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Research paper



k-Zumkeller Labeling of Graphs

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Abstract

In this paper, we mainly focus on to prove that the graphs, viz., (i)paths, (ii) comb graphs, (iii) cycles, (iv) ladder graphs and (v) $P_n \times P_n$ graphs are k-Zumkeller graphs.

Keywords: *Graphs; labeling; Zumkeller numbers; k-Zumkeller labeling.*

1. Introduction

The classes of labeled graphs can be identified through graph labeling, an interesting and potential research area of discrete mathematics. The labeled graphs finds many interesting applications in science, engineering and technology and we refer [6] for the same. The basic idea of the graph labeling is found in [11]. The collection of various graph labeling can be found in [7]. We refer the text book written by Harary [8] for notations and terminology in graph theory. There is a strong link between number theory and graph theory, the two interesting topics of mathematics. The graphs labeled with Zumkeller numbers [13], a recent development in graph labeling, are potential area of research. In [1, 2, 3, 4, 9], we have investigated two different graph labeling viz., (i) Zumkeller labeling and (ii) Strongly multiplicative Zumkeller labeling. In [5], we introduced the k-Zumkeller graphs and proved that the twig graphs are 4-Zumkeller graphs. In this paper, we further prove that the following interesting graphs, viz., (i) paths, (ii) cycles, (iii) comb graphs, (iv) ladder graphs and (v) $P_n \times P_n$ graphs are k-Zumkeller graphs.

2. Zumkeller Numbers

In this section, we see the definition of Zumkeller numbers. In the year 2003, Zumkeller had generalized the concept of perfect numbers [12] and published a sequence of positive integers known as Zumkeller numbers in A083207 integer (Sloane's) sequence. A Zumkeller number [13] n can be defined as follows.

Definition 2.1. [13]
Let n be a positive integer. The integer n is called a Zumkeller number if the two disjoint sets of positive factors of n should have the same total. That is, the sum of the positive factors of the two disjoint sets is equal. The disjoint partition is called as Zumkeller partition.
For example, the positive integer 30 is a Zumkeller number because its positive factors are partitioned into A = {6, 30} and B = {1, 2,3, 5, 10, 15} such that sum of the elements of A is equal to the sum of the elements of B.
For properties of Zumkeller Numbers, we refer [13].

3. Main Results

This section introduces the concept of the k-Zumkeller labeling of a graph introduced by us in [5] and we show that certain graphs are k-Zumkeller graphs.

Definition 3.1. [5]
Let G = (V, E) be a simple graph. A 1-1 function $f : V \to \mathbb{N}$ is called a k-Zumkeller labeling of the graph G if the induced edge function $f^* : E \to \mathbb{N}$ defined by $f^*(xy) = f(x)f(y)$, $x \in V$, $y \in V$, $xy \in E$ satisfies the following two conditions:
(i) $f^*(xy)$ is a Zumkeller number for all $xy \in E$.
(ii) The number of different Zumkeller numbers used to label the edges of G is k.

Definition 3.2. [5]
Let G = (V, E) be a simple graph. If the graph G assumes a k-Zumkeller labeling then it is known as a k-Zumkeller graph.

For example, a 4-Zumkeller labeling pattern of a graph is given in Figure 1.

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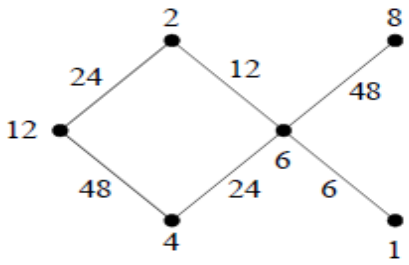


Fig1: A 4-Zumkeller graph

Lemma 3.1.
A sub graph of a k-Zumkeller graph need not be a k-Zumkeller graph.

Lemma 3.2.
The path P_n with n vertices is a 2-Zumkeller graph when n≡ 1 (mod 2) and n≥ 3.

Proof.
Let V = {v_i | 1 ≤ i ≤ n} be the vertex set and E = {$e_i = v_i v_{i+1}$ | 1 ≤ i ≤ n–1} be the edge set of the path P_n. Define a 1-1 function f : V →ℕ such that

$f(v_i) = 2^{\frac{i+1}{2}}$

$f(v_{i+1}) = p2^{\frac{n-i+2}{2}}$ for i = 1, 3, ... where p < 10, p ≠ 2 is a prime number and an induced edge function f^* : E →ℕ such that
$f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1})$ for 1 ≤ i ≤n–1.

Now $f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1}) = 2^{\frac{i+1}{2}} p2^{\frac{n-i+2}{2}} = p2^{\frac{n+3}{2}}$ for i = 1, 3, 5, ... is a Zumkeller number and constant for i = 1, 3, 5, ... and $f^*(e_{i+1}) = f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) f(v_{i+2}) = p2^{\frac{n-i+2}{2}} 2^{\frac{i+3}{2}} = p2^{\frac{n+5}{2}}$ is also a Zumkeller number and is constant for i = 1, 3, 5, ... That is, only 2-Zumkellernumbers are used to label all the edges of the path. Hence the path P_n having odd number of vertices admits a 2-Zumkeller labeling. □

Lemma 3.3.
The path P_n with n vertices is a 2-Zumkeller graph where n≡ 0 (mod 2) and n≥ 4.

Proof.
Let V = {v_i | 1 ≤ i ≤ n} and E = {$e_i = v_i v_{i+1}$ | 1≤ i ≤ n–1} be the vertex set and edge set of P_n respectively. Define a 1-1 function f : V →ℕ such that

$f(v_i) = 2^{\frac{i+1}{2}}$

$f(v_{i+1}) = p2^{\frac{n-i+1}{2}}$ for i = 1, 3, ... where p <10, p ≠ 2 is a prime number and an induced function f^* : E →ℕ such that
$f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1})$ for 1 ≤ i ≤n–1.

Now $f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1}) = 2^{\frac{i+1}{2}} p2^{\frac{n-i+1}{2}} = p2^{\frac{n+2}{2}}$ for i = 1, 3, 5, ... is a Zumkeller number and constant for i = 1, 3, 5, ... and $f^*(e_{i+1}) = f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) f(v_{i+2}) = p2^{\frac{n-i+1}{2}} 2^{\frac{i+3}{2}} = p2^{\frac{n+4}{2}}$ is also a Zumkeller number and is constant for i = 1, 3, 5, ... That is, only two Zumkeller numbers are used to label all the edges of the path. Hence the path P_n having even number of vertices admits a 2-Zumkeller labeling.

Remark 3.1.
The total number of different Zumkeller numbers used to label the edges of the path is 2, which is the degree of the non-pendant vertices of the path.

The 2-Zumkeller labeling of the paths P_8 and P_9 using Lemmas 3.3 and 3.2 are shown in Example 3.1 and Example 3.2.

Example 3.1.
For p = 3 and n = 8, the path P_8 is a 2-Zumkeller graph.

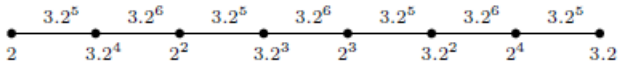


Fig. 2: 2-Zumkeller labeling of P_8

Here we use only two Zumkeller numbers viz., $3(2^5)$ and $3(2^6)$ to label the edges of the path P_8

Example 3.2.
For p = 5 and n = 9, the path P_9 is a 2-Zumkeller graph.

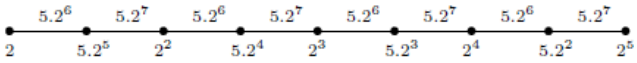


Fig. 3: 2-Zumkeller labeling of P_9

Theorem 3.1.
The path P_n with n vertices is a 2-Zumkeller graph where n≥ 3.

Theorem 3.2.
An even cycle C_n is a 3-Zumkeller graph.

Proof.
Let V = {v_i | 1 ≤ i ≤ n} and E = {$e_i = v_i v_{i+1}$ | 1≤ i ≤ n–1}∪{$e_n = v_n v_1$} be the vertex set and edge set of the even cycle C_n respectively. Now the 1-1 function f : V→ℕ is defined as in Lemma 3.3, and the edge function f^* : E →ℕ can be defined as
(i) $f^*(e_i) = f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1})$, 1 ≤ i ≤ n–1 and
(ii) $f^*(e_n) = f^*(v_n v_1) = f(v_n) f(v_1)$.
In Lemma 3.3, it is proved that $f^*(e_i)$, 1 ≤ i ≤ n–1 receive only the two Zumkeller numbers $p2^{\frac{n+2}{2}}$ and $p2^{\frac{n+4}{2}}$.
Now $f^*(e_n) = f^*(v_n v_1) = f(v_n) f(v_1) = p\, 2^1\, 2^1 = p\, 2^2$ is the third Zumkeller number.
Hence the even cycle C_n is a 3-Zumkellergraph.

Corollary 3.1.
The odd cycle C_n, n ≥ 5 is not a 3-Zumkeller graph.

Remark 3.2.
The odd cycle C_3 is a 3-Zumkeller graph.

Example 3.3.
The even cycle C_6 is a 3-Zumkeller graph for p = 3.

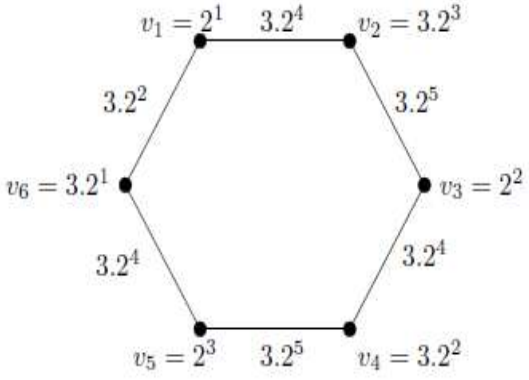


Fig. 4: 3-Zumkeller labeling of C_6

Here three distinct Zumkeller numbers viz., $3(2^4)$, $3(2^5)$,$3(2^2)$ are used to label the edges of the cycle C_6.

Theorem 3.3.
The comb graph $P_n \circ K_1$ is a 3-Zumkeller graph.

Proof.
Let $V = \{v_i, u_i \mid 1 \le i \le n\}$ be the vertex set and $E = \{v_i v_{i+1} \mid 1 \le i \le n-1\} \cup \{v_i u_i \mid 1 \le i \le n\}$ be the edge set of the comb graph. Define an injective function $f : V \to \mathbb{N}$ such that
$f(v_i) = 2^i$
$f(v_{i+1}) = p\,2^{n-i}$
$f(u_i) = p\,2^{n-i+1}$
$f(u_{i+1}) = 2^{i+1}$; for i = 1, 3, ... and p < 10 and p is a prime number not equal to 2.
and an edge function $f^* : E \to \mathbb{N}$ such that $f^*(uv) = f(u)f(v)$.
The edge labels of the comb graph are calculated as follows.
(i) $f^*(v_i v_{i+1}) = f(v_i)\, f(v_{i+1}) = 2^i\, p\, 2^{n-i} = p\, 2^n$ which is a Zumkeller number and is a constant,
(ii) $f^*(v_i u_i) = f(v_i)\, f(u_i) = 2^i\, p\, 2^{n-i+1} = p\, 2^{n+1}$ which is also a Zumkeller number and is a constant and
(iii) $f^*(v_{i+1} v_{i+2}) = f(v_{i+1}\})\, f(v_{i+2}) = p\, 2^{n-i} 2^{i+2} = p\, 2^{n+2}$ is also a Zumkeller number and is a constant.
That is, the edges of the comb graph take exactly 3 Zumkeller numbers.
Hence the comb graph $P_n \circ K_1$ is a 3-Zumkeller graph. □

Example 3.4.
The 3-Zumkeller labeling pattern of $P_6 \circ K_1$ is shown in the Figure 5.

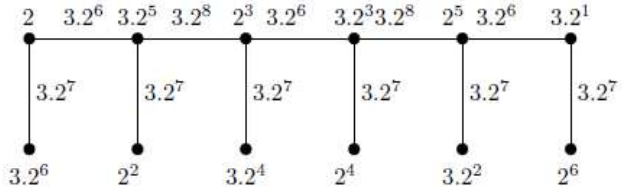


Fig. 5: 3-Zumkeller labeling of a comb graph

Remark 3.3.
The total number of different Zumkeller numbers used to label the edges of the comb graph is 3, which is the greatest degree of the non-pendant vertices of the graph.

Corollary 3.2.
For all n, the ladder graph L_n admits a 3-Zumkeller labeling.

Proof.
By joining the vertices u_i and u_{i+1} of the comb graph $P_n \circ K_1$ in the Theorem 3.3, we get the ladder graph L_n.
We follow the same labeling pattern as in the Theorem 3.3.
For the edge $u_i u_{i+1}$, we define the edge function f^* as
$f^*(u_i u_{i+1}) = f(u_i)\, f(u_{i+1}) = p\, 2^{n-i+1}\, 2^{i+1} = p2^{n+2}$ is a Zumkeller number.
Similarly, for the edge $u_{i+1} u_{i+2}$ we have
$f^*(u_{i+1} u_{i+2}) = f(u_{i+1})\, f(u_{i+2}) = 2^{i+1}\, p2^{n-(i+2)+1} = 2^{i+1}\, p\, 2^{n-i-1} = p\, 2^n$ which is also a Zumkeller number.
Hence the ladder graph L_n is a 3-Zumkeller graph. □

Example 3.5.
The 3-Zumkeller labeling of a ladder graph L_5 using the above corollary is shown in Figure 6.

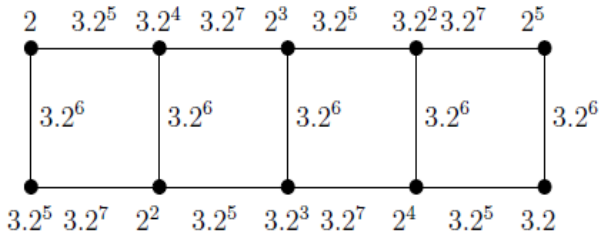


Fig. 6: 3-Zumkeller labeling of the ladder graph L_5

Remark 3.4.
In the 3-Zumkeller ladder graph, 3 represents the greatest degree of the vertices of the graph.

Theorem 3.4.
The square grid $P_n \times P_n$ for $n \equiv 0 \pmod 2$ admits (n+1) a Zumkeller labeling.

Proof.
Let $V = \{v_{ij};\ 1 \le i \le n,\ 1 \le j \le n\}$ be the vertex set and $E = \{v_{ij}\, v_{ij+1},\ v_{ij}\, v_{i+1j};\ 1 \le i \le n-1,\ 1 \le j \le n-1\}$ be the edge set.

Define a 1-1 function $f : V \to \mathbb{N}$ such that
For $i \equiv 1 \pmod 2$
$f(v_{11}) = 2$

$$f(v_{n-k\ n}) = \begin{cases} 3.2 & \text{for } k = 1 \\ 2^{k+1} f(v_{n-(k-2)\ n}) & \text{for } k \equiv 1 \pmod 2 \text{ and } 3 \le k \le n+1 \end{cases}$$

$$f(v_{1j}) = \begin{cases} 2^j f(v_{1\ j+2}) & \text{for } j \equiv 0 \pmod 2 \\ 2^{j-2} f(v_{1\ j-2}) & \text{for } j \equiv 1 \pmod 2 \end{cases}$$

$$f(v_{ij}) = \begin{cases} 2f(v_{i-1\ j+1}) & \text{for } j + j \equiv 0 \pmod 2 \\ \dfrac{f(v_{i-1\ j+1})}{2} & \text{for } i + j \equiv 1 \pmod 2 \end{cases}$$

For $i \equiv 0 \pmod 2$
$f(v_{2n}) = 2^{n-2}\, f(v_{2\ n-2})$ for $n \ge 4$
$f(v_{in}\}) = 2^{n-i+1}\, f(v_{i\ n-2})$ for $n \ge 4$, $i \ge 4$
$f(v_{nn}) = 2^{n^2/2}$
Define an induced edge function

$$f^* : E \to \left\{ 3.2^{\frac{n^2}{2}-k}; k \in \left[-\frac{n}{2}, \frac{n}{2} \right] \right\}$$

Now, the numbers of the form $3.2^{\frac{n^2}{2}-k}$ for $k \in \mathbb{N}$ are Zumkeller numbers. For completeness of the theorem, it is enough to prove that the number of Zumkeller numbers used to label the edges is n+1. Now the total number of edges in $P_n \times P_n$ square grid is 2n(n–1). Out of these edges,

(i) 2n–3 edges receive $3.2^{\frac{n^2}{2}+\frac{n}{2}}$ for $n \ge 4$

(ii)2n–3 edges receive $3.2^{\frac{n^2}{2}-\frac{n}{2}}$ for $n \ge 4$

(iii) 4m–10 edges receive $3.2^{\frac{n^2}{2} \pm \left(\frac{m-2}{2}\right)}$ for $n \ge m$ and m is even, $m > 2$ and

(iv) the edges $e_{12,1}$ and $e_{n,n-1n}$ receive $3.2^{\frac{n^2}{2}}$.

Hence the proof.

The 5-Zumkeller labeling of the square grid $P_4 \times P_4$ is given in the following example.

Example 3.6.
The total number of edges in this graph is $2 \times 4 \times 3 = 24$. From the vertex function defined in the above theorem, the range of the edge set is $\{3.2^6, 3.2^7, 3.2^8, 3.2^9, 3.2^{10}\}$. Out of these $2n-3 = 2 \times 4 - 3 = 5$ edges receive $3.2^{8 \pm 2}$. That is, 5 edges receive 3.2^{10} and 5 edges receive 3.2^6. For m =4, $4m-10 = 4 \times 4 - 10 = 6$ edges receive $3.2^{\frac{4^2}{2} \pm \left(\frac{4-2}{2}\right)} = 3.2^{8 \pm 1}$.

That is, 6 edges receive the Zumkeller number 3.2^7, 6 edges receive the Zumkeller number 3.2^9, the edges $e_{12,1}$ and $e_{4,34}$ receive

the Zumkeller number 3.2^8. In total 5+5+6+6+2 = 24 edges of $P_4 \times P_4$ receive only 5-Zumkellernumbers.

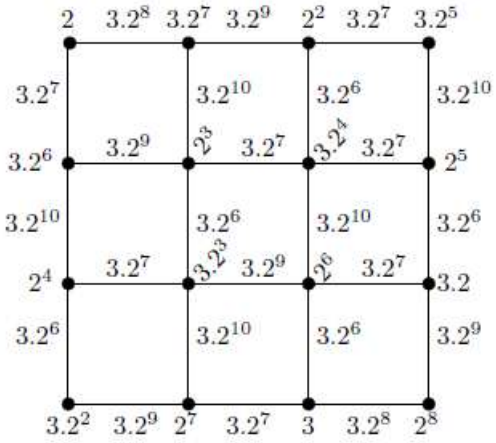


Fig. 7: 5-Zumkeller labeling of the square grid P4 × P4

4. Conclusion

In this article, we proved that the graphs viz., paths, cycles, comb graphs, ladder graphs and square grid $P_n \times P_n$ are k-Zumkeller graphs.

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