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Research paper

Some Contributions to Boolean like near Rings

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Abstract

In this paper we extend Foster's Boolean-like ring to Near-rings. We introduce the concept of a Boolean like near-ring. A near-ring N is said to be a Boolean-like near-ring if the following conditions hold: (i) a+a = 0 for all a∈ N , (ii) ab(a+b+ab) = ba for all a, b ∈ N and (iii) abc = acb for all a,b, c ∈ N (right weak commutative law). We have proved that every Boolean ring is a Boolean like near-ring. An example is given to show that the converse is not true. We prove that if N is a Boolean near-ring then conditions (i) and (ii) of the above definition are equivalent. We also proved that a Boolean near-ring with condition (iii) is a Boolean ring. As a consequence we show that a Boolean –like near-ring N is a Boolean ring if and only if it is a Boolean near-ring. Obviously, every Boolean like ring is a Boolean like near-ring. We show that if N is a Boolean-like near-ring with identity, then N is a Boolean-like ring. In addition we prove several interesting properties of Boolean-like near-rings. We prove that the set of all nilpotent elements of a Boolean –like near-ring N forms an ideal and the quotient near-ring N/I is a Boolean ring. Every homomorphic image of a Boolean like near ring is a Boolean like near ring. We further prove that every Boolean-like near-ring is a Boolean-like semiring As example is given to show that the converse of this result is not true.

***Keywords**: Boolean near ring, Boolean like ring, Boolean ring.*

1. Introduction

Definition 1.1.:
A near-ring N is said to be a Boolean like near ring if the following conditions hold:
(i) $a+a = 0$ for all a ϵ N (i.e., Characteristic of N is 2)
(ii) $ab\,(a+b+ab) = ba$, for all $a,b \in N$, and
(iii) $abc = acb$, for all $a,b,c \in N$.
(Right weak commutative law)
By (iii) of the above definition, one can prove that every Boolean-like near-ring is zero-symmetric.
According to A.L.Foster [1] a Boolean-like ring is a commutative ring R with unity 1 and is of characteristic 2 with $a(1+a)b(1+b) = 0$ for all a,b ∈ R.
It is well known that every Boolean ring with unity is a Boolean-like ring. Clearly every Boolean- like ring is Boolean- like near-ring.
Following example shows that every Boolean- like near-ring, need not, in general be a Boolean- like ring.

Example 1.2:
Let $N = \{0, a, b, c\}$ be the Klein's four group. Addition and multiplication are given in the following tables [3].
Then N is a Boolean- like near- ring but not a Boolean- like ring.

Let us recall that a near- ring N is called a Boolean near - ring if $a^2 = a$, for all $a \in N$
We show that the conditions (i) and (ii) of def 1.1 are equivalent for Boolean near-rings.

+	0	a	b	c
0	0	a	b	c
a	A	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

Theorem 1.3:
A Boolean near-ring N has characteristic 2 if and only if $ab(a+b+ab) = ba$ for $a,b \in N$.

Proof:
Hansen and Luh [4] proved that a (left) Boolean near-ring satisfies (left) weak commutative law, ie., $abc = bac$.
Suppose N has characteristic 2 then $ab(a+b+ab) = aba + abb + abab$
$= baa + abb + aabb$ (by left weak commutativity i.e., $abc = bac$)
$= ba + ab + ab = ba$ (since N has characteristic 2)

Conversly, suppose that $ab(a+b+ab) = ba$, for all $a, b \in N$
By taking $b = a$, we get that $aa\,(a + a + aa) = aa \Rightarrow aaa + aaa + aaaa$

$= aa \Rightarrow a + a + a = a \Rightarrow a + a = 0$

Therefore, N has characteristic '2'.
It is well known that a Booolean ring is a Boolean (left) near-ring satisfying the right weak commutative law. The converse is proved in the following i.e., a Boolean near-ring with condition (iii) of def 1.1 is a Boolean ring.

Lemma 1.4:
A Boolean near-ring N with right weak commutativity is a Boolean ring.

Proof:
By [4], a Boolean near- ring satisfies (left) weak commutative law i.e., $abc = bac$. Therefore, $aba = baa = ba$ by (left) weak commutative law.
Also, $ab = aab = aba$ by (right) weak commutative law.
Therefore, $ab = ba$., for all $a, b \in N$.
Hence multiplication in N is commutative.
By using two distributive laws,
we have

$$(a+b)(b+a) = (a+b)b + (a+b)a = ab + bb + aa + ba \quad (1)$$

$$(a+b)(b+a) = a(b+a) + b(b+a) = ab + aa + bb + ba \quad (2)$$

From (1) and (2),
$ab + bb + aa + ba = ab + aa + bb + ba \Rightarrow b+a = a+b$

Therefore, addition is commutative.
Hence N is a Boolean ring.

By the above lemma, we have the following:

Corollary 1.5:
A Boolean near-ring satisfies (right) weak commutative law if and only if it is a Boolean ring.
Since every Boolean- like near-ring satisfies the (right) weak commutative law, we get the following:

Corollary 1.6:
A Boolean near-ring is a Boolean- like near-ring if and only if it is a Boolean ring.

2. Elementary Result

Lemma 2.1:
If $a \in N$, then a^2 is an idempotent.

Proof:
By def, $ab(a+b+ab) = ba$ for $a, b \in N$.
Substituting $b = a$, we get that $aa(a+a+aa) = aa$ i.e., $aaaa = aa$ and so $a^4 = a^2$ Therefore, $a^2 = a^4 = (a^2)^2$ Thus a^2 is an idempotent.

Corollary 2.2:
If $a \in N$, then a is nilpotent iff $a^2 = 0$

Proof:
If a is nilpotent, then $a^k = 0$ for some $k > 1$.
Choose an even integer $m>1$ such that $a^m = 0$. Then $m = 2n$, for some $n \geq 1$
Therefore $0 = a^m = a^{2n} = (a^2)^n = a^2$, since a^2 is an idempotent.
Converse is trivial.

Note 2.3:
Since characteristic of N is 2, addition in N is commutative.
So, every Boolean-like near-ring is an abelian near-ring.

Lemma 2.4:
For any $a, b \in N$, $(a+b)^2 = (a+b)^2 (a^2+b^2)$

Proof:
By lemma 4.2.1 and by using the definition 4.1.1 repeatedly, we get that $(a+b)^2 = (a+b)^4 = (a+b)^2(a+b)^2 = (a+b)^2[(a+b)a + (a+b)b] = (a+b)^2[(a+b)a + b(a+b)]$ [by 1.1(iii)]
$= (a+b)^2[a(a+b)[a+a+b+a(a+b)] + (a+b)^2[(a+b)b[a+b+b+(a+b)b]$
$= (a+b)^2[(a+b)a[b+(a+b)a] + (a+b)^2[(a+b)b[a+(a+b)b]$
$= (a+b)^2[(a+b)ab+(a+b)^2a^2] + (a+b)^2[(a+b)ba+(a+b)^2b^2]$
$= (a+b)^2[(a+b)ab + (a+b)^2(a^2+b^2) + (a+b)ab]$
$= (a+b)^2[(a+b)^2(a^2+b^2)] = (a+b)^4(a^2+b^2) = (a+b)^2(a^2+b^2)$

Corollary 2.5:
If $a \in N$, then $a+a^2$ is nilpotent.

Proof:
By lemmas 2.4 and 2.1 $(a+a^2)^2 = (a+a^2)^2(a^2+a^4)$
$= (a+a^2)^2(a^2+a^2) = 0$,
Therefore $(a+a^2)$ is nilpotent.

Corollary 2.6:
If $a, b \in N$, then $(a+a^2)(b+b^2) = 0$

Proof:
We know that $xy(x+y+xy) = yx$, for all $x, y \in N$.
Therefore $(a+a^2)(b+b^2) = (b+b^2)(a+a^2)(b+b^2+a+a^2+(b+b^2)(a+a^2))$
$= (b+b^2)^2(a+a^2) + (b+b^2)(a+a^2)^2 + (b+b^2)^2(a+a^2)^2 = 0$,
Since $(c+c^2)^2 = 0$, for all $c \in N$, by corollaries 2.5 and 2.2.
We now prove that every Boolean- like near-ring with identity is a Boolean- like ring.

Corollary 2.7:
If N has the identity 1, then N is a Boolean like ring.

Proof:
By definition 1.1, $abc = acb$ for a, $b, c \in N$. Then $1bc=1cb$ and this implies that $bc = cb$.Thus multiplication is commutative. Since the characteristic of N is 2, $(N,+)$ is an abelian group. So, N is a commutative ring with 1.
For $a, b \in N$, $a(1+a)b(1+b) = (a+a^2)(b+b^2) = 0$ [by cor.4.6]
Therefore N is a Boolean- like ring.

Corollary 2.8:
If a and b are nilpotent elements in N, then ab = 0

Proof:
Since a and b are nilpotent, by corollary 2.2, $a^2 = 0$ and $b^2 = 0$
Again by corollary 2.6, $(a+a^2)(b+b^2) = 0$ and this implies that $ab=0$.

Lemma 2.9:
If N has no nonzero idempotent elements, then $ab = 0$
for all $a, b \in N$

Proof:
If $a \in N$, then by lemma 2.1, a^2 is an idempotent.
By hypothesis, $a^2 = 0$ .Therefore a is nilpotent.
Thus, every element of N is nilpotent. Hence if $a, b \in N$, then a and b are nilpotent elements and by corollary 2.8, $ab = 0$. Therefore, $ab = 0$ for $a, b \in N$.

Corollary 2.10:
If N has no nonzero idempotent elements, then N is a zero ring.

Lemma 2.11:
Every homomorphic image of a Boolean – like near-ring is a Boolean-like near-ring.

Proof:
Let $\varphi : N \to N'$ be an epimorphism of a Boolean-like near-ring N onto a near-ring N′ . If a′ ∈ N′, there exists a ∈ N such that $\varphi(a) = a'$. Then $a' + a' = \varphi(a) + \varphi(a) = \varphi(a+a) = \varphi(0) = 0'$. Let $a',b' \in N'$. Then there exists $a, b \in N$ such that $\varphi(a) = a'$ and $\varphi(b) = b'$. Hence $a'b'(a'+b'+a'b') = \varphi(a)\,\varphi(b)\,[\varphi(a) + \varphi(b) + \varphi(a)\,\varphi(b)] = \varphi\{ab(a+b+ab)\} = \varphi(ba) = \varphi(b)\,\varphi(a) = b'a'$. Also $a'b'c' = \varphi(a)\varphi(b)\varphi(c) = \varphi(abc) = \varphi(acb) = \varphi(a)\varphi(c)\varphi(b) = a'c'b'$. Therefore N′ is a Boolean-like near-ring.

3. Homomorphism in Boolean like near Rings

From now onwards, we consider nontrivial near-rings. So, by cor. 2.10, we may assume that N contains nonzero idempotent elements.

We now prove the main theorem. Before that we prove some results.

Lemma 3.1:
For any b ϵ N, b^2 is a central idempotent.

Proof:
By lemma 2.1 b^2 is an idempotent element.
By def 1.1 $ac\,(a+c+ac) = ca$, for all $a,c \in N$. Substituting $c = b^2$ in this equation,
we get that $a\,b^2\,(a+ b^2+a\,b^2) = b^2\,a$. Then $a\,b^2\,a + a\,b^4 + a\,b^2\,a\,b^2 = b^2a$,and by lemma 2.1, this implies that $a^2\,b^2 + a\,b^2 + a^2\,b^2 = b^2a$. Thus, $ab^2 = b^2a$.

Theorem 3.2:
The map $\varphi : N \to N$ defined by $\varphi(a) = a^2$ for all $a \in N$ is a near-ring homomorphism.

Proof:
For a,b ϵ N $\varphi(a+b) = (a+b)^2$.
By using lemma 2.4, lemma 3.1, def 1.1(iii) and note 2.3
We get that

$(a+b)^2 = (a+b)^2\,(a^2 + b^2) = (a+b)^2\,a^2 + (a+b)^2\,b^2 = a^2(a+b)^2+b^2(a+b)^2$
$= a^2\,[(a+b)\,(a+b)] + b^2\,[(a+b)\,(a+b)]$
$= a^2\,[(a+b)a + (a+b)b] + b^2\,[(a+b)a + (a+b)b)$
$=a[a^2(a+b)+ab(a+b)]+b[ba(a+b)+b^2(a+b]$
$= a[\,a^3 + a^2\,b + a^2b + ab^2] + b\,[ba^2+ b^2a + b^2\,a+b^3\,]$
$= [a^4 + a^2\,b^2\,] + [b^4 + b^2\,a^2\,] = [a^4 + a^2\,b^2 + b^4 + b^2\,a^2\,]$
$= a^4 + b^4 = a^2 + b^2 = \varphi(a) + \varphi(b)$
Therefore, $\varphi(a+b) = \varphi(a) + \varphi(b)$
Also $\varphi(ab)= (ab)^2 = (ab)(ab) = abab = a^2\,b^2 = \varphi(a)\,\varphi(b)$

Therefore φ is a homomorphism.

Lemma 3.3:
If a ϵ N, then a can be represented uniquely as a = n+e, where n is nilpotent and e is an idempotent.

Proof:
By corollaries 2.5 , 2.1 and def 1.1, $a = (a + a^2) + a^2$
where $(a + a^2)$ is nilpotent and a^2 is an idempotent.
Suppose $a = n+e$, where n is nilpotent and e is an idempotent.
By lemma 3.1, e is a central element.

Then, $a^2 =(n+e)^2 = (n+e)^2\,(n^2+e^2) = (n+e)^2e^2=[e(n+e)]^2=(en+e)^2$
$=(en+e)(en+e)=(en+e)en+(en+e)e= en(en+e)+e(en+e) = e^2n^2+e^2n+ e^2n + e^2 =e$

Therefore, $e = a^2$. Also, $(a +a^2)+ a^2 = n+ a^2$

and this implies that $a + a^2 = n$
Thus, the representation is unique.
The following result follows from the above lemma.

Corollary 3.4:
If N is Boolean- like near- ring without nonzero nilpotent elements, then N is a Boolean ring.

Theorem 3.5:
The set I of all nilpotent elements of a Boolean- like near-ring N forms an ideal and N/I is a Boolean ring.

Proof:
By cor. 2.2, an element $a \in N$ is nilpotent iff $a^2 = 0$.
Therefore, $I = \{ a \in N / a^2 = 0\}$
$\Rightarrow I = \ker \varphi$, where φ is the near-ring homomorphism defined in theorem 3.2. Hence I is an ideal . Since N is a Boolean- like near-ring, *N/I* is also a Boolean like near-ring, by lemma 2.11. Further *N/I* has no nonzero nilpotent elements. By corollary 3.4., *N/I* is a Boolean ring.

Corollary 3.6:
The set of all idempotent elements of N forms a Boolean ring.

Proof:
By lemma 2.1, for any $a \in N$, a^2 is an idempotent. Therefore $\{ a^2 / a \in N \}$ is the set of all idempotents . Hence $\{ a^2 / a \in N\} =\varphi(N)$, where φ is the homomorphism defined in theorem 3.2. But $N/I \cong \varphi(N)$. By theorem 4.3.5, *N/I* is a Boolean ring and so $\varphi(N)$ is also Boolean ring.

Theorem 3.7:
Every Boolean- like near- ring is a commutative ring.

Proof:
Let N be a Boolean- like near- ring.
Since $(N,+)$ is abelian, it suffices to prove that multiplication in N is commutative. Let $x, y \in N$. By lemma 3.3, $x = n +e$, $y = m + f$ where n, m are nilpotent elements and e,f are idempotent elements.

Then $xy = (n +e)\,(m +f) = (n +e)m + (n +e)f = (n +e)m+nf+ef$, since f is central (by lemma 3.1).
Similarly, $yx = (m +f)\,(n +e) = (m +f)n + (m +f)e = (m+f)n + me +fe$

Clearly $ef = fe$, since e, f are central elements.
To show $xy = yx$, it suffices to show that $(n +e)m +nf = (m+f)n + me$
Consider, $(\,n +e)m = m(n+e)\,[\,m+(n+e) + m\,(n+e)]$
$=m^2(n+e) + m\,(n+e)^2 + m^2\,(n+e)^2$ (by def 3.1(iii)) $= m(n+e)^2$, since $m^2 = 0$, by theorem 2.2

Similarly $(m+f)n = nf$. Therefore $(n + e)m + nf = me + nf = nf + me = (m+f)n +me= me$, because by the proof of lemma 3.3, $(n+e)^2 = e$.
This completes the proof.

According to venkateswarlu,k., et al[7] a Boolean like semiring is a near-ring N such that (i) $a+a = 0$ for all $a \in N$, and (ii) $ab(a+b+ab) = ab$
for all $a,b \in N$

Corollary 3.8:
Every Boolean- like near-ring is a Boolean like semiring.

Proof: If N is a Boolean- like near-ring, then N is a commutative ring, by theorem 5.7
Therefore, $ab(a+b+ab) = ba$ (by definition)
$= ab$ (since multiplication is commutative)

Therefore, N is a Boolean like semiring.

But every Boolean like semiring need not, in general, be a Boolean- like near-ring. This can be seen from the following example.

Example 3.9:
Let $K =\{ 0, a, b , c \}$ be the klein's four group Addition and multiplication operations + and . are defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	a	b	c

Fig: K is a Boolean- like semiring but not a Boolean- like near-ring, since $cab \neq cba$.

Corollary 3.10:
Every Boolean-like semiring is a Boolean ring if it is a Boolean near-ring.

Proof:
Let N be a Boolean-like semiring. Since $a+a =0$ for all $a \in N$, $(N,+)$ is abelian. By theorem 4.1.3, $ab(a+b+ab) = ba$ for any $a,b \in N$. By the definition of a Boolean semiring, $ab(a+b+ab) = ab$. Therefore $ab = ba$ for any $a,b \in N$. Hence N is a Boolean ring.
As a consequence we prove the following interesting result.

Corollary 3.11:
A Boolean near-ring N is a Boolean ring if and only if $ab(a+b+ab) = ab$ for all a, b $\in$ N.

Proof:
Assume that $ab(a+b+ab) = ab$ for all $a,b \in N$ - (i)
By taking $b =a$ in (i), we get that $a^3 +a^3 +a^4 = a^2 \Rightarrow a+a+a = a \Rightarrow a+a = 0$.
Therefore N is a Boolean like semiring. By corollary 3.10, N is a Boolean ring. Converse is trivial.
By lemma 1.4 and corollary 3.11, we get the following

Theorem 3.12:
A Boolean near-ring is a Boolean ring in each of the following cases
(i) $abc=acb$ for all $a,b,c \in N$
(ii) $ab(a+b+ab) = ab$ for all $a,b \in N$.

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