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Research paper



Oscillation Criteria for Generalized Second Kind Sublinear Delay Difference Equations

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Abstract

The Using Riccati transformation techniques, we present some new oscillation criteria for generalized second kind nonlinear difference equation

$\Delta_{\ell_2}(p(k\ell_2+j_2)(\Delta_{\ell_1}(u(k\ell_1+j_1)))^{\gamma})+q(k\ell_2+j_2)u^{\beta}((k-\sigma)\ell_2+j_2)=0,$

when $0<\beta<1$ is a quotient of odd positive integers, $k\in[0,\infty)$, $\ell_1,\ell_2\in(0,\infty)$, $j=k-\left[\frac{k}{\ell}\right]\ell$.

Keywords: *Generalized Difference Equation; Second Kind; Oscillation.*

1. Introduction

Difference equations is a captivating mathematical area on its own as well as a rich field of the applications in such various disciplines such as population dynamics, operations research, ecology, economics, biology etc. For general background as difference equations with numerous examples from various fields, one can refer to [1]. The study of difference equations is based on the operator Δ defined as $\Delta u(k)=u(k+1)-u(k), k\in \mathrm{N}=\{0,1,2,\cdots\}$. Eventhough many authors [1],[16] have suggested the definition of Δ as

$$\Delta u(k)=u(k+\ell)-u(k),\quad \ell\in(0,\infty), \tag{1}$$

no major progress have taken place on this line.
In [14], authors took up the definition of Δ as given in (1), they produced some important results. The results obtained can be found in [14]. They labeled the operator Δ defined by (1) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}. Recently, there has been increasing study in the asymptotic behavior of second order difference equations by many authors [2-12,17-24].
In the sequel we will be concerned the generalized second order sublinear delay difference equations of the form

$$\Delta_{\ell_2}(p(k\ell_2+j_2)(\Delta_{\ell_1}(u(k\ell_1+j_1)))^{\gamma})$$

$$+q(k\ell_2+j_2)u^{\beta}((k-\sigma)\ell_2+j_2)=0, \tag{2}$$

where Δ_ℓ denotes the generalized difference operator for any real valued function $u(k\ell+j)$, $k,\ell_1,\ell_2\in(0,\infty)$, $j=k-\left[\frac{k}{\ell}\right]\ell$, γ and $0<\beta<1$ are quotient of odd positive integers, σ is a fixed nonnegative integer, $p(k\ell_2+j_2)>0$, and $q(k\ell_2+j_2)\geq 0$ are real valued functions, and

$$\sum_{k=k_0}^{\infty}\left(\frac{1}{p(k\ell_2+j_2)}\right)^{1/\gamma}=\infty,\qquad \text{for some } k_0>0. \tag{3}$$

By a solution of (2) we mean a nontrivial real valued function $u(k\ell_1+j_1)$ defined for $k\geq-\sigma$, and satisfies equation (2) for $k\in[0,\infty)$. Suppose

$$u(k\ell_1+j_1)=A(k\ell_1+j_1)\text{for}k\in[-\sigma,0] \tag{4}$$

then equation (2) has a unique solution satisfying the initial condition (4).
A solution $u(k\ell+j)$ of equation (2) is said to be oscillatory if for every $k_1>0$ there exists an $k\geq k_1$ such that $u(k\ell+j)u((k+1)\ell+j)\leq 0$, otherwise it is nonoscillatory. Equation (2) is said to be oscillatory if all its solutions are oscillatory.

2. Main Results

Theorem 2.1
Assume that (3) holds. Also, assume that there exists a real valued functions $\rho(k\ell_2+j_2)$ such that for every $\eta\geq 1$ and positive number M

$$\limsup_{k\to\infty}\sum_{r=k_0}^{k}\left[\rho(r\ell_2+j_2)q(r\ell_2+j_2)\right.$$
$$\left.-\frac{(p((r-\sigma)\ell_2+j_2))^{1/\gamma}\eta^{1-\beta}((r-\sigma+1)\ell_2+j_2)^{1-\beta}(\Delta_{\ell_2}\rho(r\ell_2+j_2))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(r\ell_2+j_2)}\right] \tag{5}$$

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Then every solution of equation (2) oscillates.

Proof.

Suppose to the contrary that $u(k\ell_1 + j_1)$ is an eventually nonoscillatory solution of (2) such that $u((k-\sigma)\ell_2 + j_2) > 0$ for all $k \geq k_0 > 0$. Since the substitution $v(k\ell_1 + j_1) = -u(k\ell_1 + j_1)$ transforms equation (2) into an equation of the same form, we shall consider only this case. From equation (2), we have

$$\Delta_{\ell_2}(p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma})$$

$$= -q(k\ell_2 + j_2)u^{\beta}((k-\sigma)\ell_2 + j_2) \leq 0, \tag{6}$$

for $k \geq k_0$ and so $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma}$ is an eventually nonincreasing sequence. We first show that $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma} \geq 0$ for $k \geq k_0$. In fact, if there exists an real $k_1 \geq k_0$ such that $p(k_1\ell_2 + j_2)(\Delta_{\ell_1}u(k_1\ell_1 + j_1))^{\gamma} = c < 0$, then (6) implies that $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma} \leq c$ for $k \geq k_1$ that is

$$\Delta_{\ell_1}u(k\ell_1 + j_1) \leq \left(\frac{c}{p(k\ell_2 + j_2)}\right)^{1/\gamma}$$

and, hence as $k \to \infty$

$$u(k\ell_1 + j_1) \leq u(k_1\ell_1 + j_1) + c^{1/\gamma}\sum_{i=0}^{k-1}\left(\frac{1}{p(k_1\ell_2 - i\ell_1 + j_2)}\right)^{1/\gamma} \to -\infty \tag{7}$$

which contradicts the fact that $u(k\ell_1 + j_1) > 0$ for $k \geq k_0$, then $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma} \geq 0$. Also we claim that $\Delta^2_{\ell_1}u(k\ell_1 + j_1) \leq 0$. If not there exists $k_1 \geq k_0$ such that $\Delta^2_{\ell}u(k\ell + j) > 0$ for $k \geq k_1$ and this implies that $\Delta_{\ell_1}u((k+1)\ell_1 + j_1) > \Delta_{\ell_1}u(k\ell_1 + j_1)$, so that since $\Delta_{\ell_2}p(k\ell_2 + j_2) \geq 0$,

$$p((k+1)\ell + j)(\Delta_{\ell}u((k+1)\ell + j))^{\gamma}$$
$$> p((k+1)\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma}$$
$$\geq p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma}$$

and this contradicts the fact that $p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma}$ is nonincreasing sequence, then $\Delta^2_{\ell_1}u(k\ell_1 + j_1) \leq 0$, and therefore we have

$$u(k\ell_1 + j_1) > 0, \Delta_{\ell_1}u(k\ell_1 + j_1) \text{ and } \Delta^2_{\ell_1}u(k\ell_1 + j_1) \leq 0 \text{for} k \geq k_0. \tag{8}$$

Define the sequence $z(k\ell_2 + j_2)$ by

$$z(k\ell_2 + j_2) = \rho(k\ell_2 + j_2)\frac{p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma}}{u^{\beta}((k-\sigma)\ell_2 + j)} \tag{9}$$

then $z(k\ell + j) > 0$, and

$$\Delta_{\ell_2}z(k\ell_2 + j_2)$$
$$= p((k+1)\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + \ell_2 + j_1))\Delta_{\ell_2}\left(\frac{\rho(k\ell_2 + j_2)}{u^{\beta}((k-\sigma)\ell_2 + j_2)}\right)$$
$$+\frac{\rho(k\ell_2 + j_2)\Delta_{\ell_2}(p(k\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + j_1))^{\gamma})}{u^{\beta}((k-\sigma)\ell_2 + j_2)} \tag{10}$$

From (2) and (10), we have

$$\Delta_{\ell_2}z(k\ell_2 + j_2) =$$
$$-\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) + \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)}z((k+1)\ell_2 + j_2)$$
$$-\left(\frac{\rho(k\ell_2 + j_2)p((k+1)\ell_2 + j_2)}{u^{\beta}((k-\sigma+1)\ell_2 + j_2)} \times \frac{(\Delta_{\ell_1}u(k\ell_2 + \ell_2 + j))^{\gamma}\Delta_{\ell_2}u^{\beta}((k-\sigma)\ell_2 + j_2)}{u^{\beta}((k-\sigma)\ell_2 + j_2)}\right) \tag{11}$$

From (6) and (8), we get

$$p((k-\sigma)\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 - \sigma\ell_2 + j_1))^{\gamma}$$
$$\geq p((k+1)\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + \ell_2 + j_1))^{\gamma}$$

and $\quad u((k+1-\sigma)\ell + j) \geq u((k-\sigma)\ell + j)$ (12)

and then from (11) and (12), we have

$$\Delta_{\ell_2}z(k\ell_2 + j_2) =$$
$$-\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) + \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)}z((k+1)\ell_2 + j_2)$$
$$-\frac{\rho(k\ell_2 + j_2)p((k+1)\ell_2 + j_2)}{(u^{\beta}((k-\sigma+1)\ell_2 + j_2))^2} \times(\Delta_{\ell_1}u(k\ell_2 + \ell_2 + j))^{\gamma}\Delta_{\ell_2}u^{\beta}((k-\sigma)\ell_2 + j_2) \tag{13}$$

Now, by using the inequality in [13]

$u^{\beta} - v^{\beta} \geq \beta u^{\beta-1}(u-v)$ for all $u \neq v > 0$ and $0 < \beta \leq 1$.

Then, we have

$$\Delta_{\ell_2}u^{\beta}((k-\sigma)\ell_2 + j_2)$$
$$= \beta(u((k-\sigma+1)\ell_2 + j_2))^{\beta-1}\Delta_{\ell_2}u((k-\sigma)\ell_2 + j_2). \tag{14}$$

Substitute from (14) in (13), we have

$$\Delta_{\ell_2}z(k\ell_2 + j_2) =$$
$$-\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) + \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)}z((k+1)\ell_2 + j_2)$$
$$-\left(\frac{\rho(k\ell_2 + j_2)p((k+1)\ell_2 + j_2)\beta(u((k+1-\sigma)\ell_2 + j_2))^{\beta-1}}{\left(u^{\beta}((k-\sigma+1)\ell_2 + j_2)\right)^2}\right)$$
$$\times\Delta_{\ell_2}u((k-\sigma)\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + \ell_2 + j_1))^{\gamma} \tag{15}$$

From (12) and (15), we have

$$\Delta_{\ell_2}z(k\ell_2 + j_2) =$$
$$-\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) + \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)}z((k+1)\ell_2 + j_2)$$
$$-\frac{\beta\rho(k\ell_2 + j_2)(p((k+1)\ell_2 + j_2))^{1/\gamma}}{(p((k-\sigma)\ell_2 + j_2))^{1/\gamma}}$$
$$\times\frac{p((k+1)\ell_2 + j_2)(\Delta_{\ell_1}u(k\ell_1 + \ell_2 + j_1))^{\gamma+1}}{\left(u((k-\sigma+1)\ell_2 + j_2)\right)^{1-\beta}\left(u^{\beta}((k-\sigma+1)\ell_2 + j_2)\right)}$$

$$= -\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) + \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)} z((k+1)\ell_2 + j_2)$$

$$-\left(\frac{\beta\rho(k\ell_2 + j_2)(p((k+1)\ell_2 + j_2))^{(1/\gamma)-1}(p((k+1)\ell_2 + j_2))^2}{(p((k+1)\ell_2 + j_2))^2 (p((k-\sigma)\ell_2 + j_2))^{\frac{1}{\gamma}} \left(u((k-\sigma+1)\ell_2 + j_2)\right)^{1-\beta}} \right.$$

$$\left. \times \frac{(\rho((k+1)\ell_2 + j_2))^2 (\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^{2\gamma}}{\left(u^{\beta}((k-\sigma+1)\ell_2 + j_2)\right)^2 (\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^{\gamma-1}} \right) \quad (16)$$

From (8), we conclude that

$u(k\ell_2 + j_2) \le u(k_0\ell_2 + j_2) + \Delta_{\ell_2} u(k_0\ell_2 + j_2)((k-k_0)\ell_2 + j_2), k \ge k_0$

consequently there exists a $k_1 \ge k_0$ and appropriate constant $\eta \ge 1$ such that

$u(k\ell_2 + j_2) \le \eta(k\ell_2 + j_2)$ for $k \ge k_1$

and this implies that

$u((k-\sigma+1)\ell_2 + j_2) \le \eta((k-\sigma+1)\ell_2 + j_2)$ for $k \ge k_2 = k_1 + \sigma - 1$

and, hence

$$\frac{1}{(u((k-\sigma+1)\ell_2 + j_2))^{1-\beta}} \ge \frac{1}{(\eta((k-\sigma+1)\ell_2 + j_2))^{1-\beta}} \quad (17)$$

Since $p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^{\gamma}$ is a positive and increasing function, there exists a $k_2 \ge k_1$ sufficiently large such that $p(k\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + j_1))^{\gamma} \le 1/M$ for some positive constant M and $k \ge k_2$, and hence by (6) we have $p((k+1)\ell_2 + j_2)(\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^{\gamma} \le 1/M$, so that

$$\frac{1}{(\Delta_{\ell_1} u(k\ell_1 + \ell_2 + j_1))^{\gamma-1}} \ge (Mp((k+1)\ell_2 + j_2))^{(\gamma-1)/\gamma} \quad (18)$$

then from (9), (16),(17) and (18) we have

$\Delta_{\ell_2} z(k\ell_2 + j_2) =$

$$-\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) + \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)} z((k+1)\ell_2 + j_2)$$

$$-\frac{\beta\rho(k\ell_2 + j_2)M^{(\gamma-1)/\gamma}(z((k+1)\ell_2 + j_2))^2}{(p((k+1)\ell_2 + j_2))^2 (p((k-\sigma)\ell_2 + j_2))^{1/\gamma}(\eta\left((k-\sigma+1)\ell_2 + j_2\right))^{1-\beta}} \quad (19)$$

$$\Delta_{\ell_2} z(k\ell_2 + j_2) \le -\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) +$$

$$+\frac{(p((k-\sigma)\ell_2 + j_2))^{1/\gamma}\eta^{1-\beta}((k-\sigma+1)\ell_2 + j_2)^{1-\beta}(\Delta_{\ell_2}\rho(k\ell_2 + j_2))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell_2 + j_2)}$$

$$-\left[\frac{\sqrt{\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell_2 + j_2)}z((k+1)\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)\sqrt{\eta((k-\sigma+1)\ell_2 + j_2)^{1-\beta}p((k-\sigma)\ell_2 + j_2)}} \right.$$

$$\left. \frac{\sqrt{\eta^{1-\beta}((k-\sigma+1)\ell_2 + j_2)^{1-\beta}(p((k-\sigma)\ell_2 + j_2))^{1/\gamma}}\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{2\sqrt{\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell_2 + j_2)}} \right]^2$$

$$< -\left[\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) \right.$$

$$\left. -\frac{(p((k-\sigma)\ell_2 + j_2))^{1/\gamma}\eta^{1-\beta}((k-\sigma+1)\ell_2 + j_2)^{1-\beta}(\Delta_{\ell_2}\rho(k\ell_2 + j_2))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(k\ell_2 + j_2)} \right] \quad (20)$$

Summing (20) from k_2 to k, we obtain

$$-z(k_2\ell_2 + j_2) < z((k+1)\ell_2 + j_2) - z(k_2\ell_2 + j_2)$$

$$< -\sum_{r=k_2}^{k} \left[\rho(r\ell_2 + j_2)q(r\ell_2 + j_2) \right.$$

$$\left. -\frac{(p((r-\sigma)\ell_2 + j_2))^{1/\gamma}\eta^{1-\beta}((r-\sigma+1)\ell_2 + j_2)^{1-\beta}(\Delta_{\ell_2}\rho(r\ell_2 + j_2))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(r\ell_2 + j_2)} \right]$$

which yields

$$\sum_{r=k_2}^{k} \left[\rho(r\ell_2 + j_2)q(r\ell_2 + j_2) \right.$$

$$\left. -\frac{(p((r-\sigma)\ell_2 + j_2))^{1/\gamma}\eta^{1-\beta}((r-\sigma+1)\ell_2 + j_2)^{1-\beta}(\Delta_{\ell_2}\rho(r\ell_2 + j_2))^2}{4\beta(M)^{(\gamma-1)/\gamma}\rho(r\ell_2 + j_2)} \right] < c_1$$

for all large k, and this is contrary to (5). The proof is complete.

Theorem 2.2

Assume that (3) holds. Let $\rho(k\ell_2 + j_2)$ be a real valued function. Also, we assume that there exists a double fuction $H(m,k\ell_2 + j_2); m \ge k \ge 0$ such that

(*i*) $H(m,m) = 0$ *for* $m \ge 0$, (*ii*) $H(m,k\ell_2 + j_2) > 0$ *for* $m > k\ell_2 + j_2 \ge 0$,

(*iii*) $\Delta_{2(\ell_2)} H(m,k\ell_2 + j_2) = H(m,(k+1)\ell_2 + j_2) - H(m,k\ell_2 + j_2) \le 0$ *for* $m \ge k\ell_2 + j_2 \ge 0$. *If*

$$\limsup_{m\to\infty} \frac{1}{H(m,0)} \sum_{k=k_0}^{m-1} \left[H(m,k\ell_2 + j_2)\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) \right.$$

$$\left. -\frac{(\rho((k+1)\ell_2 + j_2))^2}{\overline{\rho(k\ell_2 + j_2)}} \left(h(m,k\ell_2 + j_2) - \frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)}\sqrt{H(m,k\ell_2 + j_2)} \right)^2 \right] = \infty, \quad (21)$$

where $h(m,k\ell_2 + j_2) = -\frac{\Delta_{2(\ell_2)} H(m,k\ell_2 + j_2)}{\sqrt{H(m,k\ell_2 + j_2)}}, m \ge k \ge 0,$

$$\overline{\rho(k\ell_2 + j_2)} = \frac{\beta\rho(k\ell_2 + j_2)M^{(\gamma-1)/\gamma}}{(p((k-\sigma)\ell_2 + j_2))^{1/\gamma}\eta^{1-\beta}\left((k-\sigma+1)\ell_2 + j_2\right)^{1-\beta}}$$

Then every solution of equation (2) oscillates.

Proof. We proceed as in Theorem 2.1, we assume that equation (2) has a nonoscillatory solution, say $u((k-\sigma)\ell_2 + j_2) > 0$ for all $k \ge k_0$. From the proof of Theorem 2.1, we obtain (18) for all $k \ge k_2$. From (18) we have for $k \ge k_2$,

$$\rho(k\ell_2 + j_2)q(k\ell_2 + j_2) \le -\Delta_{\ell_2} z(k\ell_2 + j_2)$$

$$+\frac{\Delta_{\ell_2}\rho(k\ell_2 + j_2)z((k+1)\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)} - \frac{\overline{\rho(k\ell_2 + j_2)}z^2((k+1)\ell_2 + j_2)}{(\rho((k+1)\ell_2 + j_2))^2}. \quad (22)$$

Therefore, we have

$$\sum_{k=n}^{m-1} H(m,k\ell_2 + j_2)\rho(k\ell_2 + j_2)q(k\ell_2 + j_2)$$

$$\le -\sum_{k=n}^{m-1} H(m,k\ell_2 + j_2)\Delta_{\ell_2} z(k\ell_2 + j_2)$$

$$+\sum_{k=n}^{m-1} \frac{H(m,k\ell_2 + j_2)\Delta_{\ell_2}\rho(k\ell_2 + j_2)z((k+1)\ell_2 + j_2)}{\rho((k+1)\ell_2 + j_2)}$$

$$-\sum_{k=n}^{m-1}\frac{H(m,k\ell_2+j_2)\overline{\rho(k\ell+j_2)}z^2((k+1)\ell_2+j_2)}{(\rho((k+1)\ell_2+j_2))^2}.$$

which yields, after summing by parts the first term in the right hand side.

$$\sum_{k=n}^{m-1}H(m,k\ell_2+j_2)\rho(k\ell_2+j_2)q(k\ell_2+j_2)\le H(m,k\ell_2+j_2)z(k\ell_2+j_2)$$

$$+\sum_{k=n}^{m-1}z((k+1)\ell_2+j_2)\Delta_{2(\ell_2)}H(m,k\ell_2+j_2)$$

$$+\sum_{k=n}^{m-1}H(m,k\ell_2+j_2)\frac{\Delta_{\ell_2}\rho(k\ell_2+j_2)}{\rho((k+1)\ell_2+j_2)}z((k+1)\ell_2+j_2)$$

$$-\sum_{k=n}^{m-1}\frac{H(m,k\ell_2+j_2)\overline{\rho(k\ell_2+j_2)}}{(\rho((k+1)\ell_2+j_2))^2}z^2((k+1)\ell_2+j_2)$$

$$=H(m,k\ell_2+j_2)z(k\ell_2+j_2)-\sum_{k=n}^{m-1}\Big[z((k+1)\ell_2+j_2)$$

$$+\frac{\rho((k+1)\ell_2+j_2)}{2\sqrt{H(m,k\ell_2+j_2)\overline{\rho(k\ell_2+j_2)}}}$$

$$\times\left(h(m,k\ell_2+j_2)\sqrt{H(m,k\ell_2+j_2)}-\frac{\Delta_{\ell_2}\rho(k\ell_2+j_2)H(m,k\ell_2+j_2)}{\rho((k+1)\ell_2+j_2)}\right)\Big]^2$$

$$+\frac{1}{4}\sum_{k=n}^{m-1}\frac{(\rho((k+1)\ell_2+j_2))^2}{\overline{\rho(k\ell_2+j_2)}}$$

$$\times\left(h(m,k\ell_2+j_2)-\frac{\Delta_{\ell_2}\rho(k\ell_2+j_2)}{\rho((k+1)\ell_2+j_2)}\sqrt{H(m,k\ell_2+j_2)}\right)^2$$

$$\sum_{k=k_1}^{m-1}\Big[H(m,k\ell_2+j_2)\rho(k\ell_2+j_2)q(k\ell_2+j_2)$$

$$-\frac{(\rho((k+1)\ell_2+j_2))^2}{4\overline{\rho(k\ell_2+j_2)}}$$

$$\times\left(h(m,k\ell_2+j_2)-\frac{\Delta_{\ell_2}\rho(k\ell_2+j_2)\sqrt{H(m,k\ell_2+j_2)}}{\rho((k+1)\ell_2+j_2)}\right)^2\Big]$$

$$<H(m,k_1\ell_2+j_2)z(k_1\ell_2+j_2)\le H(m,0)z(k_1\ell_2+j_2)$$

$$<H(m,0)\left(z(k_1\ell_2+j_2)+\sum_{k=0}^{k_1-1}\rho(k\ell_2+j_2)q(k\ell_2+j_2)\right)<\infty.$$

Contradicts the assumption. Hence the proof.

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