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Research paper



Some New CARISTI Type Results in Metric Spaces with an Application to Graph Theory

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Abstract

In this paper we proved some new Caristi type common fixed point theorems for four maps in a metric space and we gave an applications to Graph theory.

Keywords: *Metric spaces , W - compatible maps, fixed point, Caristi type mapping.*

1. Introduction

The concept of standard metric space is a fundamental tool in topology, functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric space.
Let us recall the standard metric space.

Definition 1.1
Let X be a non empty set. A mapping
$d: X \times X \to X$ is said to be a metric on X if it satisfies the following conditions.
$(d_1) d(x,y) \geq 0$ for all $x, y \in X$
$(d_2) d(x,y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$
$(d_3) d(x,y) = d(y,x)$ for all $x, y \in X$
$(d_4) d(x,y) \leq d(x,z) + d(z,y)$
for all $x, y, z \in X$. Then the pair (X, d) is called a metric space.

Example 1.2
Define $d: R \times R \to R$ by $d(x,y) = |x - y|$. Then d is a metric on R.
One of the most important tools of modern mathematics is undoubtedly Fixed point theory. It is widely used in different areas like ordinary and partial differential equations, economics, logic programming, convex optimization, control theory, etc.
In metric fixed point theory, successive approximations are rooted in the work of Cauchy, Fredholm, Liouville, Lipschitz, Peano and Picard. It is well accepted among experts that Banach is responsible for laying the ground for an abstract framework well beyond the scope of elementary differential and integral equations. Since Banach Introduced this theory in 1922([11]), it has been extended and generalized by several authors. Caristi type fixed point theorem is one of these generalizations. It is a modified ε - variation principle of Ekeland([4]).
In 1976, Caristi proved the following famous fixed point theorem.

Theorem 1.3
[3] Let (X, d) be complete metric space and $f: X \to R$ be lower semi continuous function and bounded below function. A mapping $T: X \to X$ is said to be Caristi type map on X dominated by f if T satisfies $d(x, Tx) \leq f(x) - f(Tx)$ for each $x \in X$. Then T has a fixed point.
It is well-known that the Caristi's fixed point theorem is one of the most valuable generalization of the Banach contraction principle. The aim of this paper to prove some new Caristi type common fixed point theorems for four maps in metric space.

Definition 1.4
[11] An element $x \in X$ is called a fixed point of the mapping $T: X \to X$ if $x = Tx$

Definition 1.5
[1] An element $x \in X$ is called
1. a coincident point of mappings $F, f: X \to X$ if $fx = Fx$.
2. a common fixed point of mappings $F, f: X \to X$
if $x = fx = Fx$.

Definition 1.6
[1] The mappings $F: X \to X$ and $f: X \to X$ are called w - compatible if $f(Fx) = F(fx)$ whenever $fx = Fx$.

2. Main Results

Theorem 2.1:
In a metric space (X, d) for any $S, T, f, g: X \to X$ satisfies the following conditions
(2.1.1) $d(Sx, Tu) \leq \psi(\alpha(fx))\alpha(fx) - \alpha(Sx)$
$+\psi(\gamma(gu))\gamma(gu) - \gamma(Tu)$
where $\alpha, \gamma, : X \to [0, \infty)$ are lower semi continuous functions and $\psi: (-\infty, \infty) \to (0,1)$ be continuous function.
(2.1.2) $S(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$
(2.1.3) (S, f) and (T, g) are w-compatible

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(2.1.4) either f(X) or g(X) is complete;
Then S, T, f, g have unique CCFP.

Proof:
For an arbitrary x_0, y_0 be in X and from (2.1.2), define the following sequences in X as
$z_{2n} = gx_{2n+1} = Sx_{2n}$,

$z_{2n+1} = fx_{2n+2} = Tx_{2n+1}$, $n = 0,1,2,\cdots$
Now from (2.1.1), for $z_{2n} \neq z_{2n+1}$ we have that
$0 \leq d(z_{2n}, z_{2n+1})$
$= d(Sx_{2n}, Tx_{2n+1})$
$\leq \psi\big(\alpha(fx_{2n})\big)\alpha(fx_{2n}) - \alpha(Sx_{2n}) + \psi\big(\gamma(gx_{2n+1})\big)\gamma(gx_{2n+1}) - \gamma(Tx_{2n+1})$
$\leq \psi\big(\alpha(z_{2n-1})\big)\alpha(z_{2n-1}) - \alpha(z_{2n}) + \psi\big(\gamma(z_{2n})\big)\gamma(z_{2n}) - \gamma(z_{2n+1})$

Therefore
$d(z_{2n}, z_{2n+1}) < \alpha(z_{2n-1}) - \alpha(z_{2n}) + \gamma(z_{2n}) - \gamma(z_{2n+1})$ (2.1)
and
$\alpha(z_{2n}) + \gamma(z_{2n+1})$
$\leq \psi\big(\alpha(z_{2n-1})\big)\alpha(z_{2n-1}) + \psi\big(\gamma(z_{2n})\big)\gamma(z_{2n})$
$< \alpha(z_{2n-1}) + \gamma(z_{2n})$ (2.2)

This shows that the sequences $\{\alpha(z_{2n})\}, \{\gamma(z_{2n+1})\}$ are non-increasing sequences of non-negative real numbers. So they must converges to $l_1, l_2 \geq 0$ respectively.
Suppose $l_1 > 0$ or $l_2 > 0$.
Letting $n \to \infty$ in equation (2.2), we get contradiction.

Therefore
$$\lim_{n\to\infty} \alpha(z_{2n}) = \lim_{n\to\infty} \gamma(z_{2n+1}) = 0. \quad (2.3)$$

Now, from (2.1) and for any positive integers n and m with $m > n$,
$d(z_{2n}, z_{2m})$
$\leq d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2}) + \cdots + d(z_{2m-1}, z_{2m})$
$\leq \alpha(z_{2n-1}) - \alpha(z_{2n}) + \gamma(z_{2n}) - \gamma(z_{2n+1})$
$+\alpha(z_{2n}) - \alpha(z_{2n+1}) + \gamma(z_{2n+1}) - \gamma(z_{2n+2})$
$+ \cdots + \alpha(z_{2m-2}) - \alpha(z_{2m-1}) + \gamma(z_{2m-1}) - \gamma(z_{2m})$
$\leq \alpha(z_{2n-1}) + \gamma(z_{2n}) - \alpha(z_{2m-1}) - \gamma(z_{2m})$.
as $m, n \to \infty$, and from (2.3) we have $d(z_{2n}, z_{2m}) = 0$. This shows $\{z_{2n}\}$ is a Cauchy sequence in X.
Therefore $\lim_{m,n\to\infty} d(z_{2n}, z_{2m}) = 0$.
Since f(X) is a complete subspace of (X, d) so $\{z_{2n}\} \subseteq f(X)$ is convergent in the complete metric space $(f(X), d)$.
Therefore $\lim_{n\to\infty} d(z_{2n}, u) = 0$ where $u \in X$. Since $f: X \to X$ and $u \in f(X)$ so there exists $t \in X$ such that $ft = u$.
Since α, γ are lower semi continuous functions, $z_{2n} \to u$ and as $n \to \infty$ from (2.3) we have $\alpha(u) = \gamma(u)$=0.
From (2.1.1), we have
$d(St, u) = d(St, z_{2n+1}) + d(z_{2n+1}, u)$
$= d(St, Tx_{2n+1}) + d(z_{2n+1}, u)$
$\leq \psi\big(\alpha(ft)\big)\alpha(ft) - \alpha(St) + \psi\big(\gamma(gx_{2n+1})\big)\gamma(gx_{2n+1}) - \gamma(Tx_{2n+1}) + d(z_{2n+1}, u)$
$< \alpha(ft) - \alpha(St) + \gamma(gx_{2n+1}) - \gamma\big(T(x_{2n+1}, y_{2n+1})\big) + d(z_{2n+1}, u)$.
Letting $n \to \infty$, we have
$d(St, u) < \alpha(u) - \alpha(St) + \gamma(u) - \gamma(u) + d(u, u)$
$< \alpha(u) - \alpha(St) < \alpha(u)$
= 0.
Therefore $d(St, u) = 0$, so we have $St = u = fs$.
Since (S, f) are w-compatible mappings, we have that $Su = fu$.
Now to prove that $fu = u$.
$d(fu, z_{2n}) = d(Su, Tx_{2n})$
$\leq \psi(\alpha(fu))\alpha(fu) - \alpha(Su) + \psi(\gamma(gx_{2n}))(\gamma(gx_{2n}) - \gamma(Tx_{2n})$
$< \alpha(fu) - \alpha(Su) + \gamma(gx_{2n}) - \gamma(Tx_{2n})$
$< \alpha(fu) - \alpha(fu) + \gamma(z_{2n-1}) - \gamma(z_{2n})$.
Letting $n \to \infty$, we have that
$d(fu, u) \to \gamma(u) - \gamma(u) = 0$.
Therefore $fu = u$. Hence $Su = fu = u$.
Since $S(X) \subseteq g(X)$, so there exist $a \in X$ such that $Su = ga$.
Therefore $u = Su = ga$.
Now
$d(u, Ta) = d(Su, Ta)$
$\leq \psi\big(\alpha(fu)\big)\alpha(fu) - \alpha(Su) + \psi\big(\gamma(ga)\big)\gamma(ga) - \gamma(Ta)$
$< \alpha(fu) - \alpha(Su) + \gamma(ga) - \gamma(Ta)$
$< \alpha(u) - \alpha(u) + \gamma(u) - \gamma(Ta)$
$< \gamma(u)$
$= 0$.
Therefore $d(u, Ta) = 0$.
Therefore $u = Ta$.
Since (T, g) are w - compatible mappings so we have $Tu = gu$.
Now to prove that $gu = u$.
$d(u, gu) = d(Su, Tu)$
$\leq \psi(\alpha(fu))\alpha(fu) - \alpha(Su) + \psi(\gamma(gu))\gamma(gu) - \gamma(Tu)$
$< \alpha(fu) - \alpha(Su) + \gamma(gu) - \gamma(Tu)$
$< \alpha(u) - \alpha(u) + \gamma(u) - \gamma(u)$
$= 0$.
Therefore $d(u, gu) = 0$ so $u = gu$.
So we have $u = gu = Tu$.
This shows that u is the CCFP of the functions S, T, f and g.
Now to prove the uniqueness of u.
Suppose u^* be an another coupled fixed point of S, T, f and g.
$d(u, u^*) = d(Su, Tu^*)$
$\leq \psi\big(\alpha(fu)\big)\alpha(fu) - \alpha(Su) + \psi\big(\gamma(gu^*)\big)\gamma(gu^*) - \gamma(Tu^*)$
$< \alpha(u) - \alpha(u) + \gamma(u^*) - \gamma(u^*)$
$= 0$.
Therefore $u = u^*$.
This shows that u is the unique common fixed point of S, T, f and g.
This completes the proof.

Example 2.2
Let $X = [0,1]$ and define $d: X \times X \to R^+ \cup \{0\}$ as $d(x, y) = |x - y|$ and define S, T, f, g: $X \to X$ as $Sx = \frac{1}{3}x$, $Tx = \frac{1}{4}x$, $fx = \frac{1}{2}x$ and $gx = \frac{3}{4}x$.
Define two lower semi continuous functions $\alpha, \gamma: X \to [0, \infty)$ as $\alpha(a) = a$ and $\gamma(b) = \frac{b}{2}$. Define $\psi: (-\infty, \infty) \to (0,1)$ as $\psi(x) < x$
Then clearly S, T, f, g satisfies the (ii),(iii),(iv) conditions of Theorem 2.1
Then S, T, f and g have a unique common fixed point if they satisfies the condition (i) of Theorem 2.1. Now for any $x, y \in X$
L.H.S = $d(Sx, Ty) = \left|\frac{1}{3}x - \frac{1}{4}y\right|$.
R.H.S = $\psi(\alpha(fx))\alpha(fx) - \alpha(Sx) + \psi(\gamma(gy))\gamma(gy) - \gamma(Ty)$
$= \psi(\alpha(\frac{1}{2}x))\alpha(\frac{1}{2}x) - \alpha(\frac{1}{3}x) + \psi(\gamma(\frac{3}{4}y))\gamma(\frac{3}{4}y) - \gamma(\frac{1}{4}y)$
$< \frac{2x^2+y^2}{8} - \frac{8x+3y}{24}$.
Therefore L.H.S < R.H.S for all $x, y, u, v \in [0,1]$.
Hence condition (i) of Theorem (2.1) holds, therefore S, T, f and g have a unique common fixed point (0,0).

Corollary 1
Let (X, d) be a metric space and let □ = $\{f_i: X \to X / i \in \Delta\}$ and □ = $\{S_i: X \to X / i \in \Delta\}$ be a family of mappings and let □ = $\{\alpha_i: X \to X / i \in \Delta\}$ be a family of lower semi continuous function satisfies the following condition.
for any $S_1, S_2 \in$ □ , there exists $f_1, f_2 \in$ □ satisfies
$d(S_1x, S_2u) \leq \psi(\alpha_1(f_1x))\alpha_1(f_1x) - \alpha_1(S_1x) + \psi\big(\alpha_2(f_2x)\big)\alpha_2(f_2x) - \alpha_2(S_2y)$

Suppose □, □, □ satisfies the conditions of Theorem 2.1. Then the family of mappings □, □ have a unique common fixed point.

Proof:

Suppose $S_1, S_2 \in$ □ then there exists $f_1, f_2 \in$ □ and there exist four lower semi continuous function satisfies the above contraction. Then by Theorem 2.1 S_1, S_2, f_1, f_2 have a unique common fixed point say u.

Suppose $S_3 \in$ □ then there exists $f_3 \in$ □ such that S_1, S_3, f_1, f_3 have a unique common fixed point say v.

Since u is the unique coupled fixed point of S_1 so $u = v$.

Hence the family of mappings □, □ have a unique common fixed point.

Corollary 2

Let (X, d) be a metric space and let $T: X \to X$ and $f, g: X \to X$ be mappings satisfies

(2.1) $d(Tx, Tu)$

$\leq \psi(\alpha(fx))\alpha(fx) - \alpha(Tx) + \psi(\gamma(gu))\gamma(gu) - \gamma(Tu)$

where $\alpha, \gamma: X \to [0, \infty)$ are lower semi continuous functions and $\psi: (-\infty, \infty) \to (0,1)$ be continuous function.

(2.2) $T(X \times X) \subseteq g(X)$ and $T(X \times X) \subseteq f(X)$

(2.3) either (T, f) or (T, g) are w-compatible

(2.4) either $f(X)$ or $g(X)$ is complete. Then T, f, g have unique common fixed point.

Corollary 3

Let (X, d) be a metric space and let $T, f: X \to X$ be mappings satisfies

(3.1) $d(Tx, Tu) \leq \psi\big(\alpha(fx)\big)\alpha(fx) - \alpha(Tx)$

$+\psi\big(\gamma(fu)\big)\gamma(fu) - \gamma(Tu)$

where $\alpha, \gamma, : X \to [0, \infty)$ are lower semi continuous functions and $\psi: (-\infty, \infty) \to (0,1)$ be continuous function.

(3.2) $T(X \times X) \subseteq f(X)$

(3.3) (T, f) are w-compatible

(3.4) Suppose $f(X)$ is complete.

Then T, f have a unique commomn fixed point.

Theorem 2.3

Let (X, d) be a metric space and let $S, T, f, g: X \to X$ be mappings satisfies

(2.3.1) $d(Sx, Tu) \leq \alpha(\psi(fx, gu))\psi(fx, gu) - \psi(Sx, Tu)$

where $\psi: X \times X \to [0, \infty)$ is a lower semi continuous functions and $\alpha: (-\infty, +\infty) \to (0,1)$ continuous function.

(2.3.2) $S(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$

(2.3.3) (S, f) and (T, g) are w-compatible

(2.3.4) either $f(X)$ or $g(X)$ is complete. Then S, T, f, g have unique common fixed point.

3. Application to Graph Theory

Theorem 3.1

Let G be an oriented graph on the set X with $E(G)$ containing all loops. Let (X, d) be a metric space.

Let $S, T, f, g: X \times X \to X$ are edge preserving and satisfies the Caristi mappings (2.1.1) and suppose $S(X) \subseteq g(X)$ or $T(X) \subseteq f(X)$. Either $f(X)$ or $g(X)$ is complete.

Then S, T, f, g have a unique common couple fixed point if and only if there exists $x_0 \in X$ with $(fx_0, Sx_0) \in E(G)$ and $(gx_0, Tx_0) \in E(G)$.

Proof:

Let G be an oriented graph on the set X with $E(G)$ containing all loops.

Case(i): Let $x_0 \in X$ be a unique common fixed point of S, T, f, g.

that is, $x_0 = fx_0 = Sx_0$ and $x_0 = gx_0 = Tx_0$

This shows that $(fx_0, Sx_0) \in E(G)$ and $(gx_0, Tx_0) \in E(G)$.

Case (ii): Let $x_0 \in X$ with $(fx_0, Sx_0) \in E(G)$ and $(gx_0, Tx_0) \in E(G)$.

Since $S(X) \subseteq g(X)$ or $T(X) \subseteq f(X)$

Define

$z_{2n} = gx_{2n+1} = S(x_{2n}, y_{2n})$,

$z_{2n+1} = fx_{2n+2} = T(x_{2n+1}, y_{2n+1})$, $n = 0,1,2, \cdots$

Since $x_{2n+1} \in X$ so by case(ii) assumption

$(fx_{2n+1}, Sx_{2n+1}) \in E(G)$, $(gx_{2n+1}, Tx_{2n+1}) \in E(G)$

which imples $(z_{2n}, z_{2n+1}) \in E(G)$.

Then by Theorem (2.1) S, T, f, g have a unique common fixed point.

Since $x_0 \in X$ with $(fx_0, Sx_0) \in E(G)$ and $(gx_0, Tx_0) \in E(G)$ and S, T, f, g have unique common fixed point implies that $x_0 \in X$ is the fixed point of S, T, f, g.

4. Conclusion

In this paper first we defined a Caristi type contraction on four self maps and proved that all the four maps have a unique common fixed point. Also we gave application to graph theory.

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