



Natural Decomposition Approximation Solution for First Order Nonlinear Differential Equations

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Abstract

In this research paper, we propose the Natural decomposition method (NDM) to solve nonlinear first order differential equations. We compare the results obtained by NDM with the exact solutions. This method is a combination of the natural transform method and adomian decomposition method. By showing the less error one can be concluded that the NDM is easy to use and efficient.

Keywords: Adomian polynomial; Logistic Equation; Natural transform; sumudu transforms.

1. Introduction

In the last several years with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. Nonlinear evolution equations are widely used to describe many important phenomena and dynamic process in physics, mechanics, chemistry, biology and so on. Most of the differential equations arise in a large number of mathematical and engineering applications.

In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear differential equations. Different researchers introduced integral transforms and explored their application in solving both ordinary and partial differential equation problems arising from different disciplines. Recently many researchers have been proposed many approximation methods like variational iteration methods, adomian decomposition methods, homotopy perturbation methods, and etc. But we present the integral transform method called natural transform method combination with adomian polynomial which is an efficient method to solve the nonlinear differential equations. For the nonlinear models, the NDM shows the reliable results in supplying exact solutions that converges rapidly.

2. Definitions And Properties of the N-Transform

The natural transform of the function $f(t)$ for $t \in (-\infty, \infty)$, then the general integral transform is defined by [1, 2]:

$$\mathfrak{N}[f(t)](s) = \int_{-\infty}^{\infty} k(s,t)f(t)dt$$

where $k(s,t)$ represent the kernel of the transform, s is the real or complex number which is independent of t . In this paper we employ this transform to solve initial value problems f constant coefficients.

The Natural transform of the function $f(t)$ for $t \in (-\infty, \infty)$ is defined by [3]

$$N[f(t)] = R(s,u) = \int_{-\infty}^{\infty} f(ut)e^{-st} dt \quad ; \quad s, u \in (-\infty, \infty) \quad (1)$$

provided the integral on the right side exists.

Here $N[f(t)]$ is called the Natural transform of time function. The variables s and u are the Natural transform variables.

Equation (1) can be written as

$$N[f(t)] = \int_{-\infty}^{\infty} f(ut)e^{-st} dt \quad ; \quad s, u \in (-\infty, \infty)$$

$$= \int_{-\infty}^0 f(ut)e^{-st} dt + \int_0^{\infty} f(ut)e^{-st} dt$$

$$= N^-[f(t)] + N^+[f(t)]$$

Then we define the Natural transform (N-transform)

$$N[f(t)] = N^+[f(t)] = \int_0^{\infty} f(ut)e^{-st} dt, \quad s, u \in (-\infty, \infty) \quad (2)$$

The original function $f(t)$ in equation (1) is called the inverse transform which is denoted by

$$f(t) = N^{-1}\{R(s,u)\} \quad (3)$$

Table 1: Special N-Transforms and the Conversion to Sumudu [4-6] and Laplace Transform

$f(t)$	$N[f(t)]$	$S[f(t)]$	$L[f(t)]$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{u}{s^2}$	$\frac{u}{s^2}$	$\frac{u}{s^2}$
e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{u^{n-1}}{s^n}$	u^{n-1}	$\frac{1}{s^n}$
$\sin t$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{1+s^2}$
$\cos t$	$\frac{s}{s^2+u^2}$	$\frac{1}{1+u^2}$	$\frac{s}{1+s^2}$
$t^n e^{at}$	$\frac{n! u^n}{(s-au)^{n+1}}$	$\frac{n! u^n}{(1-au)^{n+1}}$	$\frac{n!}{(s-a)^{n+1}}$

Table 2: Properties of N-Transforms

Functional Form	Natural Transform
$N[af(t) + bg(t)]$	$aF(s, u) + bG(s, u)$
$N[f(at)]$	$\frac{1}{a} F(s, u)$
$N[f'(t)]$	$\frac{s}{u} F(s, u) - \frac{f(0)}{u}$
$N[f''(t)]$	$\frac{s^2}{u^2} F(s, u) - s \frac{f(0)}{u^2} - \frac{f'(0)}{u}$
$N\left[\int_0^t f(p) dp\right]$	$\frac{u}{s} F(s, u)$

3. Methodology of the Natural Decomposition Method

Consider the general nonlinear ordinary differential equation of the form:

$$L(v) + R(v) + F(v) = g(t) \tag{4}$$

subject to the initial condition

$$v(0) = h(t) \tag{5}$$

Where L is an operator of the highest derivative, R is the reminder of the differential operator, $g(t)$ is nonhomogeneous term and $F(v)$ is the nonlinear term.

Suppose L is a differential operator of the first order, then by taking the natural transform i.e., N - Transform of eq. (3), we have,

$$\frac{sV(s, u)}{u} - \frac{V(0)}{u} + N^+[R(v)] + N^+[F(v)] = N^+[g(t)] \tag{6}$$

By substituting Eq.(5) into Eq. (6),we obtain

$$V(s, u) = \frac{h(t)}{s} + \frac{u}{s} N^+[g(t)] - \frac{u}{s} N^+[R(v) + F(v)] \tag{7}$$

Taking the inverse of the N-Transform of Eq.(7), we have

$$v(t) = G(t) - N^{-1}\left[\frac{u}{s} N^+[R(v) + F(v)]\right] \tag{8}$$

where $G(t)$ is the source term.

We now assume an infinite series solution of the unknown function $v(t)$ of the form,

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \tag{9}$$

Then by using Eq.(9), we can re-write Eq. (8) in the form,

$$\sum_{n=0}^{\infty} v_n(t) = G(t) - N^{-1}\left[\frac{u}{s} N^+\left[R \sum_{n=0}^{\infty} v_n(t) + \sum_{n=0}^{\infty} A_n(t)\right]\right] \tag{10}$$

where $A_n(t)$ is an Adomian Polynomial [7-10] which represent the nonlinear term and is defined by

$$A_n(t) = \frac{1}{n!} \frac{d^n}{d\lambda^n} N\left[\sum_{i=0}^n \lambda^i v_i\right]_{\lambda=0} \tag{11}$$

Comparing both sides of Eq. (10), we can easily build the recursive relation as follows,

$$\begin{aligned} v_0(t) &= G(t) \\ v_1(t) &= -N^{-1}\left[\frac{u}{s} N^+[Rv_0(t) + A_0(t)]\right] \\ v_2(t) &= -N^{-1}\left[\frac{u}{s} N^+[Rv_1(t) + A_1(t)]\right] \\ v_3(t) &= -N^{-1}\left[\frac{u}{s} N^+[Rv_2(t) + A_2(t)]\right] \end{aligned}$$

Eventually, we have the general recursive relation as follows:

$$v_{n+1}(t) = -N^{-1}\left[\frac{u}{s} N^+[Rv_n(t) + A_n(t)]\right], n \geq 0 \tag{12}$$

Hence, the exact or approximate solution is given by

$$v(t) = \sum_{n=0}^{\infty} v_n(t). \tag{13}$$

4. Worked Examples

4.1 Example-1

Consider a Logistic or Verhulst equation

$$y' = y^2 - y, \tag{14}$$

$$y(0) = -1 \tag{15}$$

Taking Natural transform to both sides of eq. (14), we obtain

$$\frac{s}{u} Y(s, u) - \frac{1}{u} y(0) = N^+[y^2(t)] - Y(s, u) \tag{16}$$

By substituting eq. (15) in (16) we obtain,

$$\begin{aligned} \left[\frac{s}{u} + 1\right] Y(s, u) &= N^+[y^2(t)] - \frac{1}{u} \\ \Rightarrow Y(s, u) &= \frac{u}{s+u} N^+[y^2(t)] - \frac{1}{s+u} \end{aligned} \tag{17}$$

Then by taking inverse natural transform of eq. (17), we have

ries solution f the unknown function $y(t)$ of

$$y(t) = -e^{-t} + N^{-1} \left[\frac{u}{s+u} N^+ [y^2(t)] \right] \tag{18}$$

We now assume an infinite se the form

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{19}$$

By using eq.(19), we can write eq. (18)

$$\sum_{n=0}^{\infty} y_n(t) = -e^{-t} + N^{-1} \left[\frac{u}{s+u} \left[N^+ \left[\sum_{n=0}^{\infty} A_n(t) \right] \right] \right] \tag{20}$$

where $A_n(t)$ is the Adomian polynomial representing the nonlinear term $y^2(t)$.

Then by comparing both sides of eq. (20), we can derive the general recursive relation as follows:

$$\begin{aligned} y_0(t) &= -e^{-t} \\ y_1(t) &= N^{-1} \left[\frac{u}{s+u} \left[N^+ [A_0(t)] \right] \right] \\ y_2(t) &= N^{-1} \left[\frac{u}{s+u} \left[N^+ [A_1(t)] \right] \right] \\ y_3(t) &= N^{-1} \left[\frac{u}{s+u} \left[N^+ [A_2(t)] \right] \right] \end{aligned}$$

Therefore, the general recursive relation is given by

$$y_{n+1}(t) = N^{-1} \left[\frac{u}{s+u} \left[N^+ [A_n(t)] \right] \right], \tag{21}$$

$n \geq 0$

Then by using the recursive relation derived in eq.(21), we can easily compute the remaining components of the unknown function $y(t)$ as follows

$$\begin{aligned} y_1(t) &= N^{-1} \left[\frac{u}{s+u} \left[N^+ [A_0(t)] \right] \right] \\ &= N^{-1} \left[\frac{u}{s+u} \left[N^+ [y_0^2(t)] \right] \right] \\ &= N^{-1} \left[\frac{u}{s+u} \left[N^+ [e^{-2t}] \right] \right] \\ &= N^{-1} \left[\frac{u}{s+u} \times \frac{1}{s+2u} \right] \\ &= N^{-1} \left[\frac{1}{s+u} - \frac{1}{s+2u} \right] = e^{-t} - e^{-2t} \end{aligned}$$

Similarly, we can find $y_2(t) = -e^{-3t} + 2e^{-2t} - e^{-t}$,

$$y_3(t) = -e^{-4t} + 3e^{-3t} - 3e^{-2t} + e^{-t} \text{ and so on.}$$

Then the approximate solution

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \dots = \sum_{n=0}^{\infty} y_n(t) \text{ leads to the}$$

exact solution of the form $y(t) = \frac{1}{1-2e^t}$

Table 3: Comparison Results between Approximate Solution by NDM and Exact Solution

Time	Approximate solution by NDM	Exact solution	Error
$t=1$	-0.0422	-.07257	0.0187
$t=2$	-0.00067	-.02552	0.0082
$t=3$	-0.00097	-.00092	0.0032
$t=4$	-0.00013	-.00338	0.0012
$t=5$	-0.00001	-.00124	0.0004

In Table 3, we take time periods from $t=1$ to 5 sec. for which the approximate solution is rapidle convergent to the exact solution. The numerical results of the approximate solution obtained by NDM and the exact solution is shown in Figure 1.

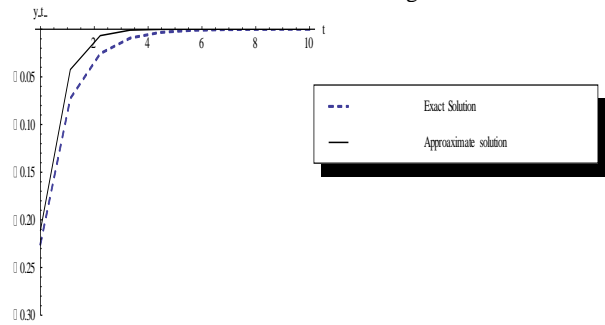


Fig. 1.: Comparison graph between Exact solution and NDM solution of logistic equation

4.2: Example-2

Consider the non-linear ODE

$$y'(t) = 1 + 2y - y^2, \quad y(0)=0 \tag{22}$$

Taking the Natural transformation both sides we get,

$$N^+ [y'(t)] = N^+ [1 + 2y - y^2]$$

Using the differentiation property of the Natural transform and above initial conditions, we have

$$R(s, u) = \frac{u}{(s-2u)s} - \frac{u}{s-2u} N^+ [y^2] \tag{23}$$

Applying the inverse Natural transform on both sides of equation (23), we get $y(t)$

$$= \frac{1}{2} (e^{2t} - 1) N^{-1} \left\{ \frac{u}{s-2u} N^+ [y^2] \right\}$$

For non-linearity part assume by Adomian's polynomial $y^2 = \sum_{n=0}^{\infty} A_n$, $y(t) = \sum_{n=0}^{\infty} y_n(t)$ where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{k=0}^{\infty} y_k \lambda^k \right) \right]_{\lambda=0}, \quad n \geq 0$$

Using

$$\sum_{n=0}^{\infty} y_n(t) = \frac{1}{2} (e^{2t} - 1) - N^{-1} \left\{ \frac{u}{s-2u} N^+ [\sum_{n=0}^{\infty} A_n] \right\}$$

Then $y_0(t) = \frac{1}{2} (e^{2t} - 1)$ and

$$y_{n+1}(t) = -N^{-1} \left\{ \frac{u}{s-2u} N^+ [\sum_{n=0}^{\infty} A_n] \right\} \text{ for } n \geq 0$$

$$y_1(t) = -N^{-1} \left\{ \frac{u}{s-2u} N^+ [A_0] \right\},$$

where, $A_0 = y_0^2$

$$= \frac{1}{4} (e^{4t} - 2e^{2t} + 1)$$

$$y_1(t) = -\frac{1}{8} e^{4t} + \frac{1}{2} t e^{2t} + \frac{1}{8}$$

Similarly, $y_2(t) = -N^{-1} \left\{ \frac{u}{s-2u} N^+ [A_1] \right\}$, where $A_1 = 2y_0y_1 = -\frac{1}{8} e^{6t} + \frac{1}{2} \left(t + \frac{1}{4} \right) e^{4t} + \frac{1}{2} \left(\frac{1}{4} - t \right) e^{2t} - \frac{1}{8}$

$$y_2(t) = \frac{1}{32} e^{6t} - \frac{1}{2} e^{4t} + \left(\frac{1}{4} t^2 - \frac{5}{8} t + \frac{27}{32} \right) e^{2t} + \frac{3}{16}$$

Then, the approximate solution becomes

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \dots$$

The exact solution of the equation is $y = 1 + \sqrt{2} \tanh \left[\sqrt{2}t + \frac{1}{2} \ln \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right]$. The exact solution is in closed agreement with the results obtained by NDM.

5. Conclusion

In this paper, we successfully found exact solution using NDM to all nonlinear models. The NDM introduces significant improvement in the fields over existing technique. Our aim in the future is to apply NDM for fractional differential equations arise in other areas of science and engineering.

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