substitute $G_3(z)$ in (3.13).

Then

$$[(\lambda z(1-z)-\mu(1-z)+\alpha z)(\lambda(1-z)+\gamma)(\lambda(1-z)+\eta)-\eta z\gamma\alpha]G_1(z)$$
$$=(\theta zG_0(z)-Az)(\lambda(1-z)+\gamma)(\lambda(1-z)+\eta) \quad (3.16)$$

From equation (3.3)

$$\xi(1-z)G_0^1(z)-(\lambda(1-z)+\theta)G_0(z)=-A. \quad (3.17)$$

Substitute z=1 in this equation and apply L 'Hospital's rule to get

$$G_0^{|}(1)=\left(\frac{\lambda}{(\xi+\theta)}\right)G_0(1)\cdot \quad (3.18)$$

Differentiate equation (3.17) with respect to z two times and substitute z=1. Then

$$G_0^{||}(1)=\frac{2\lambda}{(2\xi+\theta)}G_0^{|}(1)\cdot \quad (3.19)$$

Differentiate equation (3.16) with respect to z and substitute z=1 to obtain

$$G_1(1)=\frac{\lambda\theta}{\mu(\xi+\theta)(1-r)}G_0(1), \quad (3.20)$$

where $r=\frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)$

Substitute z=1 in (3.14) and (3.15) to obtain

$$G_2(1)=\frac{\alpha}{\gamma}G_1(1), \text{and } G_3(1)=\frac{\alpha}{\eta}G_1(1)\cdot \quad (3.21)$$

Use the normalization condition

$G_0(1)+G_1(1)+G_2(1)+G_3(1)=1$, to obtain

$$P_{0,0}=\frac{\theta K}{\xi}\left(1+\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\frac{\lambda\theta}{\mu(\xi+\theta)(1-r)}\right)^{-1}. \quad (3.22)$$

Differentiate the equation (3.16) with respect to z two times and substitute z=1.

Then

$$G_1^{|}(1)=\frac{\lambda}{\mu(1-r)}\left[\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)+(\mu-\lambda)\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)-\frac{\alpha\lambda}{\gamma\eta}\right]G_1(1)$$
$$+\frac{\lambda\theta}{\mu(1-r)(\xi+\theta)}\left[1+\frac{\lambda}{2\xi+\theta}-\lambda\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)\right]G_0(1) \quad (3.23)$$

Differentiate equation (3.13) with respect to z and substitute z=1. Then

$$G_2^{|}(1)=\frac{\lambda\alpha}{\gamma^2}+\frac{\alpha}{\gamma}G_1^{|}(1) \quad (3.24)$$

Differentiate equation (3.14) with respect to z and substitute z=1.Then

$$G_3^{|}(1)=\frac{\lambda\alpha}{\eta^2}+\frac{\lambda\alpha}{\eta\gamma}+\frac{\alpha}{\eta}G_1^{|}(1). \quad (3.25)$$

Performance measures

(i) Expected number of customers waiting in the queue during the vacation period of the server is given by

$$E(L_0)=G_0^{|}(1)=\frac{\lambda\,\xi}{(\xi+\theta)\theta K}P_{0,0}. \quad (3.26)$$

(ii)Expected number of customers waiting in the queue during the busy period of the server is given by

$$E(L_1)=G_1^{|}(1)=\frac{\lambda}{\mu(1-r)}\left[\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)+(\mu-\lambda)\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)-\frac{\alpha\lambda}{\gamma\eta}\right]G_1(1)$$
$$+\frac{\lambda\theta}{\mu(1-r)(\xi+\theta)}\left[1+\frac{\lambda}{2\xi+\theta}-\lambda\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)\right]G_0(1). \quad (3.27)$$

(iii)Expected number of customers waiting in the queue during the breakdown and waiting for repair periods is given by

$$E(L_2)=G_2^{|}(1)=\frac{\lambda\alpha}{\gamma^2}+\frac{\alpha}{\gamma}G_1^{|}(1)\cdot \quad (3.28)$$

(iv)Expected number of customers waiting in the queue during the server repair period is given by

$$E(L_3)=G_3^{|}(1)=\frac{\lambda\alpha}{\eta^2}+\frac{\lambda\alpha}{\eta\gamma}+\frac{\alpha}{\eta}G_1^{|}(1). \quad (3.29)$$

(v) The mean system size is given by

$$E(L)=E(L_0)+E(L_1)+E(L_2)+E(L_3). \quad (3.30)$$

Let S be the total sojourn time of a customer in the system measured from the moment of arrival until departure, either after completion of a service or as a result of abandonment.

(vi) By Little's formula

$$E(S)=\frac{E(L)}{\lambda}. \qquad (3.31)$$

(vii)The proportion of customers served per unit time is given by

$$P_s=\frac{\mu G_1(1)}{\lambda}=\frac{\theta}{(\xi+\theta)(1-r)}G_0(1)\cdot \qquad (3.32)$$

(viii) The proportion of customers reneged per unit time is given by

$$R_n=\sum_{n=0}^{\infty} n\xi P_{0,n}\ . \qquad (3.33)$$

4. Single Vacation Model with Impatient Customers.

We consider an M/M/1 queuing model where the server takes single vacation if there are no customers in the system to serve. After returning from vacation if there are no customers to serve he stays idle until the customer arrives for the service. Otherwise, if the queue is not empty immediately it starts the busy period.

The set of steady sate difference equations governing this model are as follows:

$$(\lambda+\theta)P_{0,0}=\xi P_{0,1}+\mu P_{1,1} \qquad (4.1)$$

$$(\lambda+n\xi+\theta)P_{0,n}=\lambda P_{0,n-1}+(n+1)\xi P_{0,n+1}, n\ge 1 \qquad (4.2)$$

$$\lambda P_{1,0}=\theta P_{0,0} \qquad (4.3)$$

$$(\lambda+\mu+\alpha)P_{1,n}=\lambda P_{1,n-1}+\mu P_{1,n+1}+\theta P_{0,n}+\eta P_{3,n}, n\ge 1 \qquad (4.4)$$

$$(\lambda+\gamma)P_{2,1}=\alpha P_{1,1} \qquad (4.5)$$

$$(\lambda+\gamma)P_{2,n}=\alpha P_{1,n}+\lambda P_{2,n-1}, n\ge 2 \qquad (4.6)$$

$$(\lambda+\eta)P_{3,1}=\gamma P_{2,1} \qquad (4.7)$$

$$(\lambda+\eta)P_{3,n}=\gamma P_{2,n}+\lambda P_{3,n-1}, n\ge 1 \qquad (4.8)$$

5. The Steady State Solution

We use the same probability generating functions defined earlier to solve the steady state equations.

Multiply equation (4.2) by z^n and summing over n from 1 to ∞

$$\xi(1-z)G_0^1(z)=(\lambda(1-z)+\theta)G_0(z)-\mu P_{1,1}\ . \qquad (5.1)$$

Put z=1 in this equation and use (4.1) to get

$$\theta G_0(1)=(\lambda+\theta)P_{0,0}-\xi P_{0,1}. \qquad (5.2)$$

Multiply the equation (5.1) by $e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\theta}{\xi}}$ and integrate from 0 to z.
Then we have

$$e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\theta}{\xi}}G_0(z)=-\frac{\mu P_{1,1}}{\xi}\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx+c \qquad (5.3)$$

Put z=0 in this equation to get $c=G_0(0)$.
Substitute the value of c in (5.3) to get

$$G_0(z)=e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}}G_0(0)-\frac{\mu P_{1,1}}{\xi}e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}}\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx \qquad (5.4)$$

Since $G_0(1)>0$ and $\underset{Z\to 1}{Lt}(1-z)^{\frac{\theta}{\xi}}=0$, we must have

$$G_0(0)=\frac{\mu P_{1,1}}{\xi}\int_0^1(1-x)^{\frac{\theta}{\xi}-1}e^{-\frac{\lambda}{\xi}x}dx. \qquad (5.5)$$

Let $K=\int_0^1(1-x)^{\frac{\theta}{\xi}-1}e^{-\frac{\lambda}{\xi}x}dx$. (5.6)

Then $G_0(0)=\frac{\mu P_{1,1}}{\xi}K$. (5.7)

From equation (5.4),

$$G_0(z)=e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}}G_0(0)\left[1-\frac{\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx}{\int_0^1 e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx}\right] \qquad (5.8)$$

The probability that the sever is on vacation is given by $\sum_{n=0}^{\infty}P_{0,1}=G_0(1)$

put z=1 in equation (5.8) and apply L'Hospital's rule.Then

$$G_0(1)=e^{\frac{\lambda}{\xi}}G_0(0)\underset{z\to 1}{Lt}\frac{\left[1-\frac{\int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx}{\int_0^1 e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx}\right]}{(1-z)^{\frac{\theta}{\xi}}}=\frac{\xi}{\theta K}P_{0,0}. \qquad (5.9)$$

Multiply equation (4.4) by z^n sum over n from 1 to ∞ and add (4.3) to get

$$[\lambda z(1-z)-\mu(1-z)+\alpha z]G_1(z)=\theta zG_0(z)+\eta zG_3(z)-\left[\frac{\mu\theta}{\lambda}(1-z)+\frac{\xi}{K}z+\frac{\alpha\theta}{\lambda}\right]P_{0,0}\cdot \qquad (5.10)$$

Multiply equation (4.6) by $z^n\ (n\ge 2)$ sum over n from 1 to ∞ and add (4.5) to get

$$[\lambda(1-z)+\gamma]G_2(z)=\alpha G_1(z)-\frac{\alpha\theta}{\lambda}P_{0,0}. \quad (5.11)$$

Multiply equation (4.8) by z^n sum over n from 1 to ∞ and add (4.7) to get

$$[\lambda(1-z)+\eta]G_3(z)=\gamma G_2(z). \quad (5.12)$$

Substitute $G_2(z)$ from (5.11) in (5.12) and then substitute $G_3(z)$ in (5.10)

$$\begin{bmatrix}(\lambda z(1-z)-\mu(1-z)+\alpha z)\\(\lambda(1-z)+\gamma)(\lambda(1-z)+\eta)-\eta z\gamma\alpha\end{bmatrix}G_1(z)$$
$$=\begin{bmatrix}\left(\theta z G_0(z)-\left(\frac{\mu\theta}{\lambda}(1-z)+\frac{\xi}{K}z+\frac{\alpha\theta}{\lambda}z\right)P_{0,0}\right)\\(\lambda(1-z)+\gamma)(\lambda(1-z)+\eta)-\frac{\alpha\gamma\eta\theta}{\lambda}zP_{0,0}\end{bmatrix}. \quad (5.13)$$

Differentiate the above equation with respect to z and substitute z=1. Then

$$G_1(1)=\frac{\theta}{(1-r)}\left[\frac{1}{\mu}G_0^{|}(1)+\left(\frac{1}{\lambda}-\frac{1}{\mu}\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\right)P_{0,0}\right], \quad (5.14)$$

where $r=\frac{\lambda}{\mu}\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)$.

From equation (5.1)

$$\xi(1-Z)G_0^1(z)-(\lambda(1-z)+\theta)G_0(z)=-\mu P_{1,1}. \quad (5.15)$$

Differentiate this equation with respect to z and substitute z=1 to get

$$G_0^{|}(1)=\left(\frac{\lambda}{(\xi+\theta)}\right)G_0(1) \quad (5.16)$$

Differentiate (5.15) with respect to z two times and put z=1. Then

$$G_0^{||}(1)=\frac{2\lambda}{(2\xi+\theta)}G_0^{|}(1). \quad (5.17)$$

use (5.16) in (5.14)to obtain

$$G_1(1)=\frac{\theta}{\mu(1-r)}\left[\left(\frac{\lambda}{(\xi+\theta)}\right)G_0(1)+\left(\frac{\mu}{\lambda}-\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\right)P_{0,0}\right] \quad (5.18)$$

Substitute z=1 in (5.11) and (5.12) to obtain

$$G_2(1)=\frac{\alpha}{\gamma}G_1(1)-\frac{\alpha\theta}{\lambda\gamma}P_{0,0} \text{ and } G_3(1)=\frac{\alpha}{\eta}G_1(1)-\frac{\alpha\theta}{\lambda\eta}P_{0,0} \quad (5.19)$$

The normalizing condition is

$$G_0(1)+G_1(1)+G_2(1)+G_3(1)=1.$$

use this to get

$$P_{0,0}=\begin{bmatrix}\frac{\xi}{\theta K}\left(1+\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\frac{\lambda\theta}{\mu(1-r)(\xi+\theta)}\right)\\+\frac{\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\theta}{\mu(1-r)}\left(\frac{\mu}{\lambda}-\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\right)-\frac{\theta}{\lambda}\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)\end{bmatrix}^{-1} \quad (5.20)$$

Differentiate equation (5.13) with respect to z two times and substitute z=1.Then

$$G_1^{|}(1)=\frac{\lambda}{\mu(1-r)}\left[\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)+(\mu-\lambda)\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)-\frac{\alpha\lambda}{\gamma\eta}\right]G_1(1)$$
$$+\frac{\lambda\theta}{\mu(1-r)(\xi+\theta)}\left[1+\frac{\lambda}{2\xi+\theta}-\lambda\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)\right]G_1(1)$$
$$+\frac{\theta}{\mu(1-r)}\left[\frac{\lambda\alpha}{\gamma\eta}-(\mu+\alpha)\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)\right]P_{0,0} \quad (5.21)$$

Differentiate equation (5.11) with respect to z and substitute z=1, Then

$$G_2^{|}(1)=\frac{\alpha\lambda}{\gamma^2}G_1(1)-\frac{\alpha\theta}{\gamma^2}P_{0,0}+\frac{\alpha}{\gamma}G_1^{|}(1). \quad (5.22)$$

Differentiate equation (5.12) with respect to z and substitute z=1. Then

$$G_3^{|}(1)=\frac{\lambda}{\eta}\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)-\frac{\theta}{\eta}\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)P_{0,0}+\frac{\alpha}{\eta}G_1^{|}(1). \quad (5.23)$$

Performance measures

(i) Expected number of customers waiting in the queue during the vacation period of the server is given by

$$E(L_0)=G_0^{|}(1)=\left(\frac{\lambda}{(\xi+\theta)}\right)G_0(1). \quad (5.24)$$

(ii)Expected number of customers waiting in the queue during the busy period of the server is given by

$$E(L_1)=G_1^{|}(1)=\frac{\lambda}{\mu(1-r)}\left[\left(1+\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)+(\mu-\lambda)\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)-\frac{\alpha\lambda}{\gamma\eta}\right]G_1(1)$$
$$+\frac{\lambda\theta}{\mu(1-r)(\xi+\theta)}\left[1+\frac{\lambda}{2\xi+\theta}-\lambda\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)\right]G_1(1)$$
$$+\frac{\theta}{\mu(1-r)}\left[\frac{\lambda\alpha}{\gamma\eta}-(\mu+\alpha)\left(\frac{1}{\gamma}+\frac{1}{\eta}\right)\right]P_{0,0} \quad (5.25)$$

(iii)Expected number of customers waiting in the queue during the breakdown period while the server is waiting for repair is given by

$$E(L_2)=G_2^{|}(1)=\frac{\alpha\lambda}{\gamma^2}G_1(1)-\frac{\alpha\theta}{\gamma^2}P_{0,0}+\frac{\alpha}{\gamma}G_1^{|}(1). \quad (5.26)$$

(iv)Expected number of customers waiting in the queue during the repair period is given by

$$E(L_3)=G_3^{|}(1)=\frac{\lambda}{\eta}\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)-\frac{\theta}{\eta}\left(\frac{\alpha}{\gamma}+\frac{\alpha}{\eta}\right)P_{0,0}+\frac{\alpha}{\eta}G_1^{|}(1). \quad (5.27)$$

(v) The mean system size is given by

$$E(L)=E(L_0)+E(L_1)+E(L_2)+E(L_3). \tag{5.28}$$

(vi) The mean sojourn time of a customer in the system is given by

$$E(S)=\frac{E(L)}{\lambda}. \tag{5.29}$$

(vii)The proportion of customers served per unit time is given by

$$P_s=\frac{\mu G_1(1)}{\lambda}=\frac{\theta}{(\xi+\theta)(1-r)}G_0(1)\cdot \tag{5.30}$$

6. Conclusions

In this paper, we analyzed the M/M/1 queuing systems with customer reneging during single and multiple vacation periods for an unreliable server. We have obtained the steady state solutions for the number of customers in the system using probability generating functions when the server is on vacation, busy, breakdown and repair periods. We also derived the closed form expressions for various performance measures for both the vacation models.

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