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Research paper

Barzilai-Borwein gradient method for solving fuzzy nonlinear equations

Ibrahim Mohammed Sulaiman[1], Mustafa Mamat[1*], Muhammad Yusuf Waziri[2], Nor Shamsidah Amir Hamzah[3]

[1]*Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu, Malaysia*
[2]*Bayero University Kano, Nigeria*
[3]*Faculty of Applied Sciences and Technology, Universiti Tun Hussien Onn Malaysia, Malaysia*
**Corresponding author: must@unisza.edu.my*

Abstract

In this paper, we employ a two-step gradient method for solving fuzzy nonlinear equations. This method is Jacobian free and only requires a line search for $k = 0$. The fuzzy coefficients are presented in parametric form. Numerical experiments on well-known benchmark problems have been presented to illustrate the efficiency of the proposed method.

Keywords: *Barzilai and Borwein method; gradient; Fuzzy nonlinear equations; parametric form,*

1. Introduction

Consider the following nonlinear equation

$$F(x) = 0 \qquad (1)$$

where F is a mapping from R^n to R^n. The above equation arises quit often in natural sciences, engineering and social sciences. Numerous methods have been proposed and applied to obtain its solution. However, in many cases, the parameters of (1) are usually presented by fuzzy numbers rather than crisp number; thus, the solution depends on the root of fuzzy equation. Fuzzy set theory was first incorporated by [2]. However, some standard known analytical techniques for solving fuzzy equations where proposed by [6, 7, 14]. Investigation indicated that these techniques are only able to solve fuzzy linear and quadratic equations. Thus, for fuzzy nonlinear equations such as

$$(\text{I}).\, ax^3 + bx^2 + cx - d = 0$$
$$(\text{II}).\, ix^2 + jcosx = e$$

where $x, a, b, c, d,$ and e are all fuzzy numbers, there is need to develop numerical methods for obtaining the root of such equations. The steepest descent method for solving fuzzy nonlinear equations was considered in [9]. This method is simple and requires less computational task in obtaining the solution of any given problem. On the other hand, this method moves in orthogonal steps and thus, performs poorly [16]. It is also badly affected by ill-conditioning and converges linearly to the solution [4]. The Newton method which can also be regarded as a steepest descent method under the ellipsoid norm $\|\cdot\|G_k$ [8] was employ to obtain the solution of fuzzy nonlinear equation by [10]. This method converges rapidly with a good initial guess. The main drawback of the Newton's method is Jacobian computation and storage in every iteration. Recently, a bracketing method known as Regula Falsi method was devised for solving fuzzy nonlinear equations [11]. This method is less expensive computationally and it's bound to converge since it brackets the root of any given problem. However, the convergence is linear and very slow towards the solution point due to lack of gradient information. For more details on numerical methods for solving fuzzy nonlinear equations, please refer to [3, 4, 12, 21]. In this paper, a gradient-base method also known as Barzilai-Borwein gradient method is proposed to solve fuzzy nonlinear equations. This method derived it step size from a two-point approximations to the secant equation underlying quasi-Newton method [5]. The Barzilai-Borwein method requires less computations since the only line searches required is for $k = 0$; thus, it speeds up convergence of the gradient method.

For recent findings and further references on Barzilai and Borwein method, please refer to [17-20].

In section 2, we present brief overview and some definitions of the fuzzy nonlinear equations. The Barzilai-Borwein gradient method for solving fuzzy nonlinear equation is presented in section 3. Numerical results of some examples are illustrated in section 4, and conclusions in the last section.

2. Preliminaries

We present some useful definitions of fuzzy numbers as follows.

Definition 1. [1, 13]. A fuzzy number is a set like $u: R \to I = [0,1]$ which satisfy the following

(1). u is upper semi-continuous,

(2). $u(x) = 0$ outside some interval $[c, d]$,

(3). there are real numbers a, b such that $c \le a \le b \le d$ and

(3.1) $u(x)$ is monotonic increasing on $[c, a]$

(3.2) $u(x)$ is monotonic deceasing on $[b, d]$

(3.3) $u(x) = 1, a \le x \le b$.

The set all these fuzzy numbers is denoted by E. An equivalent parametric form is also given in [15].

Definition 2. [1, 13]. Fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of function $\underline{u}(\alpha)$, $\overline{u}(\alpha), 0 \leq \alpha \leq 1$, which satisfies the following requirement:

(1) $\underline{u}(\alpha)$ is a bounded monotonic increasing left continuous function,
(2) $\overline{u}(\alpha)$ is a bounded monotonic decreasing left continuous function,
(3) $\underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1$.

A popular fuzzy number is the Trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness σ and right fuzziness β where the membership function is

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \leq x \leq x_0, \\ & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \leq x \leq y_0 + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Its parametric form is

$$\underline{u}(r) = x_0 - \sigma + \sigma r, \qquad \overline{u}(r) = y_0 + \beta - \beta r$$

Let $TF(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows [14].
For arbitrary $u = (\underline{u}, \overline{u}), v = (\underline{v}, \overline{v})$, and $k > 0$, the addition $(u + v)$ and multiplication by scalar k are defined as

$$(\underline{u+v})(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha), \qquad (\overline{u+v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha),$$
$$(\underline{ku})(\alpha) = k\underline{u}(\alpha), \qquad (\overline{ku})(\alpha) = k\overline{u}(\alpha)$$

3. The Barzilai-Borwein gradient method for solving fuzzy nonlinear equation

For fuzzy nonlinear equation $F(x) = 0$, the parametric form is defined as

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = \underline{c}(r), \\ \overline{F}(\underline{x}, \overline{x}, r) = \overline{c}(r), \end{cases} \quad \forall\, r \in [0,1]$$

The idea is to obtain the solution of the parameterized equation above using Barzilai-Borwein gradient method.
Now, we define a function $G_r: \mathbb{R}^2 \to \mathbb{R}$ as follows [9]

$$G_r(\underline{x}, \overline{x}, r) = \left[\underline{F}(\underline{x}, \overline{x}, r) + \overline{F}(\underline{x}, \overline{x}, r)\right]^2 \forall\, r \in [0,1]$$

whose gradient $\nabla G_r(x)$ at point $x(r) = (\underline{x}(r), \ \overline{x}(r))$ is given as

$$\nabla G_r(x) = \left(\frac{\partial G_r}{\partial \underline{x}}, \frac{\partial G_r}{\partial \overline{x}}\right)$$

We define an appropriate two-point step size method for $x_{k+1}(r) = (\underline{x}_{k+1}(r), \overline{x}_{k+1}(r))$ as

$$x_{k+1}(r) = x_k(r) - \alpha_k \nabla G_r(x_k) \tag{2}$$

Suppose $D_k = \alpha_k I$, where I is an identity matrix, then in (2) is rewritten as

$$x_{k+1}(r) = x_k(r) - D_k \nabla G_r(x_k)$$

Since the step size is from a two-point approximation, we compute α_k such that

$$\min\|s_k - D_{k+1} y_k\|, \tag{3}$$

From (3), it is obvious that D_k will possess the quasi-Newton property. Thus,

$$\alpha_k = \frac{s_k^T y_k}{\|y_k\|^2}$$

where $s_k(r) = x_{k+1}(r) - x_k(r)$ and $y_k = g_{k+1} - g_k$ for $g_k = \nabla G_r(x_k)$, $k = 0,1,2, \dots$. However, by symmetry in (3) can be written as

$$\min\|D_{k+1}^{-1} s_k - y_k\|,$$

Hence, we get

$$\alpha_k = \frac{\|s_k\|^2}{s_k^T y_k}.$$

After solving the fuzzy nonlinear equations and results obtained satisfies the inequality $\underline{x}(0) \leq \overline{x}(1) \leq \overline{x}(0)$, one can choose the following fuzzy number

$$x_0 = (\underline{x}(0), \overline{x}(1), \overline{x}(0)) \tag{4}$$

as the initial guess [3, 4, 9], and its parametric form is given as

$$\underline{x}(r) = \underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r,$$

And

$$\overline{x}(r) = \overline{x}(0) + (\underline{x}(1) - \overline{x}(0))r$$

The Barzilai-Borwein algorithm follows from the above description.

Algorithm 1. Barzilai-Borwein algorithm for solving fuzzy nonlinear equation

Step 1. Transform the given fuzzy nonlinear equation to its para metric form and solve for $r = 0$ and $r = 1$ to obtain the Initial guess x_0.
Step 2. Evaluate G_r at $(\underline{x}(0), \overline{x}(0))$ and compute the gradient $\nabla G_r(x_0)$.
Step 3. If $\|\nabla G_r(x_0)\| \leq 0$, stop. Else, let $d_0 = -\nabla G_r(x_0)$.
Step 4. If $k = 0$, solve for α_k using the line search procedure; otherwise, for $k > 0$ compute α_k by (8) or (10).
Step 5. Update the new value
$$x_{k+1}(r) = x_k(r) + \alpha_k d_k.$$
Step 6. Repeat step 1 through 5 with $k := k + 1$ until tolerance is satisfied.

4. Numerical experiment

This section presents the numerical solution of some examples to illustrate the Barzilai-Borwein method. The computations are carried out on MATLAB 7.0 using a double precision computer.

Example 1: [8]. Consider the fuzzy nonlinear equation

$$(3,4,5)x^2 + (1,2,3)x - (1,2,3) = 0$$

Without loss of generality, let's purpose x is positive, we give the parametric form of the above equation as

$$(3+r)\underline{x}^2(r)+(1+r)\underline{x}(r)-(1+r)=0,$$
$$(5-r)\overline{x}^2(r)+(3-r)\overline{x}(r)-(3-r)=0.$$

Next, we solve the above parametric equation for $r=0$ and $r=1$ to obtain the initial guess.

For $r=0$, we have

$$\begin{cases} 3\underline{x}^2(0)+\underline{x}(0)-1=0 \\ 5\overline{x}^2(0)+3\overline{x}(0)-3=0 \end{cases},$$

for $r=1$, we have

$$\begin{cases} 4\underline{x}^2(1)+2\underline{x}(1)-2=0 \\ 4\overline{x}^2(1)+2\overline{x}(1)-2=0 \end{cases}$$

after computation, we have $\underline{x}(0)=0.434$, $\overline{x}(0)=0.681$ and $\underline{x}(1)=\overline{x}(1)=0.5$. From (11), we have $x_0=(0.434,0.5,0.681)$. With the tolerance of 10^{-5}, the solution was obtained after 6 iterations. Please refer Figure 1 for details of the solution.

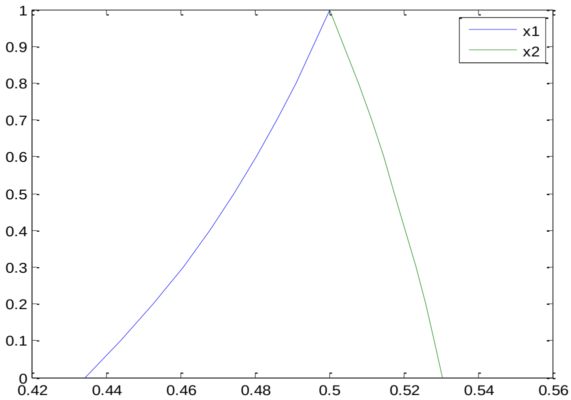

Fig. 1: Solution of Barzilai-Borwein method for Example 1.

Example 2. Consider the fuzzy nonlinear equation

$$(4,6,8)x^2+(2,3,4)x-(8,12,16)=(5,6,7).$$

Without loss of generality, let's assume x is positive, the parametric form of the above equation is as follows.

$$(4+2r)\underline{x}^2(r)+(2+r)\underline{x}(r)-(3+3r)=0,$$
$$(8-2r)\overline{x}^2(r)+(4-r)\overline{x}(r)-(9-3r)=0.$$

We solve the parametric equation for $r=0$ and $r=1$ to obtain the initial guess i.e. For $r=0$, we have

$$\begin{cases} 4\underline{x}^2(0)+2\underline{x}(0)-3=0 \\ 8\overline{x}^2(0)+4\overline{x}(0)-9=0 \end{cases}, \text{ and}$$

for $r=1$, we have

$$\begin{cases} 6\underline{x}^2(1)+3\underline{x}(1)-6=0 \\ 6\overline{x}^2(1)+3\overline{x}(1)-6=0 \end{cases}$$

which implies $\underline{x}(0)=0.6514$, $\overline{x}(0)=0.8397$, and $\underline{x}(1)=\overline{x}(1)=0.7808$. By (11), we have

$$x_0=(0.6514,0.7808,0.8397).$$

The solution of the above problem was obtained after 6 iterations with the tolerance error less than 10^{-5}. Refer to figure 2 for details of the solution.

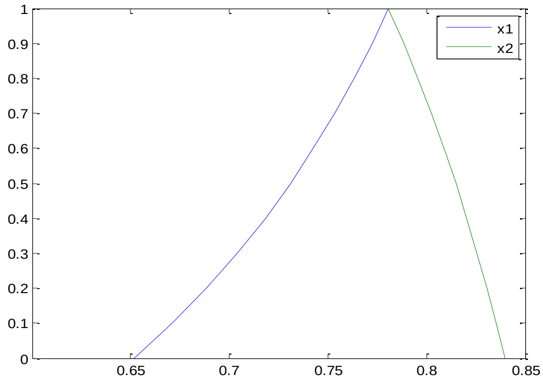

Fig. 2: Solution of Barzilai-Borwein method for Example 1.

5. Conclusion

In this paper, we proposed a gradient-base method for solving fuzzy nonlinear equation. This method does not require any matrix computation and the only line search needed is for $k=0$. Thus, the proposed scheme reduce the computational cost in every iteration. The fuzzy quantities are transform into parametric form and then solved via Barzilai-Borwein gradient method. Numerical results of some examples are provided to illustrate the efficiency of the proposed method. Research groups working on related area can use the results to improve the usage of the method.

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