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Research paper

Some Properties of the Lattice of Path Sets of a Connected Graph

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Abstract

It is known that the set of all path sets of a finite connected graph G together with empty set partially ordered by set inclusion relation forms a lattice denoted by PATH(G). In this paper we studied some properties of PATH(G). In fact, it has been shown that an element of PATH(G) is doubly irreducible if and only if it contains a single vertex which is not a cut vertex of G. Also it is proved that PATH(G) is planar if and only if G is a chain of three or more blocks.

Keywords: *Connected graph, block, cut point.*

1. Introduction

In paper [9], lattice of convex sets of connected graphs and lattice of convex edge sets of connected graphs are studied. Motivated by the above studies in [10] it is shown that set of all path sets of a connected graph together with empty set partially ordered by set inclusion relation forms a lattice, denoted by PATH(G) and studied some of its properties . In this paper, we studied some more properties of PATH(G).

In section 2, we defined PATH(G) along with an example . In section 3 it is shown that an element $A \in \mathrm{PATH(G)}$ is irreducible if and only if $A = \{v_i\}$ where v_i is not a cut point of G. Also condition under which PATH(G) is planar and satisfies lower covering condition are studied.

In section 4 , cardinality and length of PATH(G) are found when the graph G is a path, block and chain of blocks .For terminologies and notifications used in this paper we refer to [4][5].

2. Preliminaries

Let G be a finite connected graph and $V(G)$ be the vertex set of G. A subset B of V(G) is said to be a path set of G, if for every $u, v \in B$ the vertex set of all paths between u and v is contained in B. Let $\mathrm{PATH(G)}$ be the set of all path sets of G together with empty set. Define a binary relation $\leq$ on PATH(G) by for every A,B$\in$ PATH(G) set $A \leq B$ if and only if $A \subseteq B$. Then PATH(G) from a lattice with respect to this partial order , $A \wedge B = A \cap B$ and $A \vee B = \langle A \cup B \rangle$, the smallest path set containing $A \cup B$. For example, the lattice $\langle PATH(G), \subseteq \rangle$ given in fig 2 of the connected graph G given in Fig 1.

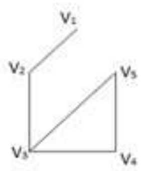

Fig 1

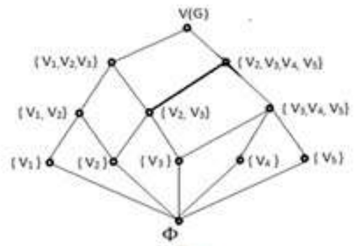

Fig 2

Throughout this paper we use PATH(G) to represents the lattice $\langle PATH(G), \subseteq \rangle$.

3. On the Lattice of $< \mathrm{PATH(G)} \subseteq >$

Remark 3.1:

PATH(G) is atomic where atoms are path sets containing only one vertex

Theorem 3.2:

An element $A \in PATH(G)$ is doubly irreducible if and only if $A = \{v_i\}$ where v_i is not a cut vertex of G

Proof:

Let $A \in PATH(G)$ be doubly irreducible. Clearly $A = \{v_i\}$ for some i, for otherwise if $A = \{v_1, v_2, v_3, \dots, v_n\}$, then $A = \bigvee_{i=1}^{n}\{v_i\}$, a contradiction. If v_i is a cut vertex of G, then let U and W be two blocks of G for which v_i is a cut vertex. Then for any $u \in \mathrm{U}$ and $w \in \mathrm{W}, v_i$ is on every $u - w$ path. Let $B \in PATH(G)$ be the shortest path from u to v_i and $C \in PATH(G)$ be the shortest path from v_i to w . Then $B \wedge C = \{v_i\} = A$, contradiction to A is doubly Irreducible. Hence v_i cannot be a cut vertex of G.

Conversely, let $A=\{v_i\}$ where v_i is not a cut vertex of G. Clearly A is join irreducible. If A is meet reducible, say $A=B\wedge C=B\cap C$ then $v_i\in B$ and $v_i\in C$. Let $v_j\in B, j\neq i$ and $v_k\in C, k\neq i$. Consider paths from v_i to v_j and v_i to v_k these paths must be distinct, Since $B\cap C=\{v_i\}$. But then v_i is a cut vertex, a contradiction.

Remark 3.3:

If a graph G of order ≤ 2 ,then clearly PATH(G) is planar. If a graph G of order ≥ 3,then we have the following result.

Theorem 3.4:

For a graph G, PATH(G) is a planar lattice if and only if G is a chain of m blocks $B_1, B_2, B_3, \dots, B_m$, where $m\geq 3$.

Proof:

Let PATH(G) be a planar lattice. If the graph G is not a chain of m blocks, then their exists at least One vertex v , with minimum three blocks incident on it as shown in fig 3. Then there exists $v_1\in B_1, \ v_2\in B_2, v_3\in B_3$ where $v\neq v_1\neq v_2\neq v_3$ and v is adjacent to $v_1 v_2, v_3$. But then PATH(G) Contains a subposet as shown in fig 4 contradicting PATH(G) is planar.

Conversely, let G be a chain of m blocks with $m\geq 3$ as shown in fig 5. We know that the sublattice of PATH(G) corresponding to a single block is a planar. Then PATH(G) will be as shown in fig 6 and hence it is planar.

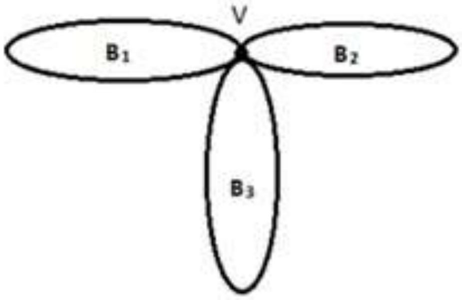

Fig 3

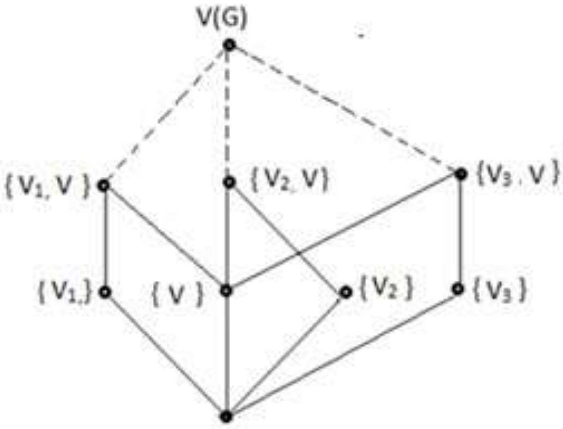

Fig 4

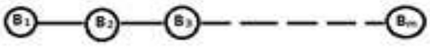

Fig 5

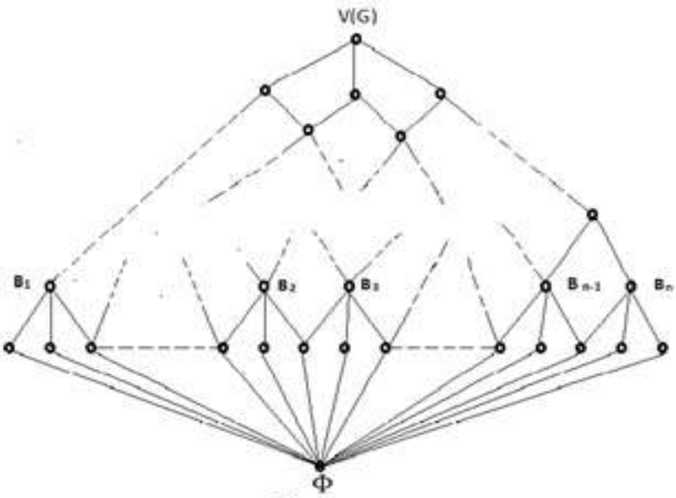

Fig 6

Theorem 3.5:

A graph G is a block if and only if PATH(G) satisfies lower covering condition.

Proof:

If a graph G is a block ,then clearly PATH(G) satisfies lower covering condition. Conversely, let PATH(G) satisfy lower covering condition. If G is not a block, then there are 3 distinct points v_1, v_2, v_3 of G, where every path joining v_1, v_3 contains v_2. But then $\emptyset=\{v_1\}\wedge\{v_3\}\prec\{v_1\}$ but $\{v_3\}<\{v_2\}\vee\{v_3\}<\{v_1\}\vee\{v_3\}$ which implies $\{v_3\}\nprec\{v_1\}\vee\{v_3\}$, a contradiction.

4. On the Cardinality of PATH(G)

Theorem 4.1:

If a graph G is a path with n number of vertices, then $|\mathrm{PATH(G)}|=\sum n+1=\frac{n(n+1)}{2}+1$ and $l(PATH(G))=n$.

Proof:

Consider a graph G which is a path with n number of vertices $v_1, v_2, v_3, \dots, v_n$. Then its path sets will be $\{v_1\},\{v_1,v_2\},\{v_1,v_2,v_3\},\dots,\{v_1,v_2,\dots,v_n\}$. So that the total number of path sets will be $\frac{n(n+1)}{2}+1$ including $\emptyset$. Then the maximum chain in PATH(G) is $\emptyset \prec \{v_1\} \prec \{v_1, v_2\} \prec \{v_1, v_2, v_3\} \prec \cdots \prec \{v_1, v_2, \dots, v_n\}$,so that length of the maximum chain is n.

Theorem 4.2:
If a graph G is a chain of m blocks, then $|\mathrm{PATH(G)}| = \frac{m(m+1)}{2} + n + 1$ and $l(PATH(G)) = m + 1$ where n is number of vertices in G.

Proof:
Consider a graph G with n number of vertices $v_1, v_2, v_3, \dots, v_n$. If G is chain of m number of blocks, then number of path set of G will be is $\frac{m(m+1)}{2} + n + 1$ including $\emptyset$ and $\{v_i\}$ $i = 1,2,3, \dots$. Then the maximum chain of the PATH(G) is $\emptyset \prec \{v_1\} \prec B_1 \prec B_1, B_2 \prec \cdots \prec V(G)$. Where $B_1, B_2, \dots, B_m$ are chains of m blocks in G.

Theorem 4.3:
If a graph G is a block ,then $|\mathrm{PATH(G)}| = n + 2$ and $l(PATH(G)) = 2$ where n is number of vertices in G.

Proof:
If a graph G is a block with n number of vertices $v_1, v_2, \dots, v_n$ Then the path sets of a G are $\emptyset, \{v_1\}, \{v_2\}, \dots, \{v_n\}, V(G)$. Hence $|\mathrm{PATH(G)}| = n + 2$. The maximum chains of PATH(G) are of the form $\emptyset \prec \{v_i\} \prec V(G)$ where $i \in n$, so that $l(PATH(G)) = 2$.

5. Conclusion

In this Paper we proved some properties of PATH (G) and cordiality PATH(G) when graph G is path, Chain and block.

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